AN EXTENDED NON-ASSOCIATIVE ALGEBRA

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Abstract. A Weyl type algebra is defined in the paper (see [2],[4], [6], [7]). A Weyl type non-associative algebra $\mathcal{W}_{n,m,s}$ and its restricted subalgebra $\mathcal{W}_{n,m,s,r}$ are defined in the papers (see [1], [14], [16]). Several authors find all the derivations of an associative (Lie or non-associative) algebra (see [3], [1], [5], [7], [10], [16]). We find $\text{Der}(\mathcal{W}_{0,0,1_n})$ of the algebra $\mathcal{W}_{0,0,1_n}$ and show that the algebras $\mathcal{W}_{0,0,1_n}$ and $\mathcal{W}_{0,0,1_n}$ are not isomorphic in this work. We show that the associator of $\mathcal{W}_{0,0,1_n}$ is zero.

1. Preliminaries

Let $\mathbb{F}$ be a field of characteristic zero (not necessarily algebraically closed). Throughout this paper, $\mathbb{N}$ and $\mathbb{Z}$ will denote the non-negative integers and the integers, respectively. Let $\mathbb{F}[x_1,\ldots,x_{m+s}]$ be the polynomial ring with the variables $x_1,\ldots,x_{m+s}$. Let $g_1,\ldots,g_n$ be given polynomials in $\mathbb{F}[x_1,\ldots,x_{m+s}]$. For $n,m,s \in \mathbb{N}$, let us define the commutative, associative $\mathbb{F}$-algebra

$$\mathbb{F}_{g_n,m,s} = \mathbb{F}[e^{\pm g_1},\ldots,e^{\pm g_n},x_1^{\pm 1},\ldots,x_m^{\pm 1},x_{m+1},\ldots,x_{m+s}]$$

in the formal power series ring $\mathbb{F}[[x_1,\ldots,x_{m+s}]]$ which is called a stable algebra in the paper (see [9]) with the standard basis

$$\{e^{a_1g_1}\ldots e^{a_ng_n}x_1^{i_1}\ldots x_{m+s}^{i_{m+s}}|a_1,\ldots,a_n,i_1,\ldots,i_m \in \mathbb{Z}, i_{m+1},\ldots,i_{m+s} \in \mathbb{N}\}$$

and with the obvious addition and the multiplication (see [9] and [14]).

$\partial_w$, $1 \leq w \leq m+s$, denotes the usual partial derivative with respect to

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$x_w$ on $\mathbb{F}_{g_n,m,s}$. For partial derivatives $\partial_u, \ldots, \partial_v$ of $\mathbb{F}_{g_n,m,s}$, the composition $\partial_u^{j_u} \circ \cdots \circ \partial_v^{j_v}$ of them is denoted $\partial_u^{j_u} \cdots \partial_v^{j_v}$ where $j_u, \ldots, j_v \in \mathbb{N}$. Let $A$ be the set \{ $\partial_u^{j_u} \circ \cdots \circ \partial_v^{j_v}$ | $j_u, \ldots, j_v \in \mathbb{N}$, $\partial_w$ is the partial derivation of $\mathbb{F}_{g_n,m,s}$ with respect to $x_w$, $1 \leq w \leq m + s$ \}.

Let us define the vector space $WN(g_n, m, s) = WN(g_n, m, s)_A$ over $\mathbb{F}$ which is spanned by the standard basis

$$\{ e^{a_1 g_{n_1}} \cdots e^{a_n g_{n_m}} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} | a_1, \ldots, a_n, i_1, \ldots, i_m \in \mathbb{Z},$$

$$i_{m+1}, \ldots, i_{m+s} \in \mathbb{N}, j_u, \ldots, j_v \in \mathbb{N}, 1 \leq u, \ldots, v \leq m + s \}$$

(1)

Thus we may define the multiplication $*$ on $WN(g_n, m, s)$ as follows:

$$e^{a_1 g_{n_1}} \cdots e^{a_n g_{n_m}} x_1^{i_{11}} \cdots x_{m+s}^{i_{1,m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} *$$

$$= e^{a_1 g_{n_1}} \cdots e^{a_n g_{n_m}} x_1^{i_{21}} \cdots x_{m+s}^{i_{2,m+s}} \partial_h^{j_h} \cdots \partial_w^{j_w}$$

(2)

$$\partial_u^{j_u} \cdots \partial_v^{j_v} (e^{a_1 g_{n_1}} \cdots e^{a_n g_{n_m}} x_1^{i_{21}} \cdots x_{m+s}^{i_{2,m+s}} \partial_h^{j_h} \cdots \partial_w^{j_w})$$

for any basis elements $e^{a_1 g_{n_1}} \cdots e^{a_n g_{n_m}} x_1^{i_{11}} \cdots x_{m+s}^{i_{1,m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v}$ and $e^{a_1 g_{n_1}} \cdots e^{a_n g_{n_m}} x_1^{i_{21}} \cdots x_{m+s}^{i_{2,m+s}} \partial_h^{j_h} \cdots \partial_w^{j_w} \in WN(g_n, m, s)$. Thus we can define the Weyl-type non-associative algebra $WN_{g_n,m,s}$ with the multiplication * in (2) and with the set $WN(g_n, m, s)$ (see [15] and [16]). For $B \subseteq A$, let us define the the non-associative subalgebra $WN_{g_n,m,s,B}$ of the non-associative algebra $WN_{g_n,m,s}$ spanned by

$$\{ e^{a_1 g_{n_1}} \cdots e^{a_n g_{n_m}} x_1^{i_1} \cdots x_s^{i_s} \partial_u^{j_u} \cdots \partial_v^{j_v} | a_1, \ldots, a_n, i_1, \ldots, i_m \in \mathbb{Z},$$

$$i_{m+1}, \ldots, i_s, j_u, \ldots, j_v \in \mathbb{N}, \partial_u^{j_u} \cdots \partial_v^{j_v} \in B, 1 \leq u, \ldots, v \leq m + s \}$$

(3)

If we take $B = \{ \partial_u^0 \cdots \partial_v^0 \}$, then the algebra $WN_{g_n,m,s,B}$ is the $\mathbb{F}$-algebra $F_{g_n,m,s}$. This implies that the algebra $WN_{g_n,m,s,B}$ contains the polynomial ring, naturally. The simplicity of $WN_{g_n,m,s,B}$ is depending on the set $B$. It is well known that the non-associative algebra $WN_{g_n,m,s}$ is simple, even though it has the right annihilator (see [7] and [8]). A non-associative algebra $WN_{g_n,m,s,B}$ is symmetric if there is a isomorphism
induced by change of variables (see [5]). Throughout the paper we put \( \partial^1 = \partial \) and \( \partial^k = \partial(\partial^{k-1}) \).

2. **Derivations of \( \overline{WN}_{0,0,13} \)**

**Lemma 1.** For any derivation \( D \) of the non-associative algebra \( \overline{WN}_{0,0,13} \)

\[ = \langle \{ x^i \partial, x^i \partial^2, x^i \partial^3 | i \in \mathbb{N} \} \rangle, \] and for any \( x^i \partial^j \), \( 1 \leq j \leq 3 \), we have the followings

\[
D(\partial) = D(\partial^2) = D(\partial^3) = 0
\]

\[
D(x^i \partial^j) = i s_1 x^{i-1} \partial^j
\]

where \( s_1 \in \mathbb{F} \).

**Proof.** Let \( D \) be the derivation in the lemma. Since \( \partial_x \) is the annihilator of itself, we have that \( \partial \ast D(\partial) = 0 \). This implies that

\[
D(\partial) = \sum_{1 \leq p \leq 3} a_p \partial^p
\]

where \( a_p \in \mathbb{F}, 1 \leq p \leq 3 \). Similarly, we can prove the followings:

\[
D(\partial^2) = \sum_{1 \leq p \leq 3} b_p \partial^p \quad \text{and} \quad D(\partial^3) = \sum_{1 \leq p \leq 3} c_p \partial^p
\]

with appropriate coefficients. Since \( x \partial \) is a right identity of \( \partial \), we have that \( \partial \ast D(x \partial) = a_2 \partial^2 + a_3 \partial^3 \). This implies that

\[
D(x \partial) = a_2 x \partial^2 + a_3 x \partial^3 + s_1 \partial + s_2 \partial^2 + s_3 \partial^3
\]

where \( s_m \in \mathbb{F}, 1 \leq m \leq 3 \). Since \( x \partial \) is an idempotent, we have that

\[
D(x \partial) = a_2 x \partial^2 + a_3 x \partial^3 + s_1 \partial
\]

Since \( D(\partial \ast x^2 \partial) = 2D(x \partial) \), we can prove that

\[
\partial \ast D(x^2 \partial) = 2a_2 x \partial^2 + 2a_3 x \partial^3 + 2s_1 \partial - 2a_2 \partial - 2a_1 x \partial
\]
Similarly, we can prove that

\[ D(x^2 \partial) = -a_1 x^2 \partial + a_2 x^2 \partial^2 + a_3 x^2 \partial^3 + 2s_1 x \partial - 2a_2 x \partial + t_1 \partial + t_2 \partial^2 + t_3 \partial^3 \]

where \( t_1, t_2, t_3 \in \mathbb{F} \). Since \( D(x \partial \ast x^2 \partial) = 2D(x^2 \partial) \), we can also prove that

\[ D(x^2 \partial) = -a_1 x^2 \partial + a_2 x^2 \partial^2 + a_3 x^2 \partial^3 + 2s_1 x \partial \]

By \( D(\partial \ast x^3 \partial) = 3D(x^2 \partial) \), we have that

\[ D(x^3 \partial) = -2a_1 x^3 \partial + a_2 x^3 \partial^2 + a_3 x^3 \partial^3 + 3s_1 x^2 \partial - 3a_2 x^2 \partial \\
\quad \quad \quad - 6a_3 x \partial + r_1 \partial + r_2 \partial^2 + r_3 \partial^3 \]

where \( r_1, r_2, r_3 \in \mathbb{F} \). Since \( D(x \partial \ast x^3 \partial) = 3D(x^3 \partial) \), we also have that

\[ D(x^3 \partial) = -2a_1 x^3 \partial + a_2 x^3 \partial^2 + a_3 x^3 \partial^3 + 3s_1 x^2 \partial \]

By \( D(x^2 \partial \ast x^2 \partial) = 2D(x^3 \partial) \), we have that

\[ -2a_1 x^3 \partial + a_2 x^3 \partial^2 + a_2 x^2 \partial + a_3 x^3 \partial^3 + 3s_1 x^2 \partial = -2a_1 x^3 \partial + a_2 x^3 \partial^2 + a_3 x^3 \partial^3 + 3s_1 x^2 \partial. \]

This implies that \( a_2 = 0 \) and we have the followings:

\[ D(x \partial) = a_3 x \partial^3 + s_1 \partial \]

\[ D(x^2 \partial) = -a_1 x^2 \partial + a_3 x^2 \partial^3 + 2s_1 x \partial \]

\[ D(x^3 \partial) = -2a_1 x^3 \partial + a_3 x^3 \partial^3 + 3s_1 x^2 \partial \]

(6)

By \( D(\partial \ast x^4 \partial) = 4D(x^3 \partial) \), we have that

\[ D(x^4 \partial) = -3a_1 x^4 \partial - 12a_3 x^2 \partial + a_3 x^4 \partial^3 + 4s_1 x^3 \partial \\
\quad \quad \quad + u_1 \partial + u_2 \partial^2 + u_3 \partial^3 \]

where \( u_1, u_2, u_3 \in \mathbb{F} \). By \( D(x \partial \ast x^4 \partial) = 4D(x^4 \partial) \), we can also prove that

(7) \[ D(x^4 \partial) = -3a_1 x^4 \partial + a_3 x^4 \partial^3 + 4s_1 x^3 \partial \]

By \( D(x^2 \partial \ast x^3 \partial) = 3D(x^4 \partial) \), we have that

(8) \[ D(x^4 \partial) = -3a_1 x^4 \partial + 2a_3 x^2 \partial + a_3 x^4 \partial^3 + 4s_1 x^3 \partial \]
By comparing (7) and (8), we have that $a_3 = 0$. The formula (6) and (7) become

\[
D(x\partial) = s_1 \partial \\
D(x^2\partial) = -a_1 x^2 \partial + 2s_1 x \partial \\
D(x^3\partial) = -2a_1 x^3 \partial + 3s_1 x^2 \partial \\
D(x^4\partial) = -3a_1 x^4 \partial + 4s_1 x^3 \partial
\]

(9)

By $D(\partial \ast x^2 \partial) = D(\partial^2)$ and $D(x\partial \ast x^2 \partial) = D(x\partial^2)$, we can also prove that

\[
D(x\partial^2) = -a_1 x \partial^2 + b_1 x \partial + b_2 x^2 \partial + b_3 x \partial^3 + s_1 \partial^2
\]

Similarly, we can prove the followings:

\[
D(x^2 \partial^2) = -2a_1 x^2 \partial^2 + b_1 x^2 \partial + b_2 x^2 \partial^2 + b_3 x^3 \partial^3 + 2s_1 x \partial^2
\]

(10)

\[
D(x^3 \partial^2) = -3a_1 x^3 \partial^2 + b_1 x^3 \partial + b_2 x^3 \partial^2 + b_3 x^3 \partial^3 + 3s_1 x^2 \partial^2
\]

Since $x\partial^2$ annihilates itself, we have that $b_1$ is zero. By $D(\partial^2 \ast x^2 \partial^2) = 2D(\partial^2)$ and $D(x\partial^2 \ast x^2 \partial^2) = 2D(x\partial^2)$, we can prove that $a_1$ and $b_2$ are zeroes. This implies that

\[
D(x\partial^2) = b_3 x\partial^3 + s_1 \partial^2
\]

\[
D(x^2 \partial^2) = b_3 x^2 \partial^3 + 2s_1 x \partial^2
\]

\[
D(x^3 \partial^2) = b_3 x^3 \partial^3 + 3s_1 x^2 \partial^2
\]

(11)

By $D(x^2 \partial^2 \ast x^3 \partial^2) = 6D(x^3 \partial^2)$, we have that $b_3 = 0$. This implies that $D(\partial) = D(\partial^2) = 0$ and

\[
D(x\partial) = s_1 \partial
\]

\[
D(x^2 \partial) = 2s_1 x \partial
\]

\[
D(x^3 \partial) = 3s_1 x^2 \partial
\]

(12)

We also have that $D(x\partial^2) = s_1 \partial^2$, $D(x^2 \partial^2) = 2s_1 x \partial^2$, and $D(x^3 \partial^2) = 3s_1 x^2 \partial^2$. Similarly, we are able to prove that $D(\partial^3) = 0$, $D(x\partial^3) = s_1 \partial^3$, $D(x^2 \partial^3) = 2s_1 x \partial^3$, and $D(x^3 \partial^3) = 3s_1 x^2 \partial^3$. By induction on $i$ of $x^i \partial$, 


we can prove that
\[ D(x^i \partial) = is_1 x^{i-1} \partial \]
Similarly, we are able to prove that
\[ D(x^i \partial^2) = is_1 x^{i-1} \partial^2 \]
\[ D(x^i \partial^3) = is_1 x^{i-1} \partial^3 \]
Therefore we have proven the lemma.  

**Note 1.** For any basis element \( x^i \partial^k, 1 \leq k \leq 3, \) of the algebra \( \overline{WN_{0,0,13}}, \)
if we define \( \mathbb{F}-\)linear map \( D_1 \) of the algebra \( \overline{WN_{0,0,13}} \) as follows:
\[ D_1(x^i \partial^k) = ix^{i-1} \partial^k \]
then the map \( D_1 \) of the algebra \( \overline{WN_{0,0,13}} \) can be linearly extended to a
derivation of the algebra \( \overline{WN_{0,0,13}}. \)  

**Theorem 1.** For any derivation \( D \) of the algebra \( \overline{WN_{0,0,13}}, D = cD_1 \)
such that \( D_1 \) is the derivation in Note 1 where \( c \in \mathbb{F}. \)

**Proof.** The proof of the theorem is straightforward by Lemma 1 and
Note 1.  

**Corollary 1.** The dimension \( \text{Dim}(\text{Der}(\overline{WN_{0,0,13}})) \) of the algebra
\( \overline{WN_{0,0,13}} \) is one and every derivation of the algebra \( \overline{WN_{0,0,13}} \) is the
inner derivation \( ad_c \partial \) where \( c \in \mathbb{F}. \)

**Proof.** The proof of the corollary is straightforward by Note 1 and
Theorem 1.  

**Lemma 2.** For any derivation \( D \) of the algebra \( \overline{WN_{0,0,1}} = \langle x^i \partial^r | i \in \mathbb{N}, 1 \leq r \leq n, n > 1 \rangle, \) for any basis element \( x^i \partial^j, 1 \leq j \leq n, \) of the
algebra \( \overline{WN_{0,0,1}} \), we have the followings
\[ D(\partial) = D(\partial^2) = \cdots = D(\partial^n) = 0 \] and \( D(x^i \partial^j) = is_1 x^{i-1} \partial^j \)
where \( s_1 \in \mathbb{F} \).

**Proof.** Let \( D \) be the derivation in the lemma. Let \( D \) be any derivation of \( D \in \text{Der}_{\text{non}}(W(0,0,1)), \) Since \( \partial^j \) annihilates \( \partial^i, 1 \leq i, j \leq n, \) we have that \( D(\partial) = \sum_{j=1}^{n} r_j \partial^j \) with \( r_j \in \mathbb{F}, 1 \leq i \leq n. \) By \( D(\partial^i * x^n \partial) = n(n-1) \cdots (n-j+1) D(x^{n-j} \partial), \) we can prove that

\[
\begin{align*}
  n r_1 x^{n-1} \partial + n(n-1) r_2 x^{n-2} \partial + \cdots + n r_n \partial + \partial^i * n s_1 x^{n-1} \partial \\
  = n(n-1) \cdots (n-j+1)(n-j) s_1 x^{n-j-1} \partial
\end{align*}
\]

Since \( \partial^i * n s_1 x^{n-1} \partial = n(n-1) \cdots (n-j+1)(n-j) s_1 x^{n-j-1} \partial, \) we have that \( r_1 = \cdots = r_n = 0. \) This implies that \( D(\partial^i) \) is zero. Since \( x \partial \) is a left identity of \( x \partial^i, \) we can prove that

\[
(13) \quad s_1 \partial^i + x \partial * D(x \partial^i) = D(x \partial^i)
\]

Let us put \( D(x \partial^i) = \sum_{i=1}^{n} b_{0,i} \partial^i + \sum_{i=1}^{n} b_{1,i} x \partial^i + \sum_{i=1}^{n} x^2 b_{2,i} \partial^i + \#_1 \) where \( \#_1 \) the sums of its remaining terms \( \sum_{i=1}^{n} x^k b_{k,i} \partial^i, \) \( k \geq 3, \) with non-zero scalars. By (13), we prove that

\[
\begin{align*}
  s_1 \partial^i + \sum_{i=1}^{n} b_{1,i} x \partial^i + 2 \sum_{i=1}^{n} x^2 b_{2,i} \partial^i + \#_2 \\
  = \sum_{i=1}^{n} b_{0,i} \partial^i + \sum_{i=1}^{n} b_{1,i} x \partial^i + \sum_{i=1}^{n} x^2 b_{2,i} \partial^i + \#_1
\end{align*}
\]

where \( \#_2 \) the sums of its remaining terms with appropriate coefficients. We can prove that \( b_{0,j} = s_1, b_{0,1} = \cdots = b_{0,j-1} = b_{0,j+1} = \cdots = b_{0,n} = 0 \) and \( b_{2,i} = \cdots = 0. \) This implies that \( D(x \partial^i) = s_1 \partial^i + \sum_{i=1}^{n} b_{1,i} x \partial^i. \)

Since \( D(\partial * x \partial^i) = 0, \) we can prove that \( b_{1,i} = 0, 1 \leq i \leq n \) and

\[
(14) \quad D(x \partial^i) = s_1 \partial^i
\]

Since \( D(x^i \partial^i) = D(x^i \partial^i), \) we have that \( i s_1 x^{i-1} \partial * x \partial^i + x^i \partial * s_1 \partial^i = D(x^i \partial^i). \) This implies that

\[
D(x^i \partial^i) = i s_1 x^{i-1} \partial^i
\]

Therefore we have proven the lemma. \( \square \)
Note 2. For any basis element $x^i \partial^k$, $1 \leq k \leq n$, $n > 1$, of the algebra $\overline{WN}_{0,0,1,n}$ if we define $\mathbb{F}$-linear map $D_1$ of $\overline{WN}_{0,0,1,n}$ as follows:

$$D_1(x^i \partial^k) = ix^{i-1} \partial^k$$

then the map $D_1$ of $\overline{WN}_{0,0,1,n}$ can be linearly extended to a derivation of $\overline{WN}_{0,0,1,n}$. We remark that if $n = 1$, then $D_0(x^i \partial) = c_1(1 - i)x^i \partial + c_2ix^{i-1} \partial$, $c_1, c_2 \in \mathbb{F}$ can be linearly extended to a derivation of $\overline{WN}_{0,0,1,1}$ [1] □

Theorem 2. For any derivation $D$ of the algebra $\overline{WN}_{0,0,1,n}$, $D = cD_1$ such that $D_1$ is the derivation in Note 2 where $c \in \mathbb{F}$.

Proof. The proof of the theorem is straightforward by Lemma 1 and Note 2. □

Corollary 2. The dimension $\text{Dim}(\text{Der}(\overline{WN}_{0,0,1,n}))$ of the algebra $\overline{WN}_{0,0,1,n}$ is one and every derivation of $\overline{WN}_{0,0,1,n}$ is the inner derivation $ad_{c\partial}$ where $ad_{c\partial}$ is induced by the element $c\partial$ for $c \in \mathbb{F}$.

Proof. The proof of the corollary is straightforward by Note 2 and Theorem 2. □

Corollary 3. The associator of the algebra $\overline{WN}_{0,0,1,n}$ is zero (see [17]).

Proof. The proof of the corollary is straightforward by Note 1 and Theorem 1. □

Corollary 4. For any $s \in \mathbb{N}$, the algebras $\overline{WN}_{0,0,1,n}$ and $\overline{WN}_{0,0,s}$ are not isomorphic.

Proof. The proof of the corollary is straightforward by Corollary 2 and Theorem 1 in the paper (see [5]). □
Proposition 1. The matrix ring $M_m(\mathbb{F})$ is not a subalgebra of the algebra $WN_{0,0,1,n}$.

Proof. Let $A$ be a finite dimensional subalgebra of the algebra $WN_{0,0,1,n}$. If $\dim(A) \geq 2$, then it is easy to prove that $A$ has no identity. This implies that the matrix ring $M_m(\mathbb{F})$ is not a subalgebra of the algebra $WN_{0,0,1,n}$. Therefore we have proven the proposition. □

References


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