ON THE SIMPLICIAL COMPLEX STEMMED FROM A DIGITAL GRAPH

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Abstract. In this paper, we give a digital graph-theoretical approach of the study of digital images with relation to a simplicial complex. Thus, a digital graph $G_k$ with some $k$-adjacency in $\mathbb{Z}^n$ can be recognized by the simplicial complex spanned by $G_k$. Moreover, we demonstrate that a graphically $(k_0, k_1)$-continuous map $f : G_{k_0} \subset \mathbb{Z}^{n_0} \to G_{k_1} \subset \mathbb{Z}^{n_1}$ can be converted into the simplicial map $S(f) : S(G_{k_0}) \to S(G_{k_1})$ with relation to combinatorial topology. Finally, if $G_{k_0}$ is not $(k_0, 3^{n_0} - 1)$-homotopy equivalent to $SC_{3^{n_0} - 1}$, a graphically $(k_0, k_1)$-continuous map (respectively a graphically $(k_0, k_1)$-isomorphism $f : G_{k_0} \subset \mathbb{Z}^{n_0} \to G_{k_1} \subset \mathbb{Z}^{n_1}$ induces the group homomorphism (respectively the group isomorphism) $S(f)_* : \pi_1(S(G_{k_0}), v_0) \to \pi_1(S(G_{k_1}), f(v_0))$ in algebraic topology.

1. Introduction

A digital image can be viewed as a graph whose points are 1’s or 0’s of an image and whose edges define nearness and connectedness. A graph homotopy was recently introduced with relation to the graph contractibility and the simple homotopy for cell complexes [4]. And Graham homotopy for hypergraph was also investigated in order to characterize
acyclic hypergraph and acyclic relational database schemes [4]. Furthermore, a digital graph homotopy and a fundamental group of digital graph were investigated by Kong [20] and restated by Malgouyres in terms of the elementary $L(X)$-deformation [23]. For a digital image $X$, Kong’s fundamental group of $X$ and Malgouyres’ fundamental group of $X$ are equal, which are different from Boxer’s fundamental group of $X$. The study of digital image in the digital topological point of view has been studied via digital covering theory [10, 11, 12, 16, 17], digital homotopy theory [6, 7, 8, 9, 20, 23] and the digital surface point of view [13, 14, 15]. Meanwhile, A. Bretto as well as D. Nogly and M. Schladt dealt with the topology on digital graphs and they gave interesting results about digital topology on digital graphs [2, 18, 24] with relation to Khalimsky topology [19]. It is helpful to remind that a digital graph on $\mathbb{Z}^n$ need not be a Hausdorff space.

Let $\mathbb{N}$ (resp. $\mathbb{Z}$) represent the set of natural numbers (resp. integers), let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{Z}^n$ be the set of points in the Euclidian $n$-dimensional space with integer coordinates. Recently, U. Eckhardt and L. J. Latecki [5] and Kong [21] investigated the special kinds of topologies with relation to several adjacency relations in $\mathbb{Z}^n$, $2 \leq n \leq 4$. In [3] the impossibility of topologizing the digital plane was shown while retaining 8-connectedness contrary to 4-connectedness. Moreover, it turned out that a circle with an odd number vertices $n \geq 5$ has no topology [2, 3, 22, 24].

We consider a digital graph in a quadruple $(\mathbb{Z}^n, k, \bar{k}, G)$, where $n \in \mathbb{N}$, $G$ in $\mathbb{Z}^n$ is the set we regard as belonging to the digital graph depicted, $k$ represents an adjacency relation for $G$, and $\bar{k}$ represents an adjacency relation for $\mathbb{Z}^n - G$. We say that the pair $(G, k)$ is a digital $k$-graph and denote it $G_k$.

Furthermore, we show that a graph with some $k$-adjacency in $\mathbb{Z}^n$ can be recognized in $\mathbb{R}^n$ by the simplicial complex spanned by some
digital graph. Moreover, we demonstrate that a graphically \((k_0, k_1)\)-continuous map \(f : G_{k_0} \subset \mathbb{Z}^n_0 \rightarrow G_{k_1} \subset \mathbb{Z}^n_1\) can be converted into the simplicial map \(S(f) : S(G_{k_0}) \subset \mathbb{R}^n_0 \rightarrow S(G_{k_1}) \subset \mathbb{R}^n_1\) with relation to combinatorial topology, if \(G_{k_0}\) is not \((k_0, 3^{n_0} - 1)\)-homotopy equivalent to \(SC_{3^{n_0} - 1}\).

All graphs in this paper are undirected without multiple edges and with no loops.

2. Technical notations

In this section, we briefly summarize the basic concepts of the general graph in [27]. We say that \(G = (V, E)\) is a connected graph in \(\mathbb{Z}^n\), for every pair \(p, q \in V\), if there is a finite sequence \((p = v_0, v_1, \ldots, q = v_n)\) such that \(v_iv_{i+1} \in E, i \in \{0, 1, \ldots, n - 1\}\). The finite sequence above is called a path with the length \(l(p, q) = n\). And further, the graph \(G = (V, E)\) is called a circle if \(v_0 = v_n\). For some graph \(G = (V, E)\), a graph \(G' = (V', E')\) is called an induced subgraph of \(G\) generated by \(V' \subset V\) if \(E' = \{pq \in E | p, q \in V'\}\).

Since the notion of general \(k\)-connectivity for a digital graph in \(\mathbb{Z}^n\) is absolutely necessary for studying digital \(k\)-graphs in \(\mathbb{Z}^n, n \geq 1\), In this paper, we will use the general \(k\)-adjacency relations in \(\mathbb{Z}^n\), i.e. \(k \in \{3^n - 1 | n \geq 2\}, 3^n - \sum_{t=0}^{r-2} C^n_t 2^{n-r} - 1 | 2 \leq r \leq n - 1, n \geq 3\}, 2n(n \geq 1)\)[8, 9, 15]. For example, in \(\mathbb{Z}^2\) each graph is considered with \(k\)-adjacency, where \(k \in \{8, 4\}\); in \(\mathbb{Z}^3\) \(k \in \{26, 18, 6\}\); in \(\mathbb{Z}^4\), \(k \in \{80, 64, 32, 8\}\); in \(\mathbb{Z}^5\), \(k \in \{242, 210, 130, 50, 10\}\); in \(\mathbb{Z}^6\), \(k \in \{728, 664, 472, 232, 72, 12\}\); and so forth[6, 7, 8, 9]. For \(\{a, b\} \subset \mathbb{Z}\) with \(a \leq b\), \(\mathbb{Z}^n = \{a \leq n \leq b| n \in \mathbb{Z}\}\) is called a digital interval [1].

From now on for a digital graph \(G_k = (V_k, E_k)\) and \(u, v \in V_k\) if \(u \in N_k(v) = \{v'|v\ is\ k\text{-adjacent\ to\ }v'\}\), then we call the edge \(uv \in E_k\) a \(k\)-edge. For a \(k\)-graph \(G_k = (V_k, E_k)\), two vertices \(p(\neq)q \in V_k\) are called \(k\)-connected if there is a \(k\)-path in \(G_k\) [25]: a finite sequence of vertices
(v₀, v₁, v₂, · · · , vₙ) in Vₖ such that p = v₀, q = vₙ and vᵢvᵢ₊₁ ∈ Eₖ, i ∈ [0, n − 1]Z, i.e., vᵢ and vᵢ₊₁ are k-adjacent. And the length of a k-path is the number n above.

3. Digital graph (k₀, k₁)-continuity

In this section, we need the notions of induced k-subgraph and graph k-neighborhood to establish the notion of graph (k₀, k₁)-continuity. Basically, the notion of graph (k₀, k₁)-continuity comes from the following: For two digital kᵢ-graphs Gₖᵢ = (Vₖᵢ, Eₖᵢ) in Zⁿᵢ, i ∈ {0, 1}, a map f : Gₖ₀ → Gₖ₁ is graphically (k₀, k₁)-continuous map if and only if for any (u, v) ∈ Eₖ₀, (f(u), f(v)) ∈ Eₖ₁ or f(u) = f(v).

Definition 3.1. [18] In a digital graph G = (V, E) in Zⁿ, for the subset of some vertices V′ ⊂ V, the induced k-subgraph generated by V′ is the subgraph of G which consists of vertices V′ and only k-edges joining vertices from V′.

Example 3.2. Let G = (V, E) in Z² where V = {p₀ = (−1, 2), p₁ = (−1, 1), p₂ = (0, 0), p₃ = (1, 0), p₄ = (1, 1), p₅ = (2, 1), p₆ = (3, 2)} and E = {p₂p₄, p₃p₅, pᵢpᵢ₊₁| i ∈ [0, 5]Z}.

Then the induced 4-subgraph generated by V′ = {p₂, p₃, p₄, p₅} is the 4-graph G₄ = (V₄, E₄), where V₄ = V′ and E₄ = {pᵢpᵢ₊₁| i ∈ [2, 4]Z}. Meanwhile, the induced 8-subgraph generated by V′ = {p₂, p₃, p₄, p₅} is the 8-graph G₈ = (V₈, E₈), where V₈ = V′ and E₈ = {p₂p₄, p₃p₅, pᵢpᵢ₊₁| i ∈ [2, 4]Z}.

Remark 3.3. For the set of some vertices V′ in a k-graph G = (V, E), the induced subgraph generated by V′ is different from the induced k-subgraph generated by V′. For example, in Example 3.2, the induced subgraph generated by V′ = {p₂, p₃, p₄, p₅} is (V′, E₄ ⊔ {p₂p₄, p₃p₅}) which is different from the induced 4-subgraph generated by V′.
The graph $k$-neighborhood of a vertex with radius is taken from the notion of induced $k$-subgraph in Definition 3.1 as follows.

**Definition 3.4.** [18] For a digital graph $G = (V, E)$ in $\mathbb{Z}^n$, let $k$ be some general adjacency relation in $\mathbb{Z}^n$. The graph $k$-neighborhood of $v_0 \in V$ with radius $\varepsilon$ is defined in $G$ as follows: $GN_k(v_0, \varepsilon) := (V_k(v_0, \varepsilon), E_k(v_0, \varepsilon))$ is an induced $k$-subgraph generated by $\{v \in V \mid l_k(v_0, v) \leq \varepsilon\} \cup \{v_0\}$, where $l_k(v_0, v)$ is the length of a shortest $k$-path from $v_0$ to $v$ and $\varepsilon \in \mathbb{N}$.

If there is no $k$-path from $v_0$ to $v_1$, then we assume $l_k(v_0, v_1) = \infty$. Hereafter, we use $G_k(v_0, \varepsilon)$ instead of $GN_k(v_0, \varepsilon)$.

**Example 3.5.** Let $G = (V, E)$ in $\mathbb{Z}^2$ where $V = \{p_0 = (-2, -1), p_1 = (-1, -1), p_2 = (0, 0), p_3 = (0, 1), p_4 = (0, 2), p_5 = (1, 2), p_6 = (2, 1)\}$ and $E = \{p_ip_{i+1}, p_ip_5 | i \in [0, 5]_2\}$. Using 4-adjacency for $G$, we have $l_4(p_2, (0, 1)) = 1$, $l_4(p_2, (0, 2)) = 2$, $l_4(p_2, (1, 2)) = 3$ and $l_4(p_2, (-1, -1)) = \infty = l_4(p_2, (-2, -1)) = l_4(p_2, (2, 1))$. Thus $G_4(p_2, 1)$ is an induced 4-subgraph generated by $\{p_2, (0, 1)\}$; $G_4(p_2, 2)$ is an induced 4-subgraph generated by $\{p_2, (0, 1), (0, 2)\}$; $G_4(p_2, 3)$ is an induced 4-subgraph generated by $\{p_2, (0, 1), (0, 2), (1, 2)\}$; and for $4 \leq \varepsilon \leq \infty$, $G_4(p_2, 3) = G_4(p_2, \varepsilon)$. For 8-adjacency, we have $l_8(p_2, (0, 1)) = 1 = l_8(p_2, (-1, -1))$, $l_8(p_2, (0, 2)) = 2 = l_8(p_2, (1, 2)) = l_8(p_2, (-2, -1))$, $l_8(p_2, (2, 1)) = 3$ and so forth. Thus $G_8(p_2, 1)$ is an induced 8-subgraph generated by $\{p_2, (0, 1), (-1, -1)\}$; $G_8(p_2, 2)$ is an induced 8-subgraph generated by $\{p_2, (0, 1), (0, 2), (1, 2), (-1, -1), (-2, -1)\}$; and for $3 \leq \varepsilon \leq \infty$, $G_8(p_2, \varepsilon) = G$.

Until now, there has been no approach to establish graph $(k_0, k_1)$-continuity as a digital graph analogue of the continuity in topology. We now give an introduction of graph $(k_0, k_1)$-continuity. Definitions 3.6 and 3.7 are the digital graph version of digital continuity of [7, 8, 9].
Definition 3.6. [18] Let $G_{ki} = (V_{ki}, E_{ki})$ be a $k_i$-graph in $\mathbb{Z}^{n_i}$, $i \in \{0, 1\}$. A map $f : G_{k0} \to G_{k1}$ is (graphically) $(k_0, k_1)$-continuous if for every $v_0 \in V_{k0}$, $\varepsilon \in \mathbb{N}$, and $G_{k1}(f(v_0), \varepsilon) \subset G_{k1}$, there is a $\delta$ such that the corresponding $G_{k0}(v_0, \delta) \subset G_{k0}$ satisfies $f(G_{k0}(v_0, \delta)) \subset G_{k1}(f(v_0), \varepsilon)$.

We recall a general graph isomorphism: For the two general graphs $X = (V_0, E_0)$ and $Y = (V_1, E_1)$, $X$ is isomorphic to $Y$ if there exists a bijective map $f$ between $V_0$ and $V_1$ and further, two vertices $u$ and $v$ are adjacent in $E_0$ if and only if $f(u)$ and $f(v)$ are adjacent in $E_1$.

We now establish a graph $(k_0, k_1)$-isomorphism as the digital graph analogue of a graph isomorphism.

Definition 3.7. [18] Let $G_{ki} = (V_{ki}, E_{ki})$ be a digital graph in $\mathbb{Z}^{n_i}$, $i \in \{0, 1\}$. We say that $h : G_{k0} \to G_{k1}$ is a graph $(k_0, k_1)$-isomorphism if

- the restriction map on $V_{k0}$, $h|_{V_{k0}}$, is bijective;
- $h$ is graphically $(k_0, k_1)$-continuous; and
- $h^{-1}$ is graphically $(k_1, k_0)$-continuous.

Then we use the notation, $G_{k0} \cong_{(k_0, k_1)} G_{k1}$. If $k_0 = k_1$, we call it a graph $k_0$-isomorphism.

Here is the digital graph version of the digital $(k_0, k_1)$-covering space in [11, 12, 16].

Definition 3.8. [18] Let $G_{ki} = (V_{ki}, E_{ki})$ be a digital $k_i$-graph on $\mathbb{Z}^{n_i}$, $i \in \{0, 1\}$, and $p : G_{k0} \to G_{k1}$ be a graphically $(k_0, k_1)$-continuous surjection.

Suppose for any $v_1 \in V_{k1}$ there exists $\varepsilon \in \mathbb{N}$ such that

- (GC 1) for some $\delta \in \mathbb{N}$ and some index set $M$,
  
  $p^{-1}((G_{k1}(v_1, \varepsilon)) = \bigcup_{i \in M} G_{k0}(v_{0,i}, \delta)$ with $v_{0,i} \in p^{-1}(v_1)$;

- (GC 2) if $i, j \in M$ and $i \neq j$, then $G_{k0}(v_{0,i}, \delta) \cap G_{k0}(v_{0,j}, \delta) = \emptyset$; and

- (GC 3) the restriction map $p$ on $G_{k0}(v_{0,i}, \delta), p|_{G_{k0}(v_{0,i}, \delta)} : G_{k0}(v_{0,i}, \delta) \to G_{k1}(v_1, \varepsilon)$, is a graph $(k_0, k_1)$-isomorphism for all $i \in M$. 
Then we say that the map \( p \) is a graph \((k_0,k_1)\)-covering map and \( G_{k_0} \) is a graph \((k_0,k_1)\)-covering over \( G_{k_1} \). And we call \((G_{k_0}, p, G_{k_1})\) a graph \((k_0,k_1)\)-covering. And further, we say that \( G_{k_0} \) is a total \( k_0 \)-graph and \( G_{k_1} \) is a base \( k_0 \)-graph. If the map \( p : (G_{k_0}, v_0) \to (G_{k_1}, v_1) \) is a graph \((k_0,k_1)\)-covering with \( p(v_0) = v_1 \), then we say that it is the pointed graph \((k_0,k_1)\)-covering map.

The graph \( k_1 \)-neighborhood of \( v_1 \in V_{k_1} \) with radius \( \varepsilon \), \( G_{k_1}(v_1, \varepsilon) \subset G_{k_1} \), is called an elementary graph \( k_1 \)-neighborhood.

The collection \( \{G_{k_0}(v_0,i, \delta) \mid i \in M \} \) is a partition of \( p^{-1}(G_{k_1}(v_1, \varepsilon)) \).

Let \( SC_{k}^{n,l} \) be a simple \( k \)-circle with distinct \( l \) vertices on \( \mathbb{Z}^n \). Namely, we assume \( SC_{k}^{n,l} = (V_k, E_k) \), where \( V_k = \{v_0, v_1, v_2, \cdots, v_{l-1}\} \subset \mathbb{Z}^n \) and \( E_k = \{v_i v_{i+1}, v_{i-1}v_0 \mid i \in [0, l - 2]_\mathbb{Z}\} \) such that \( v_i \) and \( v_j \) are \( k \)-adjacent if and only if \( j = i \pm 1 \pmod{l} \) for any \( i \in [0, l - 1]_\mathbb{Z} \). Then we get the following examples:

**Example 3.9.** \( (\mathbb{Z}, p, SC_{k}^{n,l}) \) is a graph \((2,k)\)-covering, where \( p : \mathbb{Z} \to SC_{k}^{n,l} \) is the graphically \((2,k)\)-continuous map with \( p(i) = v_i(\mod{l}) \) and \( p([i - 1, i]_\mathbb{Z}) = e_i(\mod{l}) \) for all \( i \in \mathbb{Z} \), \( SC_{k}^{n,l} = (V_k, E_k) \) with \( V_k = \{v_0, v_1, v_2, \cdots, v_{l-1}\} \) and \( E_k = \{e_i = v_i v_{i+1}, e_{i-1} = v_{i-1}v_0 \mid i \in [0, l - 2]_\mathbb{Z}\} \).

**Theorem 3.10.** Let \( G_{k_1} = (V_{k_1}, E_{k_1}) \) be a \( k_1 \)-connected graph with vertex set \( V_{k_1} \), let \( (X, p) \) be a graph \((k_0,k_1)\)-covering of \( G_{k_1} \), and \( V_{k_0} = p^{-1}(V_{k_1}) \). Then \( X = G_{k_0} = (V_{k_0}, E_{k_0}) \).

**Proof.** First, we can see easily that \( V_{k_0} \) is a \( k_0 \)-connected and a discrete space. If not, it leads to a contradiction to the graph \((k_0,k_1)\)-covering of \( p \). Next, for any \( e_i \in E_{k_1} \), each component of \( p^{-1}(e_i) \) is a covering graph of \( e_i \). And further, for each component \( e'_i \in p^{-1}(e_i) \), \( p|_{e'_i} : e'_i \to e_i \) is a \((k_0,k_1)\)-isomorphism. Moreover, for any \( e_i \in E_{k_1} \), with \( v_i v_{i+1} = e_i \), there is a \( k_0 \)-edge \( v'_i v'_{i+1} = e'_i \), where \( v'_i \in p^{-1}(v_i) \), \( v'_{i+1} \in p^{-1}(v_{i+1}) \) and \( e'_i \in p^{-1}(e_i) \). And further, \((G'_{k_0}, p|_{G'_{k_0}}, G'_{k_1})\) is also a graph \((k_0,k_1)\)-covering, where \( G'_{k_j} = (V'_{k_j}, E'_{k_j}) \), \( j \in \{0,1\} \) and
\[ V'_{k_0} = \{ v'_i, v'_{i+1} \}, \quad V'_{k_1} = \{ v_i, v_{i+1} \}, \quad E'_{k_0} = \{ e'_i \} \text{ and } E'_{k_1} = \{ e_i \}. \]

By induction, we can get the graph \((k_0, k_1)\)-covering \((G_{k_0} = X, p, G_{k_1})\) induced from \(p^{-1}(V_{k_1})\), as required. \(\square\)

In fact, due to some properties of the graph \((k_0, k_1)\)-covering, digital graphical invariant can be investigated.

### 4. Graph \((k_0, k_1)\)-homotopy

The digital graph homotopy was introduced [4, 18] and further, the digital \(k\)-fundamental group of a digital graph was calculated in terms of \(L(X)\)-deformation [23].

In this paper, the current notion is the digital graph version of the digital \((k_0, k_1)\)-homotopy of [1] with our digital graph \((k_0, k_1)\)-continuity and general \(k_i\)-adjacency relations in \(\mathbb{Z}^n, i \in \{0, 1\}\).

Roughly, a graph \((k_0, k_1)\)-homotopy represents the following: Graphically \((k_0, k_1)\)-continuous maps \(f, g : G_{k_0} \to G_{k_1}\) are \((k_0, k_1)\)-homotopic if there is a continuous deformation of \(f\) in \(G_{k_1}\) and finally, the deformed map coincides with \(g\). Moreover, we remind that the mapping of edges in graph \((k_0, k_1)\)-continuity is absolutely determined by that of vertices.

The following Definitions 4.1 and 4.2 are the digital graph version of digital \((k_0, k_1)\)-homotopy of [1].

**Definition 4.1.** Let \(G_{k_i} = (V_{k_i}, E_{k_i})\) be a \(k_i\)-graph in \(\mathbb{Z}^n, i \in \{0, 1\}\), and \(f, g : G_{k_0} \to G_{k_1}\) be graphically \((k_0, k_1)\)-continuous maps. And suppose that there is a positive integer \(m\) and a map \(F : G_{k_0} \times [0, m]_{\mathbb{Z}} \to G_{k_1}\) such that

- for all \(x \in V_{k_0}, e_i \in E_{k_0}\), the map \(F\) is defined by \(F(x, 0) = f(x), F(e_i, 0) = f(e_i)\) and \(F(x, m) = g(x), F(e_i, m) = g(e_i)\);

- for all \(x \in V_{k_0}\), the induced map \(F\) determined by \(F_x : [0, m]_{\mathbb{Z}} \to V_{k_1}\) with \(F_x(t) = F(x, t)\) is graphically \((2, k_1)\)-continuous for all \(t \in [0, m]_{\mathbb{Z}}\); and
A graph is taken as follows: \( f \) of \( \pi_k \) extension [1], which depends on the \( =p \) \( f \) based at \( \text{notation} \). Then we say that \( f \) and \( g \) are graphically \((k_0,k_1)\)-homotopic each other and the map \( F \) is called a graph \((k_0,k_1)\)-homotopy and we use the notation \( f \simeq_{(k_0,k_1)} g \).

**Definition 4.2.** A graph \( G_k = (V_k,E_k) \) is graphically \( k \)-contractible if and only if \( 1_{G_k} \simeq_{k} c_{\{v_0\}} \) for a constant map \( c_{\{v_0\}} \) and for some \( v_0 \in V_k \).

We now make a digital graph analogue of the fundamental group of [1] without any difficulties in dimension and \( k \)-adjacency relations of graphs in \( \mathbb{Z}^n \).

Let \( (G_k,\{p\}) \) be a \( k \)-graph with base vertex \( \{p\} \) in \( \mathbb{Z}^n \). A \( k \)-loop \( f \) based at \( p \) is a \( k \)-path, \( f : [0,m_f]\mathbb{Z} \to (G_k,\{p\}) \) such that \( f(0) = \{p\} \). The number \( m_f \) is not fixed from the notion of trivial extension [1], which depends on the \( k \)-loop on \( (G_k,\{p\}) \). And we put \( F^k(G_k,\{p\}) = \{f | f \text{ is a } k \text{-loop based at } p \} \). For maps \( f,g \in F^k(G_k,\{p\}) \), i.e., \( f : [0,m_f]\mathbb{Z} \to (G_k,\{p\}) \) with \( f(0) = p = f(m_f) \) and \( g : [0,m_g]\mathbb{Z} \to (G_k,\{p\}) \) with \( g(0) = p = g(m_g) \), a map \( f * g : [0,m_f + m_g]\mathbb{Z} \to (G_k,\{p\}) \) is taken as follows: \( f * g : [0,m_f + m_g]\mathbb{Z} \to (G_k,\{p\}) \) is defined by

\[
(f * g)(t) = \begin{cases} 
  f(t) & \text{if } 0 \leq t \leq m_f; \\
  g(t - m_f) & \text{if } m_f \leq t \leq m_f + m_g.
\end{cases}
\]

Then \( f * g \in F^k(G_k,\{p\}) \). We denote by \( [f] \) the graph \( k \)-homotopy class of \( f \). Obviously, the homotopy class \( [f * g] \) only depends on the homotopy classes \( [f] \) and \( [g] \). Furthermore, if \( f_1,f_2,g_1,g_2 \in F^k(G_k,\{p\}) \) with \( f_1 \in [f_2], g_1 \in [g_2] \), then \( [f_1 * g_1] = [f_2 * g_2] \) [1, 19]. Finally, the \( k \)-fundamental group for digital graphs is established with the operation \( [f] \cdot [g] = [f * g] \) [1] as follows.

\[
\pi^k_1(G_k,\{p\}) = \{[f] | f \in F^k(G_k,\{p\})\}.
\]

In the following, we use the notation \( \pi^k_1(G_k,p) \) (resp. \((G_k,p)\)) instead of \( \pi^k_1(G_k,\{p\}) \) (resp. \((G_k,\{p\})\)).
Obviously, for any graphically \((k_0, k_1)\)-continuous map \(h : G_{k_0} \to G_{k_1}\), there is a group homomorphism \(h_* : \pi_1^{k_0}(G_{k_0}, p) \to \pi_1^{k_1}(G_{k_1}, h(p))\) with \(h_*([f]) = [h \circ f]\), where \([f] \in \pi_1^{k_0}(G_{k_0}, p)\) and \(\circ\) means the composition. And further, if a graph \(G_k\) is \(k\)-connected, then \(\pi_1^{k}(G_k, p)\) is isomorphic to \(\pi_1^{k}(G_k, q)\) for any \(p, q \in G_k\) [1].

**Theorem 4.3.** A graph \((k_0, k_1)\)-isomorphism \(h : G_{k_0} \to G_{k_1}\) induces a group isomorphism \(h_* : \pi_1^{k_0}(G_{k_0}, p) \to \pi_1^{k_1}(G_{k_1}, h(p))\) with \(h_*([f]) = [h \circ f]\), where \([f] \in \pi_1^{k_0}(G_{k_0}, p)\).

**Proof.** First, \(h_*\) is well-defined. If \(f' \in [f] \in \pi_1^{k_0}(G_{k_0}, p)\), let \(F : (G_{k_0}, p) \times [0, m]_Z \to (G_{k_0}, p)\) be a graph \(k_0\)-homotopy between \(f\) and \(f'\). Then \(h \circ F\) is a graph \(k_1\)-homotopy between the \(k_1\)-loops \(h \circ f\) and \(h \circ f'\). Thus \(h \circ f' \in [h \circ f]\).

Second, the induced map \(h_*\) is a homomorphism.

For any maps \(f, g \in F_1^{k_0}(G_{k_0}, p)\), the graphically \((2, k_0)\)-continuous maps \(f : [0, m_f]_Z \to (G_{k_0}, p)\) and \(g : [0, m_g]_Z \to (G_{k_0}, p)\),

the map \(h \circ (f \ast g) : [0, m_f + m_g]_Z \to (G_{k_1}, q)\) is defined as follows:

\[
 h \circ (f \ast g)(t) = \begin{cases} h(f(t)) & 0 \leq t \leq m_f; \\ h(g(t - m_f)) & m_f \leq t \leq m_f + m_g. \end{cases}
\]

Thus \(h \circ (f \ast g) = (h \circ f) \ast (h \circ g)\) and \(h_*([f] \cdot [g]) = h_*([f] \ast [g]) = [h \circ (f \ast g)] = [h \circ f] \cdot [h \circ g] = h_*([f]) \cdot h_*([g])\).

The induced map \(h_*\) depends not only on the graphically \((k_0, k_1)\)-continuous map \(h : (G_{k_0}, p) \to (G_{k_1}, q)\) but also on the choice of the base points \(p\) and \(q\).

Second, \(h_*\) is bijective. First of all for any \([g] \in \pi_1^{k_1}(G_{k_1}, q)\), we get \(g : [0, m]_Z \to (G_{k_1}, q)\) is a \((2, k_1)\)-continuous map such that \(g(0) = q = g(m)\). Because \(h\) is a \((k_0, k_1)\)-isomorphism, there is a \((2, k_0)\)-continuous map: \(f_1 : [0, m]_Z \to (G_{k_0}, p)\) such that \(f_1(0) = p = f_1(m)\) and \(h \circ f_1 = g\).

Thus \(h_*([f_1]) = [h \circ f_1] = [g]\).
Next, \( h_2 \) is injective: if \( h_2([f_1]) = [h \circ f_1] = c_{(q)} \in \pi_1^{k_1}(G_{k_1}, q) \), we only prove that \( f_1 \simeq_{d k_0} c_{(p)}. \) Since \( h \circ f_1 \simeq_{d k_1} c_{(q)} \), there is a \((2, k_0)\)-continuous map \( f_1 : [0, m]_\mathbb{Z} \to (G_{k_0}, p) \) such that \( f_1(0) = p = f_1(m) \) and \( f_1 \simeq_{d k_0} c_{(p)}. \)

Third, \( h_2 \) is a homomorphism. For any \([f_1], [f_2] \in \pi_1^{k_0}(G_{k_0}, p), h_2([f_1] \cdot [f_2]) = h_2([f_1 * f_2]) = [h \circ ((f_1 * f_2))] = [(h \circ f_1) \cdot (h \circ f_2)] = h_2[f_1] \cdot h_2[f_2]. \) ☐

**Example 4.4.** For example, let \( G_8 = (V_8, E_8) \), where \( V_8 = \{v_0 = (x_1, y_1), v_1 = (x_1 - 1, y_1 + 1), v_2 = (x_1 - 2, y_1), v_3 = (x_1 - 1, y_1 - 1)\} \) and \( E_8 = \{v_i v_{i+1}, v_3 v_0 | i \in \{0, 1, 2\}\} \). While Kong’s and Malgouyres’ fundamental group \( \pi_1^8(G_8) \) are not trivial, Boxer’s fundamental group \( \pi_1^8(G_8) \) is trivial \([1, 16]\).

We now state the graph \((k_0, k_1)\)-homotopy equivalence.

**Definition 4.5.** For a \( k_0 \)-graph \( G_{k_0} \) in \( \mathbb{Z}^{n_0} \) and a \( k_1 \)-graph \( G_{k_1} \) in \( \mathbb{Z}^{n_1} \), if there is a graphically \((k_0, k_1)\)-continuous map \( h : G_{k_0} \to G_{k_1} \) and a graphically \((k_1, k_0)\)-continuous map \( l : G_{k_1} \to G_{k_0} \) such that \( l \circ h \simeq_{k_0} 1_{G_{k_0}} \) and \( h \circ l \simeq_{k_1} 1_{G_{k_1}} \), then the map \( h : G_{k_0} \to G_{k_1} \) is called a graph \((k_0, k_1)\)-homotopy equivalence. And we use the notation, \( G_{k_0} \simeq_{h, (k_0, k_1)} e G_{k_1} \). Furthermore, if \( k_0 = k_1 \), we call \( h \) a graph \( k_0 \)-homotopy equivalence and denote it by \( G_{k_0} \simeq_{h, k_0} e G_{k_1} \).

**Example 4.6.** Here is an example of a graph 8-homotopy equivalence. Put \( G_8 = (V_8, E_8), V_8 = \{v_1 = (x_1, y_1), v_2 = (x_1, y_1 + 1), v_3 = (x_1, y_1 + 2), v_4 = (x_1 - 1, y_1 + 2), v_5 = (x_1 - 2, y_1 + 1), v_6 = (x_1 - 2, y_1), v_7 = (x_1 - 2, y_1 - 1), v_8 = (x_1 - 1, y_1 - 1)\}, E_8 = \{v_2 v_4, v_6 v_8, v_8 v_1, v_1 v_{i+1} | i \in [1, 7]_\mathbb{Z}\}. \) Meanwhile, \( G_8' = (V_8', E_8'), V_8' = \{w_1 = v_1, w_2 = v_2, w_3 = v_4, w_4 = v_5, w_5 = v_6, w_6 = v_8\}, E_8' = \{w_6 w_1, w_i w_{i+1} | i \in [1, 5]_\mathbb{Z}\}. \) Then consider the map \( l : G_8' \to G_8 \) as the inclusion map. And \( h : G_8 \to G_8' \) is defined as follows: \( h(v_1) = w_i, i \in \{1, 2\}, h(v_i) = w_3, i \in \{3, 4\}, h(v_i) = w_{i-1}, i \in \{5, 6\} \) and \( h(v_1) = w_6, i \in \{7, 8\}. \) Then the maps \( h \) and \( l \) are graphically 8-continuous. Finally \( G_8 \simeq_{h, 8, e} G_8'. \)
5. A simplicial complex spanned by a digital graph

A digital graph-theoretical approach has some advantages to study digital images with relation to the study of a simplicial complex in combinatorial topology. And further, it turns out that a digital image with $k$-adjacency in $\mathbb{Z}^n$ can be recognized by the simplicial complex spanned by some digital graph in $\mathbb{Z}^n$. Moreover, a graphical $(k_0, k_1)$-continuous map $f : G_{k_0} \to G_{k_1}$ is proved to be converted into the simplicial map $S(f) : S(G_{k_0}) \to S(G_{k_1})$ with relation to combinatorial topology.

We now recall the notions of a simplicial complex and a simplicial map\cite{26} with relation to the study of digital graphs in $\mathbb{Z}^n$.

For a digital $k$-graph $G_k = (V_k, E_k)$ in $\mathbb{Z}^n$, the simplicial complex spanned by $G_k$ is established in $\mathbb{R}^n$, whose set of vertices is $V_k$ and set of edges is $E_k$. Consequently, a combinatorial geometric model from $G_k$ is determined. Thus the study of digital graph is very related to that of combinatorial geometry.

**Example 5.1.** Consider the digital 8-graph $G_8 = (V_8, E_8)$ in $\mathbb{Z}^2$, where $V_8 = \{v_1 = (x_1, y_1), v_2 = (x_1 - 1, y_1 - 1), v_3 = (x_1, y_1 - 1), v_4 = (x_1 + 1, y_1 - 2), v_5 = (x_1 + 1, y_1 - 3), v_6 = (x_1 + 2, y_1 - 2)\}$ and $E_8 = \{v_1v_2, v_2v_3, v_1v_3, v_3v_4, v_4v_5, v_5v_6, v_4v_6\}$. Then the simplicial complex spanned by $G_8$ consists of the two triangles, $v_1v_2v_3$, $v_4v_5v_6$ and a line $v_3v_4$ in $\mathbb{R}^2$.

Throughout this section $S(G_k)$ stands for the simplicial complex in $\mathbb{R}^n [26]$ spanned by $G_k \subset \mathbb{Z}^n$.

A digital image with some $k$-adjacency can be presented by the digital $k$-graphs in $\mathbb{Z}^n$ and further, recognized by the simplicial complex in $\mathbb{R}^n$ spanned by some digital graph. Moreover, a graphical $(k_0, k_1)$-continuous map $f : G_{k_0} \to G_{k_1}$ can be converted into the simplicial map $S(f) : S(G_{k_0}) \to S(G_{k_1})$(Theorem 5.3).
The following proposition characterizes the relationship between digital $k$-graph $G_k$ and the simplicial complex spanned by $G_k$. The following Proposition is stemmed from Definitions 3.1 and 3.4.

**Proposition 5.2.** A digital $k$-graph $G_k$ in $\mathbb{Z}^n$ determines the simplicial complex $S(G_k)$ in $\mathbb{R}^n$.

**Theorem 5.3.** Let $G_{k_i} = (V_{k_i}, E_{k_i})$ be a digital graph in $\mathbb{Z}^{n_i}$, $i \in \{0, 1\}$. Then a graphically $(k_0, k_1)$-continuous map $f : G_{k_0} \to G_{k_1}$ determines a simplicial map $S(f) : S(G_{k_0}) \to S(G_{k_1})$.

**Proof.** The proof is completed from Proposition 5.2 and Definition 3.6. □

**Theorem 5.4.** Let $G_{k_i} = (V_{k_i}, E_{k_i})$ be a digital graph in $\mathbb{Z}^{n_i}$, $i \in \{0, 1\}$, and $f : G_{k_0} \to G_{k_1}$ be a graphically $(k_0, k_1)$-continuous map (respectively a graphically $(k_0, k_1)$-isomorphism) such that $G_{k_0}$ is not $(k_0, 3^{n_0} - 1)$-homotopy equivalent to $SC_{3^{n_0}-1}^{n_0, 4}$. Then the map $f$ induces the group homomorphism (respectively the group isomorphism) $(S(f))_* : \pi_1(S(G_{k_0}), v_0) \to \pi_1(S(G_{k_1}), f(v_0))$ in algebraic topology via $S(f)_*(\alpha) = [S(f)(\alpha)]$. But the converse does not hold.

**Proof.** First, we show the need for the assumption that $G_{k_0}$ is not $(k_0, 3^{n_0} - 1)$-homotopy equivalent to $SC_{3^{n_0}-1}^{n_0, 4}$. We saw in Example 4.4 that the digital graph $G_8$ is pointed graphically 8-contractible. Even though the digital 8-fundamental group $\pi_1^8(G_8, v_0) = 0$, the typical fundamental group $\pi_1(S(G_8), v_0)$ is a cyclic group in algebraic topology. Thus, the conclusion of Theorem 5.4 may fail if $\pi_1^{k_0}(G_{k_0}, v_0) = 0$. Next, the proof is completed from Proposition 5.2, Theorem 5.3 and Definition 3.6. We can see easily that the converse does not hold in general. □

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