SYMMETRIC BI-$f$-MULTIPLIERS OF INCLINE ALGEBRAS

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ABSTRACT. In this paper, we introduce the concept of a symmetric bi-$f$-multiplier in incline algebras and give some properties of incline algebras. Also, we characterize $\text{Ker}(D)$ and $\text{Fix}_a(D)$ by symmetric bi-$f$-multipliers in incline algebras.

1. Introduction

Z. Q. Cao, K. H. Kim and F. W. Roush [2] introduced the notion of incline algebras in their book. Some authors studied incline algebras and application. N. O. Alshehri [1] introduced the notion of derivation in incline algebras. In this paper, we introduce the concept of a symmetric bi-$f$-derivation in incline algebra and give some properties of incline algebras. Also, we characterize $\text{Ker}_D(K)$ and $\text{Fix}_D(K)$ by symmetric bi-$f$-derivations in incline algebras.

2. Incline algebras

An incline algebra is a set $K$ with two binary operations denoted by “+” and “∗” satisfying the following axioms:

(K1) $x + y = y + x$,
(K2) $x + (y + z) = (x + y) + z$,
(K3) $x ∗ (y ∗ z) = (x ∗ y) ∗ z$,
(K4) $x ∗ (y + z) = (x ∗ y) + (x ∗ z)$,
(K5) $(y + z) ∗ x = (y ∗ x) + (z ∗ x)$,
(K6) $x + x = x$,

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(K7) $x + (x \ast y) = x,$  
(K8) $y + (x \ast y) = y$
for all $x, y, z \in K$.

For convenience, we pronounce “+” (resp. “$$\ast$$”) as addition (resp. multiplication). Every distributive lattice is an incline algebra. An incline algebra is a distributive lattice if and only if $x \ast x = x$ for all $x \in K$. Note that $x \leq y \iff x + y = y$ for all $x, y \in K$. It is easy to see that “$$\leq$$” is a partial order on $K$ and that for any $x, y \in K$, the element $x + y$ is the least upper bound of $\{x, y\}$. We say that $$\leq$$ is induced by operation $+$.

In an incline algebra $K$, the following properties hold.

(K9) $x \ast y \leq x$ and $y \ast x \leq x$ for all $x, y \in K$,
(K10) $y \leq z$ implies $x \ast y \leq x \ast z$ and $y \ast x \leq z \ast x$, for all $x, y, z \in K$,
(K11) If $x \leq y$ and $a \leq b$, then $x + a \leq y + b$, and $x \ast a \leq y \ast b$ for all $x, y, a, b \in K$.

Furthermore, an incline algebra $K$ is said to be commutative if $x \ast y = y \ast x$ for all $x, y \in K$. A map $f$ is isotone if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in K$.

A subincline of an incline algebra $K$ is a non-empty subset $M$ of $K$ which is closed under the addition and multiplication. A subincline $M$ is said to be an ideal if $x \in M$ and $y \leq x$ then $y \in M$. An element “0” in an incline algebra $K$ is a zero element if $x + 0 = x = 0 + x$ and $x \ast 0 = 0 = 0 \ast x$ for any $x \in K$. An non-zero element “1” is called a multiplicative identity if $x \ast 1 = 1 \ast x = x$ for any $x \in K$. A non-zero element $a \in K$ is said to be a left (resp. right) zero divisor if there exists a non-zero $b \in K$ such that $a \ast b = 0$ (resp. $b \ast a = 0$) A zero divisor is an element of $K$ which is both a left zero divisor and a right zero divisor. An incline algebra $K$ with multiplicative identity 1 and zero element 0 is called an integral incline if it has no zero divisors. By a homomorphism of inclines, we mean a mapping $f$ from an incline algebra $K$ into an incline algebra $L$ such that $f(x + y) = f(x) + f(y)$ and $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in K$. A map $f : K \to K$ is regular if $f(0) = 0$. A subincline $I$ of an incline algebra $K$ is said to be $k$-ideal if $x + y \in I$ and $y \in I$, then $x \in I$. Let $K$ be an incline algebra. An element $a \in K$ is called a additively cancellative if for all $a, b \in K$, $a + b = a + c \Rightarrow b = c$. If every element of $K$ is additively cancellative, it is called additively cancellative.
Definition 2.1. Let $K$ be an incline algebra. A mapping $D(., .) : K \times K \to K$ is called symmetric if $D(x, y) = D(y, x)$ holds for all $x, y \in K$.

Definition 2.2. Let $K$ be an incline algebra and $x \in K$. A mapping $d(\cdot) = D(\cdot, \cdot)$ is called trace of $D(., .)$, where $D(., .) : K \times K \to K$ is a symmetric mapping.

Definition 2.3. Let $K$ be an incline algebra and let $D : K \times K \to K$ be a symmetric mapping. We call $D$ a symmetric bi-derivation on $K$ if it satisfies the following condition

$$D(x \ast y, z) = (D(x, z) \ast y) + (x \ast D(y, z))$$

for all $x, y, z \in K$.

3. *-Symmetric bi-$f$-multipliers of incline algebras

In what follows, let $K$ denote an incline algebra with a zero-element unless otherwise specified.

Definition 3.1. Let $K$ be an incline algebra and let $D : K \times K \to K$ be a symmetric mapping. We call $D$ a *-symmetric bi-$f$-multiplier on $K$ if there exists a function $f : K \to K$ such that

$$D(x \ast y, z) = D(x, z) \ast f(y)$$

for all $x, y, z \in K$.

Obviously, a *-symmetric bi-$f$-multiplier $D$ on $K$ satisfies the relation

$$D(x, y \ast z) = D(x, y) \ast f(z)$$

for all $x, y, z \in K$.

Example 3.2. Let $K$ be a commutative incline algebra. Define a mapping on $K$ by $D(x, y) = f(x) \ast f(y)$ where $f : K \to K$ satisfies $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in K$. Then we can see that $D$ is a *-symmetric bi-$f$-multiplier on $K$.

Example 3.3. Let $K$ be a commutative incline algebra and $a \in K$. Define a mapping on $K$ by $D(x, y) = (f(x) \ast f(y)) \ast a$ where $f : K \to K$ satisfies $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in K$. Then we can see that $D$ is a *-symmetric bi-$f$-multiplier on $K$.

Example 3.4. Let $K = \{0, a, b, 1\}$ be a set in which “+” and “*” is defined by

$$0 \ast 0 = 0, \quad 0 \ast a = a \ast 0 = a, \quad 0 \ast b = b \ast 0 = b, \quad 0 \ast 1 = 1 \ast 0 = 1$$

$$a \ast a = a, \quad a \ast b = b \ast a = 0, \quad a \ast 1 = 1 \ast a = 1$$

$$b \ast b = b, \quad b \ast 0 = 0 \ast b = 0, \quad b \ast 1 = 1 \ast b = 1$$

$$1 \ast 0 = 0 \ast 1 = 0, \quad 1 \ast a = a \ast 1 = a, \quad 1 \ast b = b \ast 1 = b, \quad 1 \ast 1 = 1$$

These operations make $K$ an incline algebra.

By the above definitions, $D(x, y) = f(x) \ast f(y)$ is a *-symmetric bi-$f$-multiplier on $K$. We can see that $D$ satisfies the relation $D(x, y \ast z) = D(x, y) \ast f(z)$ for all $x, y, z \in K$.
Then it is easy to check that \((K, +, \ast)\) is an incline algebra. Define a map \(D : K \times K \to K\) by
\[
D(x, y) = \begin{cases} 
  b & \text{if } (x, y) \in \{(b, b), (b, 1), (1, b), (1, 1)\} \\
  0 & \text{otherwise}
\end{cases}
\]
and \(f : K \to K\) by
\[
f(x) = \begin{cases} 
  b & \text{if } x \in \{b, 1\} \\
  0 & \text{otherwise}
\end{cases}
\]

Then it is easily checked that \(D\) is a \(\ast\)-symmetric bi-\(f\)-multiplier of an incline algebra \(K\).

**Proposition 3.5.** Let \(K\) be an incline algebra and let \(D\) be a \(\ast\)-symmetric bi-\(f\)-multiplier on \(K\). Then the following identities hold.

(i) \(D(x \ast y, z) \leq f(y)\), for all \(x, y, z \in K\),

(ii) \(D(x, y) = D(x, y) \ast f(1)\), for all \(x, y \in K\),

(iii) \(D(x \ast y, z) \leq D(x, z) + f(y)\), for all \(x, y \in K\).

**Proof.** (i) Let \(x, y, z \in K\). By using (K9), we have \(D(x \ast y, z) = D(x, z) \ast f(y) \leq f(y)\).

(ii) Let \(x, y \in K\). Then we have \(D(x, y) = D(x \ast 1, y) = D(x, y) \ast f(1)\).

(iii) Let \(x, y, z \in K\). Then we have \(D(x \ast y, z) = D(x, z) \ast f(y) \leq D(x, z)\). Also, we get \(D(x, z) \ast f(y) \leq f(y)\). Therefore, we have \(D(x \ast y, z) \leq D(x, z) + f(y)\).

**Proposition 3.6.** Every \(\ast\)-symmetric bi-\(f\)-multiplier on \(K\) with \(f(0) = 0\) is regular.

**Proof.** Let \(D\) be a \(\ast\)-symmetric bi-\(f\)-multiplier on \(K\) with a zero element. Then we have
\[
D(0, 0) = D(x \ast 0, 0) = D(x, 0) \ast f(0) = D(x, 0) \ast 0 = 0
\]
for all \(x \in K\).

**Proposition 3.7.** Let \(D\) be a \(\ast\)-symmetric bi-\(f\)-multiplier on \(K\). If \(K\) is a distributive lattice, we have \(D(x, y) \leq f(x)\) and \(D(x, y) \leq f(y)\) for all \(x, y \in K\).
Proof. Let $D$ be a $*$-symmetric bi-$f$-multiplier on $K$ and let $K$ be a distributive lattice. Then $D(x, y) = D(x * x, y) = D(x, y) * f(x)$, and so by using (K9), we get $D(x, y) \leq f(x)$. Similarly, we have $D(x, y) \leq f(y)$.

**Proposition 3.8.** Let $D$ be a $*$-symmetric bi-$f$-multiplier on $K$ and let $K$ be a distributive lattice. Then we have $d(x) \leq f(x)$ for all $x \in K$.

**Proof.** Let $D$ be a $*$-symmetric bi-$f$-multiplier on $K$ and let $K$ be a distributive lattice. Then we have

$$d(x) = D(x, x) = D(x * x, x) = D(x, x) * f(x) = D(x, x) * f(x) \leq f(x)$$

for all $x \in K$.

**Theorem 3.9.** Let $K$ be an integral incline with a multiplicative identity and let $D$ be a $*$-symmetric bi-$f$-multiplier on $K$ where $f$ is a function satisfying $f(1) = 1$ and $a \in K$. Then for all $x, y \in K$, we have $D(x, y) * a = 0$ implies $a = 0$ or $D = 0$.

**Proof.** Let $D(x, y) * a = 0$ for all $x, y \in K$. Since $K$ is an integral incline, that is, it has no zero-divisors, we have $a = 0$ or $D(x, y) = 0$ for all $x, y \in K$. Hence we get $a = 0$ or $D = 0$.

**Definition 3.10.** Let $K$ be an incline algebra. If $D : K \times K \to K$ be a symmetric mapping. We call $D$ a **additive mapping** if it satisfies

$$D(x + y, z) = D(x, z) + D(y, z)$$

for all $x, y, z \in K$.

**Proposition 3.11.** Let $d$ be a trace of additive $*$-symmetric bi-$f$-multiplier $D$ on $K$. Then the following identities hold for all $x, y \in K$,

(i) $d(x + y) = d(x) + d(y) + D(x, y)$ and $d(x) + d(y) \leq d(x + y)$,

(ii) $D(x * y, x) \leq d(x)$.

**Proof.** (i) Let $x, y \in K$. Then we have

$$d(x + y) = D(x + y, x + y) = D(x, x + y) + D(y, x + y)
= D(x, x) + D(x, y) + D(y, x) + D(y, y)
= D(x, x) + D(y, y) + D(x, y).$$

Hence we get $d(x + y) = d(x) + d(y) + D(x, y)$ and $d(x) + d(y) \leq d(x + y)$.

(ii) Let $x, y \in K$. It follows from (K7) that $d(x) = D(x, x) = D(x + (x * y), x) = D(x, x) + D(x * y, x)$, which implies $D(x * y, x) \leq d(x)$.
Proposition 3.12. Let $D$ be a trace of $*$-symmetric bi-$f$-multiplier on $K$. Then $D(x * y, y) \leq D(x, y)$ for all $x, y \in K$.

Proof. Let $x, y \in K$. Then we have

$$D(x, y) = D(x + x * y, y) = D(x, y) + D(x * y, y),$$

which implies $D(x * y, y) \leq D(x, y)$.

Definition 3.13. Let $D$ be a $*$-symmetric bi-$f$-multiplier on $K$. If $x \leq w$ implies $D(x, y) \leq D(w, y)$, $D$ is called an isotone $*$-symmetric bi-$f$-multiplier for all $x, y, w \in K$.

Theorem 3.14. Let $D$ be a additive $*$-symmetric bi-$f$-multiplier on $K$. Then $D$ is an isotone $*$-symmetric bi-$f$-multiplier on $K$.

Proof. Let $x$ and $w$ be such that $x \leq w$. Then $x + w = w$, and so

$$D(w, y) = D(w + x, y) = D(w, y) + D(x, y)$$

for all $x, y, w \in K$. This implies that $D(x, y) \leq D(w, y)$. This completes the proof.

Proposition 3.15. Let $D$ be a $*$-symmetric bi-$f$-multiplier on $K$ and let $f$ be an endomorphism on $K$. Then $Fix_a(D)$ is a subincline of $K$.

Proof. Let $x, y \in Fix_a(D)$. Then we have $D(x, a) = f(x)$ and $D(y, a) = f(y)$, and so

$$D(x * y, a) = D(x, a) * f(y) = f(x) * f(y) = f(x * y).$$

Hence we get $x * y \in Fix_a(D)(K)$. Also, we get $D(x + y, a) = D(x, a) + D(y, a) = f(x) + f(y) = f(x + y)$, and so $x + y \in Fix_a(D)$. This completes the proof.

Proposition 3.16. Let $D$ be a $*$-symmetric bi-$f$-multiplier on $K$ with $f(x * y) = f(x) * f(y)$ for all $x, y \in K$. If $x \in Fix_a(D)$ and let $f$ be an endomorphism on $K$, then $x * y \in Fix_a(D)$.

Proposition 3.17. Let $K$ be additively cancellative and let $D$ be a additive $*$-symmetric bi-$f$-multiplier on $K$ and let $f$ be an endomorphism on $K$. Then $Fix_a(D)$ is a $k$-ideal of $K$. 
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Proof. Let $x + y \in \text{Fix}_a(D)$ and $y \in \text{Fix}_D(K)$. Then we have $f(x) + f(y) = f(x + y) = D(x + y, a) = D(x, a) + D(y, a) = D(x, a) + f(y)$. Since $K$ is additively cancellative, we have $f(x) = D(x, a)$, which implies $x \in \text{Fix}_a(D)$. This completes the proof.

Definition 3.18. Let $K$ be an incline algebra and let $D : K \times K \to K$ be a symmetric mapping. Define a set $\text{Ker}(D)$ by $\text{Ker}(D) = \{ x \in K \mid D(0, x) = 0 \}$.

Proposition 3.19. Let $D$ be an additive $*$-symmetric bi-$f$-multiplier on $K$. If $x \leq y$ and $y \in \text{Ker}(D)$, then we have $x \in \text{Ker}(D)$.

Proof. Let $x \leq y$ and $y \in \text{Ker}(D)$. Then we get $x + y = y$ and $D(0, y) = 0$. Hence we get

$$0 = D(0, y) = D(0, x + y) = D(0, x) + D(0, y) = D(0, x) + 0 = D(0, x),$$

which implies $x \in \text{Ker}(D)$. This completes the proof.

Proposition 3.20. Let $D$ be a additive $*$-symmetric bi-$f$-multiplier on $K$. Then $\text{Ker}(D)$ is a subincline of $K$.

Proof. Let $x, y \in \text{Ker}(D)$. Then $D(x, 0) = 0$, and so

$$D(0, x * y) = D(x * y, 0) = D(x, 0) * f(y) = 0 * f(y) = 0,$$

which implies $x * y \in \text{Ker}(D)$. Now $D(x + y, 0) = D(x, 0) + D(y, 0) = 0 + 0 = 0$. Hence $x + y \in \text{Ker}(D)$. This completes the proof.

Theorem 3.21. Let $D$ be a additive $*$-symmetric bi-$f$-multiplier on $K$. Then $\text{Ker}(D)$ is an ideal of $K$.

Proof. By Proposition 3.10 and 11, it is obvious that $\text{Ker}(D)$ is an ideal of $K$.

4. $+-$Symmetric bi-$f$-multipliers of incline algebras

Definition 4.1. Let $K$ be an incline algebra and let $D : K \times K \to K$ be a symmetric mapping. We call $D$ a $+-$symmetric bi-$f$-multiplier on $K$ if there exists a function $f : K \to K$ such that

$$D(x, y + z) = D(x, y) + f(z)$$

for all $x, y, z \in K$. 


Example 4.2. Let $K$ be an incline algebra. Define a mapping on $K$ by $D(x, y) = x + f(y)$ where $f : K \rightarrow K$ satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in K$. Then we can see that $D$ is a $+\text{-symmetric bi-$f$-multiplier on $K$}.$

Proposition 4.3. Let $D$ be a $+\text{-symmetric bi-$f$-multiplier on $K$}.$ Then the following identities hold. 

(i) $f(y) \leq D(x, y)$, for all $x, y, z \in K$, 
(ii) $D(x, y) + f(y) \leq D(x, y)$, for all $x, y \in K$.

Proof. (i) Let $D$ be a $+\text{-symmetric bi-$f$-multiplier on $K$}.$ Then we have $D(x, y) = D(x, y + y) = D(x, 0) + f(y)$, which implies $f(y) \leq D(x, y)$.

(ii) Let $D$ be a $+\text{-symmetric bi-$f$-multiplier on $K$}.$ Then we have $D(x, y) = D(x, 0 + y) = D(x, 0) + f(y)$, which implies $D(x, 0) + f(y) \leq D(x, y)$. \qed

Proposition 4.4. Let $D$ be a $+\text{-symmetric bi-$f$-multiplier on $K$}.$ with $f(x + y) = f(x) + f(y)$ for all $x, y \in K$ and $x + y \in Fix_a(D)$, then $x \in Fix_a(D)$.

Proof. Let $D$ be a $+\text{-symmetric bi-$f$-multiplier on $K$ and $x \in Fix_a(D)$}.$ Then we have $D(a, x) = f(x)$. Hence 

$$D(a, x + y) = D(a, x) + f(y) = f(x) + f(y)$$

which implies $x + y \in Fix_D(K)$. \qed

Proposition 4.5. Let $D$ be a $+\text{-symmetric bi-$f$-multiplier on an incline algebra $K$}.$ that is additively cancellative. If $f(x + y) = f(x) + f(y)$ for all $x, y \in K$ and $x + y \in Fix_a(D)$ and $y \in Fix_a(D)$, then $x \in Fix_a(D)$.

Proof. Let $D$ be a $+\text{-symmetric bi-$f$-multiplier and $x + y \in Fix_a(D)$}.$ Then

$$f(x) + f(y) = f(x + y) = D(a, x + y)$$

$$= D(a, x) + f(y)$$

Therefore we get $D(a, x) + f(y) = f(x) + f(y)$. Since $K$ is additively cancellative, we have $D(a, x) = f(x)$, which implies $x \in Fix_a(D)$. \qed
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