PRINCIPAL FIBRATIONS AND GENERALIZED
H-SPACES

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Abstract. For a map $f : A \to X$, there are concepts of $H^f$-spaces, $T^f$-spaces, which are generalized ones of $H$-spaces [17,18]. In general, Any $H$-space is an $H^f$-space, any $H^f$-space is a $T^f$-space. For a principal fibration $E_k \to X$ induced by $k : X \to X'$ from $\epsilon : PX' \to X'$, we obtain some sufficient conditions to having liftings $H^f$-structures and $T^f$-structures on $E_k$ of $H^f$-structures and $T^f$-structures on $X$ respectively. We can also obtain some results about $H^f$-spaces and $T^f$-spaces in Postnikov systems for spaces, which are generalizations of Kahn’s result about $H$-spaces.

1. Introduction

A map $f : A \to X$ is cyclic [14] if there is a map $F : X \times A \to X$ such that $F|_X \sim 1_X$ and $F|_A \sim f$. It is clear that a space $X$ is an $H$-space if and only if the identity map $1_X$ of $X$ is cyclic. We called a space $X$ as an $H^f$-space for a map $f : A \to X$ [17] if there is a cyclic map $f : A \to X$, that is, there is an $H^f$-structure $F : X \times A \to X$ such that $Fj \sim \nabla(1 \vee f)$, where $j : X \vee A \to X \times A$ is the inclusion. We showed [17] that if a space $X$ is an $H$-space, then for any space $A$ and any map $f : A \to X$, $X$ is an $H^f$-space for a map $f : A \to X$, but the converse does not hold. In [1], Aguade introduced a $T$-space as a space $X$ having the property that the evaluation fibration $\Omega X \to X^{S^1} \to X$ is fibre homotopically trivial. It is easy to show that any $H$-space is a $T$-space. However, there are many $T$-spaces which are not $H$-spaces in [16]. Let $\Sigma X$ denotes the reduced suspension of $X$, and $\Omega X$ denotes the based loop space of $X$. Let $\tau$ be the adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$. The symbols $e$ and $e'$ denote $\tau^{-1}(1_{\Omega X})$ and $\tau(1_{\Sigma X})$.

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respectively. It is well known [1] that a space $X$ is a $T$-space if and only if the evaluating map $e : \Sigma \Omega X \to X$ is cyclic. We called a space $X$ as a $T^f$-space for a map $f : A \to X$ [18] if $e : \Sigma \Omega X \to X$ is $f$-cyclic, that is, there is a $T^f$-structure $F : \Sigma \Omega X \times A \to X$ such that $Fj \sim \nabla(e \vee f)$, where $j : \Sigma \Omega X \vee A \to \Sigma \Omega X \times A$ is the inclusion. We also showed [18] that if $X$ is a $T$-space, then for any space $A$ and any map $f : A \to X$, $X$ is a $T^f$-space for a map $f : A \to X$, but the converse does not hold. We called a space $X$ as a $G^f$-space for a map $f : A \to X$ [19] if $e : \Sigma \Omega X \to X$ is weakly $f$-cyclic, that is, $e_\#(\pi_n(\Sigma \Omega X)) \subset G_n(A, f, X)$ for all $n$. For a map $f : A \to X$, there are concepts of $H^f$-spaces, $T^f$-spaces and $G^f$-spaces which are generalized ones of $H$-spaces. In general, Any $H$-space is an $H^f$-space, any $H^f$-space is a $T^f$-space and any $T^f$-space is a $G^f$-space. In this paper, for a principal fibration $E_k \to X$ induced by $k : X \to X'$ from $\epsilon : PX' \to X'$, we obtain some sufficient conditions to having liftings $H^f$-structures and $T^f$-structures on $E_k$ of $H^f$-structures and $T^f$-structures on $X$ respectively. We can also obtain some results about $H^f$-spaces and $T^f$-spaces in Postnikov systems for spaces, which are generalizations of Kahn’s result about $H$-spaces.

2. Gottlieb sets for maps and generalized $H$-spaces

Let $f : A \to X$ be a map. A based map $g : B \to X$ is called $f$-cyclic [12] if there is a map $\phi : B \times A \to X$ such that the diagram

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\phi} & X \\
j \downarrow & & \nabla \downarrow \\
A \vee B & \xrightarrow{(f \vee g)} & X \vee X
\end{array}
$$

is homotopy commute, where $j : A \vee B \to A \times B$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. We call such a map $\phi$ an associated map of a $f$-cyclic map $g$. Clearly, $g$ is $f$-cyclic iff $f$ is $g$-cyclic. In the case, $f = 1_X : X \to X$, $g : B \to X$ is called cyclic [14]. We denote the set of all homotopy classes of $f$-cyclic maps from $B$ to $X$ by $G(B; A, f, X)$ which is called the Gottlieb set for a map $f : A \to X$. In the case $f = 1_X : X \to X$, we called such a set $G(B; X, 1, X)$ the Gottlieb set denoted $G(B; X)$. In particular, $G(S^n; A, f, X)$ will be denoted by $G_n(A, f, X)$. Gottlieb [3,4] introduced and studied the evaluation subgroups $G_n(X) = G_n(X, 1, X)$ of $\pi_n(X)$. 
In general, $G(B; X) \subset G(B; A, f, X) \subset [B, X]$ for any map $f : A \to X$ and any space $B$. However, there is an example [20] such that $G(B, X) \neq G(B; A, f, X) \neq [B, X]$.

The next proposition is an immediate consequence from the definition.

**Proposition 2.1.**

1. For any maps $f : A \to X$, $\theta : C \to A$ and any space $B$, $G(B, X) \subset G(B; A, f, X) \subset G(B; C, f\theta, X)$.
2. $G(B, X) = G(B; X, 1_X, X) \subset G(B; A, f, X) \subset G(B; A, *, X) = [B, X]$ for any spaces $X$, $A$ and $B$.
3. $G(B, X) = \cap \{G(B; A, f, X) \mid f : A \to X \text{ is a map and } A \text{ is a space}\}$.
4. If $h : C \to A$ is a homotopy equivalence, then $G(B; A, f, X) = G(B; C, fh, X)$.
5. For any map $k : X \to Y$, $k#(G(B; A, f, X)) \subset G(B; A, kf, Y)$.
6. For any map $k : X \to Y$, $k#(G(B; X)) \subset G(B; X, k, Y)$.
7. For any map $s : C \to B$, $s#(G(B; A, f, X)) \subset G(C; A, f, X)$.

**Proposition 2.2.**

1. $[9] X$ is an $H$-space $\iff G(B, X) = [B, X]$ for any space $B$.
2. $[16] X$ is a $T$-space $\iff G(\Sigma C, X) = [\Sigma C, X]$ for any space $C$.
3. $[4] X$ is a $G$-space $\iff G_n(X) = \pi_n(X)$ for all $n$.

It is clear that any $H$-space is a $T$-space and any $T$-space is a $G$-space.

**Proposition 2.3.** Let $f : A \to X$ be a map. Then

1. $[17] X$ is an $H^f$-space $\iff G(B; A, f, X) = [B, X]$ for any space $B$.
2. $[18] X$ is a $T^f$-space $\iff G(\Sigma C; A, f, X) = [\Sigma C, X]$ for any space $C$.
3. $[19] X$ is a $G^f$-space $\iff G_n(A, f, X) = \pi_n(X)$ for all $n$.

It is clear that any $H^f$-space is a $T^f$-space and any $T^f$-space is a $G^f$-space.

### 3. Principal fibrations and generalized $H$-spaces

Let $f : A \to X$, $f' : A' \to X'$, $l : A \to A'$, $k : X \to X'$ be maps. Then a pair of maps $(k, l) : (X, A) \to (X', A')$ is called a map from $f$ to
If the following diagram is commutative;

\[
\begin{array}{ccc}
A \xrightarrow{f} & X \\
\downarrow l & \downarrow k & \\
A' \xrightarrow{f'} & X'.
\end{array}
\]

It will be denoted by \((k, l) : f \to f'\).

Given maps \(f : A \to X\), \(f' : A' \to X'\), let \((k, l) : f \to f'\) be a map from \(f\) to \(f'\). Let \(PX'\) and \(PA'\) be the spaces of paths in \(X'\) and \(A'\) which begin at \(*\) respectively. Let \(\epsilon_{X'} : PX' \to X'\) and \(\epsilon_{A'} : PA' \to A'\) be the fibrations given by evaluating a path at its end point. Let \(p_k : E_k \to X\) be the fibration induced by \(k : X \to X'\) from \(\epsilon_{X'}\). Let \(p_l : E_l \to \) induced by \(l : A \to A'\) from \(\epsilon_{A'}\). Then there is a map \(f : E_l \to E_k\) such that the following diagram is commutative

\[
\begin{array}{ccc}
E_l \xrightarrow{f} & E_k \\
p_l \downarrow & \downarrow p_k \\
A \xrightarrow{f} & X,
\end{array}
\]

where \(E_l = \{(a, \xi) \in A \times PA'|l(a) = \epsilon(\xi)\}\), \(E_k = \{(x, \eta) \in X \times PX'|k(x) = \epsilon(\eta)\}\), \(f(a, \xi) = (f(a), f' \circ \xi)\), \(p_k(x, \eta) = x\), \(p_l(a, \xi) = a\).

**Definition 3.1.** Let \(X\) be an \(H^l\)-space for a map \(f : A \to X\). Then a map \((k, l) : f \to f'\) is called an \(H^l\)-primitive if there is an associated map \(F : X \times A \to X\) such that \(Fj \sim \nabla(1 \vee f)\) and \(kF(p_k \times p_l) \sim * : E_k \times E_l \to X'\), where \(j : X \vee A \to X \times A\) is the inclusion.

**Definition 3.2.** Let \(X\) be a \(T^l\)-space for a map \(f : A \to X\). Then a map \((k, l) : f \to f'\) is called a \(T^l\)-primitive if there is an associated map \(F : \Sigma\Omega X \times A \to X\) such that \(Fj \sim \nabla(e \vee f)\) and \(kF(\Sigma\Omega p_k \times p_l) \sim * : \Sigma\Omega E_k \times E_l \to X'\), where \(j : \Sigma\Omega X \vee A \to \Sigma\Omega X \times A\) is the inclusion.

**Definition 3.3.** [19] Let \(X\) be a \(G^l\)-space for a map \(f : A \to X\). Then a map \((k, l) : f \to f'\) is called a \(G^l\)-primitive if for each \(m\) and each map \(g : S^m \to X\), there is a map \(F : S^m \times A \to X\) such that \(Fj \sim \nabla(g \vee f)\), \(kF(1 \times p_l) \sim * : S^m \times E_l \to X'\), where \(j : S^m \vee A \to S^m \times A\) is the inclusion.

It is well known that any map \(g : S^m \to X\), \(g \sim e\Sigma\tau(g) : S^m \to X\). Thus we know the above definition is equivalent to one in [19].

**Proposition 3.4.**
(1) If \( X \) is an \( H' \)-space for a map \( f : A \to X \) and \( (k,l) : f \to f' \) is an \( H' \)-primitive, then \( (k,l) : f \to f' \) is a \( T' \)-primitive.

(2) If \( X \) is a \( T' \)-space for a map \( f : A \to X \) and \( (k,l) : f \to f' \) is an \( T' \)-primitive, then \( (k,l) : f \to f' \) is a \( G' \)-primitive.

Proof. (1) Since \( (k,l) : f \to f' \) is an \( H' \)-primitive, there is an associated map \( F : X \times A \to X \) such that \( F_j \sim \nabla(1 \lor f) \) and \( kF(p_k \times p_l) \sim * : E_k \times E_l \to X' \). Let \( F' = F(e_X \times 1) : \Sigma \Omega X \times A \to X \). Then \( F'j' \sim Fj(e_X \lor 1) \sim \nabla(1 \lor f)(e_X \lor 1) = \nabla(e_X \lor f) \), where \( j' : \Sigma \Omega X \lor A \to \Sigma \Omega X \times A \) is the inclusion. Moreover, since \( (p_k \times p_l)(e_{E_k} \times 1_{E_l}) \sim (e_X \times 1_A)(\Sigma \Omega p_k \times p_l) : \Sigma E_k \times E_l \to X \times A \), we have that \( kF'(\Sigma \Omega p_k \times p_l) \sim kF(e_X \times 1)(\Sigma \Omega p_k \times p_l) \sim kF(p_k \times p_l)(e_{E_k} \times 1_{E_l}) \sim * \). Thus \( (k,l) : f \to f' \) is a \( T' \)-primitive.

(2) Since \( (k,l) : f \to f' \) is a \( T' \)-primitive, there is an associated map \( F : \Sigma \Omega X \times A \to X \) such that \( F_j \sim \nabla(e \lor f) \) and \( kF(\Sigma \Omega p_k \times p_l) \sim * : \Sigma \Omega E_k \times e_l \to X' \). For each \( m \) and each \( g : S^m \to X \), let \( F' = F(\Sigma \tau(g) \times 1) : S^m \times A \to X \). Then \( F'j' \sim Fj(\Sigma \tau(g) \lor 1) \sim \nabla(e \lor f)(\Sigma \tau(g) \lor 1) \sim \nabla(g \lor f) \), where \( j' : S^m \lor A \to S^m \times A \) is the inclusion. Moreover, since \( (1 \times p_l)(\Sigma \tau(g) \times 1_{E_l}) \sim (\Sigma \tau(g) \times 1_A)(1_{S^m} \times p_l) : S^m \times E_l \to \Sigma \Omega X \times A \), we have that \( kF'(1_{S^m} \times p_l) = kF(\Sigma \tau(g) \times 1)(1_{S^m} \times p_l) \sim (kF(\Sigma \Omega p_k \times p_l)(\Sigma \tau(g) \times 1_{E_l}) \sim *(\Sigma \tau(g) \times 1_{E_l}) \sim * \). Thus \( (k,l) : f \to f' \) is a \( G' \)-primitive. □

Lemma 3.5.

(1) A map \( l : C \to X \) can be lifted to a map \( C \to E_k \) if and only if \( kl \sim * \).

(2) [5] Given maps \( g_i : A_i \to E_k \), \( i = 1, 2 \) and \( g : A_1 \times A_2 \to E_k \) satisfying \( p_k g|A_i \sim p_k g_i \), \( i = 1, 2 \), then there is a map \( h : A_1 \times A_2 \to E_k \) such that \( p_k h = p_k g \) and \( h|A_i \sim g_i \), \( i = 1, 2 \).

Theorem 3.6.

(1) If \( X \) is an \( H' \)-space for a map \( f : A \to X \) and \( (k,l) : f \to f' \) is an \( H' \)-primitive, then \( E_k \) is an \( H' \)-space for \( f' : E_l \to E_k \).

(2) If \( X \) is a \( T' \)-space for a map \( f : A \to X \) and \( (k,l) : f \to f' \) is a \( T' \)-primitive, then \( E_k \) is a \( T' \)-space for \( f' : E_l \to E_k \).

Proof. (1) Since \( (k,l) : f \to f' \) is an \( H' \)-primitive, there is a map \( F : X \times A \to X \) such that \( F_j \sim \nabla(1 \lor f) \) and \( kF(p_k \times p_l) \sim * : E_k \times E_l \to X' \), where \( j : X \lor A \to X \times A \) is the inclusion. From Lemma 3.5(1), there is a lifting \( F' : E_k \times E_l \to E_k \) of \( F(p_k \times p_l) : E_k \times E_l \to E_k \), that is, \( p_k F' = F(p_k \times p_l) \). Then \( p_k F'|E_k = F(p_k \times p_l)|E_k \sim F|X p_k \sim p_k 1_{E_k} \) and \( p_k F'|E_l = F(p_k \times p_l)|E_l \sim F|A p_l \sim p_l F \). Thus we have,
from Lemma 3.5(2), that there is a map \( \tilde{F} : E_k \times E_l \to E_k \) such that 
\[ p_k \tilde{F} = p_k F' = F(p_k \times p_l) \quad \text{and} \quad \tilde{F}|_{E_k} \sim 1_{E_k}, \quad \tilde{F}|_{E_l} \sim \tilde{f}. \]
Thus \( E_k \) is an \( H^I \)-space for \( \tilde{f} : E_l \to E_k \). This proves the theorem.

(2) Since \((k, l) : f \to f'\) is a \( T^I\)-primitive, there is a map \( F : \Sigma \Omega X \times A \to X \) such that
\[ Fj \sim \nabla(e \vee f) \quad \text{and} \quad kF(\Sigma \Omega p_k \times p_l) \sim * : \Sigma \Omega E_k \times E_l \to X', \]
where \( j : X \vee A \to X \times A \) is the inclusion. From Lemma 3.5(1), there is a lifting \( F' : \Sigma \Omega E_k \times E_l \to E_k \) of \( F(\Sigma \Omega p_k \times p_l) : \Sigma \Omega E_k \times E_l \to E_k \), that is, \( p_k F' = F(\Sigma \Omega p_k \times p_l) \). Then 
\[ F|_{\Sigma \Omega E_k} = F(\Sigma \Omega p_k \times p_l), \]
then \( F' \) is a fibration with fiber \( \Sigma \Omega E_k \). From Lemma 3.5(2), that there is a map \( \tilde{F} : \Sigma \Omega E_k \times E_l \to E_k \) such that \( p_k \tilde{F} = p_k F' = F(\Sigma \Omega p_k \times p_l) \) and \( \tilde{F}|_{\Sigma \Omega E_k} = \iota_{E_k}, \tilde{F}|_{E_l} \sim \tilde{f} \). Thus \( E_k \) is a \( T^I \)-space for \( \tilde{f} : E_l \to E_k \). This proves the theorem.

**Proposition 3.7.** [19] If \( X \) is a \( G^I \)-space for a map \( f : A \to X \) and \((k, l) : f \to f'\) is a \( G^I\)-primitive, then \( E_k \) is a \( G^I \)-space for \( \tilde{f} : E_l \to E_k \).

In 1951, Postnikov [13] introduced the notion of the Postnikov system as follows: A *Postnikov system* for \( X \) (or homotopy decomposition of \( X \)) consists of a sequence of spaces and maps satisfying:

1. \( i_n : X \to X_n \) induces an isomorphism \( (i_n)_\# : \pi_i(X) \to \pi_i(X_n) \) for \( i \leq n \).
2. \( p_n : X_n \to X_{n-1} \) is a fibration with fiber \( K(\pi_n(X), n) \).
3. \( p_n i_n \sim i_{n+1} \). It is a well known fact [11] that if \( X \) is a 1-connected space having a homotopy type of CW-complex, then there is a Postnikov system \( \{X_n, i_n, p_n\} \) for \( X \) such that \( p_{n+1} : X_{n+1} \to X_n \) is the fibration induced from the path space fibration over \( K(\pi_{n+1}(X), n + 2) \) by a map \( k_{X}^{n+2} : X_n \to K(\pi_{n+1}(X), n + 2) \). It is well known [7] that if \( A \) and \( X \) are spaces having the homotopy type of 1-connected countable CW-complexes and \( f : A \to X \) is a map, then there exist Postnikov systems \( \{A_n, i'_n, p'_n\} \) for \( A \) and \( X \) respectively and induced maps \( \{f_n : A_n \to X_n\} \) satisfying (1) for each \( n \), the following diagram is homotopy commutative

\[
\begin{array}{ccc}
A_n & \xrightarrow{f_n} & X_n \\
\downarrow{k_{A}^{n+2}} & & \downarrow{k_{X}^{n+2}} \\
K(\pi_{n+1}(A), n + 2) & \xrightarrow{\tilde{f}_\#} & K(\pi_{n+1}(X), n + 2),
\end{array}
\]

that is, \((k_{X}^{n+2}, k_{A}^{n+2}) : f_n \to \tilde{f}_\# \). (2) \( f_{n+1} : A_{n+1} \to X_{n+1} \) given by \( f_{n+1} = f_n \) satisfying commute diagram
(3) for each $n$, the following diagram is homotopy commutative

$$
\begin{array}{ccc}
A_{n+1}(=E_{k+2}^{n+2}) & \xrightarrow{f_{n+1}=f_n} & X_{n+1}(=E_{k+2}^{n+2}) \\
\downarrow p_n(=p_{k+2}^n) & & \downarrow p_n(=p_{k+2}^n) \\
A_n & \xrightarrow{f_n} & X_n.
\end{array}
$$

**Theorem 3.8.** Let $A$ and $X$ be spaces having the homotopy type of 1-connected countable CW-complexes and $f:A\to X$ a map, and $\{A_n,i'_n,p'_n\}$ and $\{X_n,i_n,p_n\}$ Postnikov systems for $A$ and $X$ respectively.

1. If $X$ is an $H^f$-space for a map $f:A\to X$, then each $X_n$ is an $H^{f_n}$-space and the all pair of $k$ invariants $(k_{X}^{n+2},k_A^{n+2}) : f_n \to \tilde{f}_n$ are $H^{f_n}$-primitive.

2. If $X_{n-1}$ is an $H^{f_{n-1}}$-space and the pair of $k$-invariants $(k_{X}^{n+1},k_A^{n+1}) : f_{n-1} \to \tilde{f}_{n-1}$ is $H^{f_{n-1}}$-primitive, then $X_n$ is an $H^{f_n}$-space, where $f_n$ is an induced map from $f$.

**Proof.** (1) Clearly $\{X_n \times A_n,i_n \times i'_n,p_n \times p'_n\}$ is a Postnikov system for $X \times A$. Then we have, by Kahn’s result [7, Theorem 2.2], that there are families of maps $f_n : A_n \to X_n$ and $F_n : X_n \times A_n \to X_n$ such that $p_n f_n = f_{n-1} p_n$ and $i_n f \sim f_n i'_n$, and $p_n F_n = F_{n-1} (p_n \times p'_n)$ and $i_n F \sim F_n (i_n \times i'_n)$ for $n = 2, 3, \cdots$ respectively, and $k_{X}^{n+2} f_n \sim \tilde{f}_n k_{A}^{n+2}$, $k_{X}^{n+2} F_n \sim \tilde{F}_n (k_{X}^{n+2} \times k_{A}^{n+2})$, where $k_{X}^{n+2} : A_n \to K(\pi_{n+1}(A),n+2)$ and $k_{A}^{n+2} : X_n \to K(\pi_{n+1}(X),n+2)$ are $k$-invariants of $A$ and $X$ respectively, $\tilde{f}_n : K(\pi_{n+1}(A),n+2) \to K(\pi_{n+1}(X),n+2)$ and $\tilde{F}_n : K(\pi_{n+1}(X),n+2) \times K(\pi_{n+1}(X),n+2) \to K(\pi_{n+1}(X \times A),n+2)$ are the induced maps by $f : A \to X$ and $F : X \times A \to X$ respectively. Since $F|_{X} \sim 1$ and $F|_{A} \sim f$, we know, from Kahn’s another result [8, Theorem 1.2], that $F_n|_{X_n} = (F|_{X})_n \sim 1$ and $F_n|_{A_n} = (F|_{A})_n \sim f_n$. Thus for each $n$, there exists an $H^{f_n}$-structure $F_n : X_n \times A_n \to X_n$ on $X_n$ such that $F_n f_n \sim \nabla (1 \vee f_n)$, where $f_n : X_n \vee A_n \to X_n \times A_n$ is the inclusion and $f_n$ is an induced map from $f$, and $X_n$ is an $H^{f_n}$-space. Moreover, since there is a lifting $F_{n+1} : X_{n+1} \times A_{n+1} \to X_{n+1}$ of $F_n$ such that $p_n+1 F_{n+1} \sim F_n (p_n+1 \times p'_n+1)$, we know, from Lemma
3.5(1), that $k_X^{n+2}F_n(p_{n+1} \times p'_{n+1}) \sim \ast$ and all the pair of $k$-invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_#$ are $H^{f_n}$-primitive, where $\tilde{f}_# : K(\pi_{n+1}(A), n + 2) \to K(\pi_{n+1}(X), n + 2)$ is the induced map by $f : A \to X$.

(2) It follows from Theorem 3.6(1).

Taking $f = 1_X$, $f' = 1_{K(\pi_{n+1}(X), n+2)}$, $l = k = k_X^{n+2}$, we can obtain, from the fact [15] $p_{n+1} : X_{n+1} \to X_n$ is an $H$-map if and only if $k_X^{n+1}$ is primitive and the above theorem, the following corollary given by Kahn [8].

**COROLLARY 3.9.** [8, Theorem 1.3] Let $X$ be space having the homotopy type of 1-connected countable CW-complexes and $\{X_n, i_n, p_n\}$ Postnikov systems for $X$.

(1) If $X$ is an $H$-space, then each $X_n$ is $H$-space and all the $k$ invariants $k_X^{n+2}$ is primitive.

(2) If $X_{n-1}$ is an $H$-space and the $k$-invariants $k_X^{n+1}$ is primitive, then $X_n$ is an $H$-space, where $f_n$ is an induced map from $f$.

**THEOREM 3.10.** Let $A$ and $X$ be spaces having the homotopy type of 1-connected countable CW-complexes and $f : A \to X$ a map, and $\{A_n, i'_n, p'_n\}$ and $\{X_n, i_n, p_n\}$ Postnikov systems for $A$ and $X$ respectively.

(1) If $X$ is a $T^l$-space for a map $f : A \to X$, then each $X_n$ is $T^{f_n}$-space and the all pair of $k$-invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_#$ are $T^{f_n}$-primitive.

(2) If $X_{n-1}$ is a $T^{f_n-1}$-space and the pair of $k$-invariants $(k_X^{n+1}, k_A^{n+1}) : f_{n-1} \to \tilde{f}_#$ is $T^{f_n-1}$-primitive, then $X_n$ is a $T^{f_n}$-space, where $f_n$ is an induced map from $f$.

**Proof.** (1) Clearly $\{\Sigma \Omega X_n \times A_n, \Sigma \Omega i_n \times i'_n, \Sigma \Omega p_n \times p'_n\}$ is a Postnikov system for $\Sigma \Omega X \times A$. Then we have, by Kahn’s result [7, Theorem 2.2], that there are families of maps $f_n : A_n \to X_n$ and $F_n : \Sigma \Omega X_n \times A_n \to X_n$ such that $p_n f_n = f_{n-1} p'_n$ and $i_n f \sim f_{n} i'_n$, and $p_n F_n = F_{n-1}(\Sigma \Omega p_n \times p'_n)$ and $i_n F \sim F_n(\Sigma \Omega i_n \times i'_n)$ for $n = 2, 3, \cdots$ respectively, and $k_X^{n+2} f_n \sim \tilde{f} k_A^{n+2}$, $k_X^{n+2} F_n \sim \tilde{F}_# (k_{\Sigma \Omega X}^{n+2} \times k_A^{n+2})$, where $k_A^{n+2} : A_n \to K(\pi_{n+1}(A), n + 2)$ and $k_X^{n+2} : X_n \to K(\pi_{n+1}(\Sigma \Omega X), n + 2)$ are $k$-invariants of $A$, $X$ and $\Sigma \Omega X$ respectively, $\tilde{f}_# : K(\pi_{n+1}(A), n + 2) \to K(\pi_{n+1}(X), n + 2)$ and $\tilde{F}_# : K(\pi_{n+1}(\Sigma \Omega X), n + 2) \times K(\pi_{n+1}(A), n + 2) \approx K(\pi_{n+1}(\Sigma \Omega X \times A), n + 2) \to K(\pi_{n+1}(X), n + 2)$ are the induced maps by $f : A \to X$ and $F : \Sigma \Omega X \times A \to X$ respectively. Since $F|_{\Sigma \Omega X} \sim e$ and $F|_A \sim f$,
we know, from Kahn’s another result [8, Theorem 1.2], that $F_n\Sigma\Omega X_n = (F|\Sigma\Omega X)_n \sim 1$ and $F_nA_n = (F|A)_n \sim f_n$. Thus for each $n$, there exists a $Tf_n$-structure $F_n : \Sigma\Omega X_n \times A_n \rightarrow X_n$ on $X_n$ such that $F_nj_n \sim \nabla(e \lor f_n)$, where $j_n : \Sigma\Omega X_n \lor A_n \rightarrow \Sigma\Omega X_n \times A_n$ is the inclusion and $f_n$ is an induced map from $f$, and $X_n$ is a $Tf_n$-space. Moreover, since there is a lifting $F_{n+1} : \Sigma\Omega X_{n+1} \times A_{n+1} \rightarrow X_{n+1}$ of $F_n$ such that $p_{n+1}F_{n+1} \sim F_n(\Sigma\Omega p_{n+1} \times p'_{n+1})$, we know, from Lemma 3.5(1), that $k_X^{n+2}F_n(\Sigma\Omega p_{n+1} \times p'_{n+1}) \sim *$ and all the pair of $k$-invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \rightarrow \tilde{f}_\#$ are $Tf_n$-primitive, where $\tilde{f}_\# : K(\pi_{n+1}(A), n + 2) \sim K(\pi_{n+1}(X), n + 2)$ is the induced map by $f : A \rightarrow X$.

(2) It follows from Theorem 3.6(2). □

In [19], the similar result with the above is known as follows.

**Proposition 3.11.** [19] Let $A$ and $X$ be spaces having the homotopy type of 1-connected countable CW-complexes and $f : A \rightarrow X$ a map, and $\{A_n, i'_n, p'_n\}$ and $\{X_n, i_n, p_n\}$ Postnikov systems for $A$ and $X$ respectively.

1. If $X$ is a $Gf'$-space for a map $f : A \rightarrow X$, then each $X_n$ is $Gf_n$-space and the all pair of $k$-invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \rightarrow \tilde{f}_\#$ are $Gf_n$-primitive.

2. If $X_{n-1}$ is a $Gf_{n-1}$-space and the pair of $k$-invariants $(k_X^{n+1}, k_A^{n+1}) : f_{n-1} \rightarrow \tilde{f}_\#$ are $Gf_{n-1}$-primitive, then $X_n$ is a $Gf_n$-space, where $f_n$ is an induced map from $f$.

Taking $f = 1_X$, $f' = 1_{K(\pi_{n+1}(X), n+2)}$, $l = k = k_X^{n+2}$, we can obtain the following corollary given by Haslam [5].

**Corollary 3.12.** [5] Let $X$ be space having the homotopy type of 1-connected countable CW-complexes and $\{X_n, i_n, p_n\}$ Postnikov systems for $X$.

1. If $X$ is a $G$-space, then each $X_n$ is $G$-space and all the $k$-invariants $k_X^{n+2}$ are $G$-primitive.

2. If $X_{n-1}$ is a $G$-space and the $k$-invariants $k_X^{n+1}$ is $G$-primitive, then $X_n$ is a $G$-space, where $f_n$ is an induced map from $f$.

**References**


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