GENERALIZED VECTOR QUASIVARIATIONAL-LIKE INEQUALITIES

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Abstract. In this paper, we introduce two kinds of generalized vector quasivariational-like inequalities for multivalued mappings and show the existence of solutions to those variational inequalities under compact and non-compact assumptions, respectively.

1. Introduction and Preliminaries

A vector variational inequality problem was firstly introduced in a finite dimensional Euclidean space with its applications by Giannessi [9]. Later, many authors [1-6, 9, 10, 13-17, 21-25] have extensively studied the problem in infinite dimensional spaces under different assumptions. In particular, vector variational-like inequalities were considered in [1-2, 10, 13, 15] and vector quasivariational inequalities were considered in [3-6, 10, 13, 14, 17, 23-25].

In this paper we introduce two kinds of generalized vector quasivariational-like inequality problems for multivalued mappings and show the existence of solutions to our inequality problems.

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Let $X$ and $Y$ be topological spaces, and $F : X \to 2^Y$ a multivalued mapping.

**Definition 1.1.** $F$ is called upper semi-continuous (in short, u.s.c.) at $x \in X$ if for each open set $V$ in $Y$ containing $F(x)$, there is an open set $U$ containing $x$ such that $F(u) \subseteq V$ for all $u \in U$; $F$ is called u.s.c. on $X$ if $F$ is u.s.c. at every point of $X$. $F$ is called lower semi-continuous (in short, l.s.c.) at $x \in X$ if for each open set $V$ in $Y$ with $F(x) \cap V \neq \emptyset$, there is an open set $U$ containing $x$ such that $F(u) \cap V \neq \emptyset$ for all $u \in U$; $F$ is called l.s.c. on $X$ if $F$ is l.s.c. at every point of $X$. $F$ is called continuous at $x \in X$ if $F$ is both u.s.c. and l.s.c. at $x \in X$.

**Lemma 1.1.** $F$ is l.s.c. at $x \in X$ if and only if for any $y \in F(x)$ and for any net $\{x_\alpha\}$ in $X$ converging to $x$, there is a net $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$ for each $\alpha$, and $\{y_\alpha\}$ converges to $y$.

**Definition 1.2.** $F$ is called closed if the graph $G, F = \{(x, y) \in X \times Y : y \in F(x)\}$ of $F$ is closed in $X \times Y$, i.e., for each $x \in X$, $\{x_\alpha\} \subset X$ with $x_\alpha \to x$ and each $\{y_\alpha\} \subset Y$ with $y_\alpha \in F(x_\alpha)$ and $y_\alpha \to y$, then we have $y \in F(x)$.

**Definition 1.3.** $F$ is called compact if $F(X)$ is contained in some compact subset of $Y$.

**Definition 1.4.** Let $F^- : Y \to 2^X$ be a multivalued mapping defined by

$$x \in F^-(y) \quad \text{if and only if} \quad y \in F(x).$$

$F$ is said to have open lower sections if for each $y \in Y$, $F^-(y)$ is open in $X$.
In an ordered Hausdorff topological vector space $Z$, usually a closed convex pointed solid proper cone $P$ in $Z$ defines partial orders $<$ and $\leq$ as
\[
x <_P y \quad \text{iff} \quad x - y \in -\text{int}P
\]
\[
x \leq_P y \quad \text{iff} \quad x - y \in -P
\]
for $x, y \in Z$. To an arbitrary subset $C$ of $Z$, the orders can be extended by setting
\[
C <_P 0 \quad \text{iff} \quad C \subseteq -\text{int}P
\]
\[
C \leq_P 0 \quad \text{iff} \quad C \subseteq -P.
\]
A point $z_0$ in a nonempty subset $C$ of $Z$ is called a vector maximal point of $C$ [27] if the set $\{z \in C : z_0 \leq_P z, z \neq z_0\} = \emptyset$, which is equivalent to
\[
C \cap (z_0 + P) = \{z_0\}.
\]
The following simple fact needed in our research was first introduced by Luc;

**Lemma 1.2** [18] Let $C$ be a nonempty compact subset of an ordered Banach space $Z$. Then $\max C \neq \emptyset$, where $\max C$ denotes the set of all vector maximal points of $C$.

2. Main results

Now we introduce $P$-convexity of a two variable function, which is an essential concept to our results.

**Definition 2.1.** Let $K$ be a nonempty convex subset of a vector space $X$, and $P$ a pointed, closed convex cone in a topological vector space $Z$, which has an apex at the origin and a nonempty interior $\text{int}P$. A multivalued mapping $H : K \times K \to 2^Z$ is said to be $P$-convex with respect to the first variable if for $x_1, x_2, y \in K$, $u_1 \in H(x_1, y)$. $u_2 \in$
$H(x_2, y)$ and $\lambda \in [0, 1]$, there exists $u \in H(\lambda x_1 + (1 - \lambda)x_2, y)$ such that

$$\lambda u_1 + (1 - \lambda)u_2 \in u + P.$$  

Throughout this section, $X$, $Y$ denote two Hausdorff topological vector spaces, and $Z$ denotes an ordered Hausdorff topological vector space. Let $K$ be a nonempty convex subset of $X$, $D$ a nonempty subset of $Y$ and $\{C(x)|x \in K\}$ a family of solid convex cones in $Z$, that is, for each $x \in K$, $\text{intC}(x)$ is nonempty and $C(x) \neq Z$. $L(X, Z)$ denotes the space of all continuous linear operators from $X$ to $Z$. Let $F : K \rightarrow 2^D$, $G : K \rightarrow 2^K$, $M : K \times D \rightarrow 2^{L(X, Z)}$ and $H : K \times K \rightarrow 2^Z$ be multivalued mappings, and $\eta : X \times X \rightarrow X$ a mapping.

We consider the following two kinds of generalized vector quasivariational-like inequalities for multivalued mappings;

(VQVLI)$_1$ Find $\bar{x} \in K$ such that for each $x \in K$ there exists $\bar{s} \in F(\bar{x})$ satisfying the following inequality;

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle + u \notin -\text{intC}(\bar{x})$$

for any $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$, where

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle \geq \max_{s \in M(\bar{x}, \bar{s})} \langle s, \eta(x, z) \rangle$$

and $\langle s, \eta(x, z) \rangle$ denotes the evaluation of a continuous linear operator $s$ from $X$ into $Z$ at $\eta(x, z)$.

(VQVLI)$_2$ Find $\bar{x} \in K$ and $\bar{s} \in F(\bar{x})$ such that

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle + u \notin -\text{intC}(\bar{x})$$

for $x \in K$, $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$.

Putting $H \equiv \emptyset$ in (VQVLI)$_1$ and (VQVLI)$_2$, we obtain the following vector quasivariational-like inequalities;
\((VQVLI)_1\) Find \(\bar{x} \in K\) such that for any \(x \in K\) there exists \(\bar{s} \in F(\bar{x})\) satisfying the following inequality;

\[
\max\langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle \notin -\text{int}C(\bar{x})
\]

for \(z \in G(\bar{x})\) and

\((VQVLI)_2\) Find \(\bar{x} \in K\) and \(\bar{s} \in F(\bar{x})\) such that

\[
\max\langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle \notin -\text{int}C(\bar{x})
\]

for \(x \in K\) and \(z \in G(\bar{x})\).

By replacing \(Y, H : K \times K \to 2^Z\) and \(M : K \times D \to 2^{L(X, Z)}\) with \(Z, H : K \times K \to Z\) and \(S : K \to 2^{L(X, Z)}\), respectively in \((VQVLI)_1\) and \((VQVLI)_2\), we obtain the following vector variational-like inequalities for multivalued mappings;

\((VVLI)\) Find \(\bar{x} \in K\) satisfying the following inequality;

\[
\max\langle S(\bar{x}), \eta(x, z) \rangle + H(x, \bar{x}) \notin -\text{int}C(\bar{x})
\]

for \(x \in K\) and \(z \in G(\bar{x})\).

Putting \(H \equiv \emptyset\) and \(G(\bar{x}) = K\) in \((VVLI)\), we obtain the following vector variational-like inequalities for multivalued mappings, introduced and studied by Chang, Thompson and Yuan [2];

\((VVLI)'\) Find \(\bar{x} \in K\) satisfying the following inequality;

\[
\max\langle S(\bar{x}), \eta(x, \bar{x}) \rangle \notin -\text{int}C(\bar{x}) \quad \text{for} \quad x \in K.
\]

Putting \(Z = Y, \eta(x, z) = x - z\) and \(H = \emptyset\), and replacing \(M : K \times D \to 2^{L(X, Z)}\) with \(S : K \to L(X, Y)\) in \((VQVLI)_1\) and \((VQVLI)_2\), we have the following variational inequality;
(VVI) Find $\bar{x} \in K$ such that
\[ \langle S(\bar{x}), x - z \rangle \not\in -\text{int}C(\bar{x}) \quad \text{for} \ x \in K \ \text{and} \ z \in G(\bar{x}). \]

Putting $C(x) \equiv C$ for $x \in K$ and $\eta(x, y) = x - y$ in (VVLII)', we obtain the following vector-valued variational inequality considered by Lee et al. [16];

Find $\bar{x} \in K$ such that for each $x \in K$, there exists $\bar{s} \in S(\bar{x})$ such that
\[ \langle \bar{s}, x - \bar{x} \rangle \not\in -\text{int}C \ 0, \]
where $x \not\in_P y$ means $x - y \not\in P$.

Putting $Z = \mathbb{R}$, $L(X, Z) = X^*$, the dual of $X$ and $C(x) \equiv \mathbb{R}^+$, the positive orthant for $x \in K$ in (VVLII)', we obtain the following scalar-valued variational inequality considered by Cottle and Yao [7], Isac [12], and Noor [19];

Find $\bar{x} \in K$ such that
\[ \sup_{u \in S(\bar{x})} \langle u, \eta(x, \bar{x}) \rangle \geq 0, \quad \text{for} \ x \in K. \]

Replacing $S : K \to 2^{\mathcal{L}(X,Z)}$ with $S : X \to L(X, Z)$ and putting $\eta(x, z) = x - g(z)$, where $g : K \to K$ is a mapping, then (VVLII)' reduces to the following vector variational inequality (VVI) considered by Siddiqi et al. [22];

(VVI)' Find $\bar{x} \in K$ such that
\[ \langle S(\bar{x}), x - g(\bar{x}) \rangle \not\in -\text{int}C(\bar{x}) \ 0, \quad \text{for} \ x \in K. \]

Putting $G(x) = \{x\}$ for $x \in K$ in (VVI) or $g(x) = x$ for $x \in K$ in (VVI)', we obtain the following vector-valued variational inequality considered by Chen [3];
Find $\tilde{x} \in K$ such that
\[ \langle S(\tilde{x}), x - \tilde{x} \rangle \not\in \text{int} C(\tilde{x}) \quad 0, \quad \text{for } x \in K. \]

Putting $C(x) \equiv C$ and $g(x) = x$ for $x \in K$ in (VVI)', we obtain the following vector-valued variational inequality considered by Chen et al. [3-5];

Find $\tilde{x} \in K$ such that
\[ \langle S(\tilde{x}), x - \tilde{x} \rangle \not\in \text{int} C \quad 0, \quad \text{for } x \in K. \]

Putting $Z = \mathbb{R}$, $X = \mathbb{R}^n$, $C(x) \equiv \mathbb{R}^+$ for $x \in K \subseteq \mathbb{R}^n$, $L(X, Z) = \mathbb{R}^n$ and $\eta(x, y) = x - y$, we obtain the following scalar-valued variational inequality considered by Hartman and Stampacchia [11]; find $\tilde{x} \in K$ such that
\[ \langle S(\tilde{x}), x - \tilde{x} \rangle \geq 0 \quad \text{for } x \in K. \]

2.1. Compact set case

When we consider the existence of solutions to (VQVLI)$_1$ for the compact set case, Ky Fan’s Section Theorem in [8] is very useful and indispensable.

**Theorem 2.1** [8]. Let $K$ be a nonempty compact convex subset of a Hausdorff topological vector space. Let $A$ be a subset of $K \times K$ having the following properties

(i) $(x, x) \in A$ for all $x \in K$;

(ii) for any $x \in K$, the set $A_x := \{y \in K : (x, y) \in A\}$ is closed in $K$;

(iii) for any $y \in K$, the set $A^y := \{x \in K : (x, y) \not\in A\}$ is convex or empty in $K$.

Then there exists $\bar{y} \in K$ such that $K \times \{\bar{y}\} \subset A$. 
The following main theorem for the existence of solutions to (VQVLI)\textsubscript{1} is for the compact set case.

**Theorem 2.2.** Let $K$ be a nonempty compact convex subset of $X$ and $D$ a nonempty subset of $Y$. Let $F : K \rightarrow 2^D$ be closed, $G : K \rightarrow 2^K$ be l.s.c. and nonempty convex-valued, $M : K \times D \rightarrow 2^{L(X,Z)}$ be nonempty compact-valued, and a multivalued mapping $W : K \rightarrow 2^Z$ defined by $W(x) = Z \setminus \{-\text{int}C(x)\}$, $x \in K$, closed. Let $\eta : X \times X \rightarrow X$ be linear, and $H : K \times K \rightarrow 2^Z$ be $P$-convex with respect to the first variable and l.s.c. with respect to the second, where $P := \bigcap_{x \in K} C(x)$.

Suppose further that

1. $\langle M(x, \cdot), \eta(x, \cdot) \rangle = 0$ and $H(x, x) = \{0\}$ for all $x \in K$;
2. $F$ is compact; and
3. $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha) \rangle$ converges to $\max\langle M(y, s), \eta(x, z) \rangle$ provided that $y_\alpha \rightarrow y$, $s_\alpha \rightarrow s$ and $z_\alpha \rightarrow z$.

Then (VQVLI)\textsubscript{1} is solvable.

**Proof.** By the assumption that $M$ is nonempty compact-valued, from the continuity of $\langle \cdot, \cdot \rangle$, $\langle M(y, s), \eta(x, z) \rangle$ is compact in $Z$. So we can define $A = \{(x, y) \in K \times K : \text{there exists } s \in F(y) \text{ such that } \max\langle M(y, s), \eta(x, z) \rangle + u \notin -\text{int}C(y) \text{ for any } z \in G(y) \text{ and } u \in H(x, y)\}$. By the condition (1), it is easily shown that $(x, x) \in A$ for all $x \in K$. Next, $A_x = \{y \in K : (x, y) \in A\}$, $x \in K$ is closed. In fact, let $\{y_\alpha\}$ be a net in $A_x$ such that $y_\alpha \rightarrow y$. Then by Lemma 1.1, for any $z \in G(y)$ there exists a net $\{z_\alpha\}$ converging to $z$ such that $z_\alpha \in G(y_\alpha)$ for each $\alpha$. Also by the lower semi-continuity of $H$ with respect to the second variable, for any $u \in H(x, y)$ there exists a net $\{u_\alpha\}$ converging to $u$ such that $u_\alpha \in H(x, y_\alpha)$ for each $\alpha$. Since $y_\alpha \in A_x$ we can choose $s_\alpha \in F(y_\alpha)$ such that

$$\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha) \rangle + u_\alpha \in W(y_\alpha)$$
for \( z_\alpha \in G(y_\alpha) \) and \( u_\alpha \in H(x, y_\alpha) \). By the condition (2) and the closedness of \( F \), we can assure the existence of limit \( s \) of \( \{s_\alpha\} \) such that \( s \in F(y) \). Hence by the condition (3) and the closedness of \( W \), we have

\[
\max\langle M(y, s), \eta(x, z) \rangle + u \in W(y)
\]

for any \( z \in G(y) \) and \( u \in H(x, y) \). Finally, \( A^y = \{x \in K : (x, y) \not\in A\} \), \( y \in K \) is convex. Indeed, let \( x_1, x_2 \in A^y \) and \( \lambda \in [0, 1] \). Then from the fact that \( (x_1, y) \not\in A \), for any \( s \in F(y) \) there exist \( z_1 \in G(y) \) and \( u_1 \in H(x_1, y) \) such that

\[
\max\langle M(y, s), \eta(x_1, z_1) \rangle + u_1 \in -\text{int}C(y)
\]

and from the fact that \( (x_2, y) \not\in A \), for any \( s \in F(y) \) there exist \( z_2 \in G(y) \) and \( u_2 \in H(x_2, y) \) such that

\[
\max\langle M(y, s), \eta(x_2, z_2) \rangle + u_2 \in -\text{int}C(y).
\]

Hence, for any \( s \in F(y) \) there exist \( u \in H(\lambda x_1 + (1 - \lambda)x_2, y) \) and \( z := \lambda z_1 + (1 - \lambda)z_2 \in G(y) \) for \( \lambda \in [0, 1] \) such that

\[
\max\langle M(y, s), \eta(\lambda x_1 + (1 - \lambda)x_2, z) \rangle + u
\]

\[
= \max\langle M(y, s), \eta(\lambda x_1 + (1 - \lambda)x_2, \lambda z_1 + (1 - \lambda)z_2) \rangle + u
\]

\[
= \max\langle M(y, s), \lambda \eta(x_1, z_1) + (1 - \lambda)\eta(x_2, z_2) \rangle + u
\]

\[
\leq \lambda \max\langle M(y, s), \eta(x_1, z_1) \rangle + (1 - \lambda) \max\langle M(y, s), \eta(x_2, z_2) \rangle + u
\]

\[
\in \lambda \max\langle M(y, s), \eta(x_1, z_1) \rangle + (1 - \lambda) \max\langle M(y, s), \eta(x_2, z_2) \rangle + \lambda u_1
\]

\[
+ (1 - \lambda)u_2 - P
\]

\[
= \lambda(\max\langle M(y, s), \eta(x_1, z_1) \rangle + u_1) + (1 - \lambda)(\max\langle M(y, s), \eta(x_2, z_2) \rangle + u_2) - P
\]

\[
\subseteq -\text{int}C(y) - \text{int}C(y) - C(y)
\]

\[
= -\text{int}C(y).
\]
Thus $\lambda x_1 + (1 - \lambda)x_2 \in A^y$, which shows that $A^y$ is convex. Hence by Ky Fan’s Section Theorem there exists $\bar{x} \in K$ such that

$$K \times \{\bar{x}\} \subset A,$$

which implies that for any $x \in K$, there exists $\bar{s} \in F(\bar{x})$ such that

$$\max \langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle + u \notin -\text{int}C(\bar{x})$$

for $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$. This completes the proof.

### 2.2. Noncompact set case

For considering the existence of solutions to $(VQVLI)_2$ for noncompact set case, we use the following particular form of the generalized Ky Fan’s Section Theorem due to Park [20].

**Theorem 2.3.** Let $K$ be a nonempty convex subset of $X$ and $A \subset K \times K$ satisfy the following conditions:

1. $(x, x) \in A$, $x \in K$;
2. $A_x = \{y \in K : (x, y) \in A\}$, $x \in K$, is closed;
3. $A^y = \{x \in K : (x, y) \notin A\}$, $y \in K$, is convex or empty;
4. there exists a nonempty compact subset $B$ of $K$ such that for each finite subset $N$ of $K$ there exists a nonempty compact convex subset $L_N$ of $K$ containing $N$ such that

$$L_N \cap \{y \in K : (x, y) \in A \text{ for any } x \in L_N\} \subset B.$$

Then there exists a $y_0 \in B$ such that $K \times \{y_0\} \subset A$.

In particular, if $K = B$, that is, $K$ is a compact convex subset of $X$, then the condition (iv) is obviously true, thus the three conditions of Ky Fan’s Section Theorem are sufficient to show the existence of $y_0 \in K$ such that $K \times \{y_0\} \subset A$. 
To show the existence of solutions to (VQVLI)$_2$ for the noncompact set case, the following lemmas are essential.

**Lemma 2.4.** Let $K$ be a nonempty convex subset of $X$ and $D$ be a nonempty subset of $Y$. Let $f : K \to D$ be a continuous function, $M : K \times D \to 2^{L(X,Z)}$ be nonempty compact-valued, and $G : K \to 2^K$ a l.s.c. mapping with nonempty convex-values. Let a multivalued mapping $W : K \to 2^Z$ defined by $W(x) = Z \setminus \{-\text{int}C(x)\}$, $x \in K$, be closed. Let $\eta : X \times X \to X$ be linear and $H : K \times K \to 2^Z$ $P$-convex with respect to the first variable and l.s.c. with respect to the second, where $P = \bigcap_{x \in K} C(x)$. Suppose further that

1. $\langle M(x, \cdot), \eta(x, \cdot) \rangle = 0$ and $H(x, x) = \{0\}$ for all $x \in K$,

2. $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha) \rangle$ converges to $\max\langle M(y, s), \eta(x, z) \rangle$ provided that $y_\alpha \to y$, $s_\alpha \to s$ and $z_\alpha \to z$;

3. there is a nonempty compact subset $B$ of $K$ such that for each nonempty finite subset $N$ of $K$, there is a nonempty compact convex subset $L_N$ of $K$ containing $N$ such that for $y \in L_N \setminus B$, there exist $x \in L_N$, $z \in G(y)$ and $u \in H(x, y)$ such that

$$\max\langle M(y, f(y)), \eta(x, z) \rangle + u \in -\text{int}C(y).$$

Then there exists $\bar{x} \in K$ such that

$$\max\langle M(\bar{x}, f(\bar{x})), \eta(x, z) \rangle + u \notin -\text{int}C(\bar{x})$$

for any $x \in K$, $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$.

**Proof.** Let $A = \{(x, y) \in K \times K : \max\langle M(y, f(y)), \eta(x, z) \rangle + u \notin -\text{int}C(y) \text{ for any } z \in G(y) \text{ and } u \in H(x, y)\}$. It is easily shown that $(x, x) \in A$ for $x \in K$ from the condition (2). And $A_x = \{y \in K : (x, y) \in A\}$, $x \in K$, is closed. In fact, for any net $\{y_\alpha\}$ in $A_x$ converging to $y$, we have $\max\langle M(y_\alpha, f(y_\alpha)), \eta(x, z_\alpha) \rangle + u_\alpha \notin -\text{int}C(y_\alpha)$ for any $z_\alpha \in G(y_\alpha)$ and $u_\alpha \in H(x, y_\alpha)$. From Lemma 2.1 and the condition
(1), \( \max (M(y, f(y)), \eta(x, z)) + u \notin -intC(y) \) for any \( z \in G(y) \) and \( u \in H(x, y) \), we have \( y \in A_x \), showing the closedness of \( A_x \) for \( x \in K \). By a similar method shown in the proof of Theorem 2.2, we can show that the set \( A^y = \{ x \in K | (x, y) \notin A \} \), \( y \in K \), is convex. Further note that the assumption (3) implies that for \( y \in L_N \setminus B \) there exists \( x \in L_N \) such that \( y \notin A_x \). Hence the condition (iv) of Theorem 2.3 is satisfied. Hence there exists \( \bar{x} \in K \) such that

\[ \max (M(\bar{x}, f(\bar{x})), \eta(x, z)) + u \notin -intC(\bar{x}) \]

for \( x \in K \), \( z \in G(\bar{x}) \) and \( u \in H(x, \bar{x}) \). This completes the proof.

**Lemma 2.5 [26].** Let \( X \) be a paracompact Hausdorff topological space and \( Y \) a topological vector space. Let \( F : X \to 2^Y \) be a multivalued mapping with nonempty convex-values. If \( F \) has open lower sections, then there exists a continuous function \( f : X \to Y \) such that \( f(x) \in F(x) \) for \( x \in X \).

Now we consider the existence of solution to \((VQVLI)_2\).

**Theorem 2.6.** Let \( K \) be a nonempty paracompact convex subset of \( X \) and \( D \) a nonempty convex subset of \( Y \). Let \( F : K \to 2^D \) have nonempty convex-values and open lower sections, \( G : K \to 2^K \) be a l.s.c. mapping with nonempty convex-values, \( M : K \times D \to 2^{L(X, Z)} \) be nonempty compact-valued, and \( W : K \to 2^Z \) defined by \( W(x) = Z \setminus \{ -intC(x) \} \), \( x \in K \), closed. Let \( \eta : X \times X \to X \) be linear and \( H : K \times K \to 2^Z \) be \( P \)-convex with respect to the first variable and l.s.c. with respect to the second, where \( P = \bigcap_{x \in K} C(x) \).

Suppose further that

1. \( \langle M(x, \cdot), \eta(x, \cdot) \rangle = 0 \) and \( H(x, x) = \{ 0 \} \) for all \( x \in K \),
2. \( \max (M(y_\alpha, s_\alpha), \eta(x, z_\alpha)) \to \max (M(y, s), \eta(x, z)) \) provided that \( y_\alpha \to y \), \( s_\alpha \to s \) and \( z_\alpha \to z \),
(3) $F$ is compact,

(4) there is a nonempty compact subset $B$ of $K$ such that for any nonempty finite subset $N$ of $K$, there is a nonempty compact convex subset $L_N$ of $K$ containing $N$ such that for any $y \in L_N \setminus B$, there exist $x \in L_N$, $z \in G(y)$ and $u \in H(x, y)$ such that

$$\max \langle M(y, s), \eta(x, z) \rangle + u \in -\text{int} C(y)$$

for any $s \in F(y)$.

Then $(\text{VQVLI})_2$ is solvable, i.e., there exist $\bar{x} \in K$ and $\bar{s} \in F(\bar{x})$ such that

$$\max \langle M(\bar{x}, \bar{s}), \eta(x, z) \rangle + u \not\in -\text{int} C(\bar{x})$$

for any $x \in K$, $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$.

**Proof.** Since $F^{-}(y)$ is open in $X$ for $y \in D$, by Lemma 2.5 there exists a continuous function $f : K \rightarrow D$ such that $f(x) \in F(x)$ for $x \in K$. So, by Lemma 2.4 there exists $\bar{x} \in K$ such that

$$\max \langle M(\bar{x}, f(\bar{x})), \eta(x, z) \rangle + u \not\in -\text{int} C(\bar{x})$$

for any $x \in K$, $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$. This completes the proof.

**References**


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