IRREDUCIBLE MODULES OVER
THE $E_6$-TYPE LIE ALGEBRA

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ABSTRACT. Let $L := L(G)$ denote the classical modular Lie algebra of $E_6$-
type over an algebraically closed field $F$ of characteristic $p > 5$ associated
with some simple and simply connected algebraic group $G$. After the proto-
types of those for $E_8$-type in [4], we shall search for irreducible (=simple)
$L$-modules in this paper.

§ 1. Introduction

We put $L :=$ the modular classical $E_6$-type Lie algebra over an algebraically
closed field $F$ of characteristic $p \geq 7$ unless otherwise stated. Referring to
chapter 5 in [4], we get the following facts without difficulty.

**Proposition (1.1).** Under the notations as above, we have $[Q(\mathcal{U}(L)) :\nQ(3)] = p^{n-\ell} = p^{2m}$, where $3 := \mathcal{U}(\mathcal{U}(L))$, $\ell = \text{rank } L = 6$, and $n = \text{dim } L = 78$.

**Corollary (1.2).** The universal enveloping algebra $\mathcal{U}(L)$ is a free $3$-module
of rank $p^{2m} = p^{72}$.

As is known in [2], we may recollect the root system $\Phi$ of $E_6$ and its base
$\Delta : \Phi = \{\epsilon_i - \epsilon_j \text{ for } i, j \leq 6 ; \frac{1}{2}(\sum_{i=1}^{6}(-1)^{k(i)}\epsilon_i \pm \sqrt{2} \epsilon_7) \text{ such that } k(i) = 0,1 \text{ and } \sum(-1)^{k(i)} = 0 ; \pm \sqrt{2} \epsilon_7\}$ for orthonormal unit vectors $\epsilon_i$'s in the
space $\mathbb{R}^7$ and $\Delta = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_4 - \epsilon_5, \epsilon_5 - \epsilon_6, \frac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \sqrt{2} \epsilon_7)\}$. Let $V$ be any finite dimensional irreducible
$L$-module $V$; its associated irreducible representation $\rho : \mathcal{U}(L) \to \text{End}_F(V)$ is
uniquely determined by $3/3 \cap \ker \rho$ up to isomorphisms. We also have relations

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\[ Q(\mathcal{U}(L)) : Q(3) = p^{2m} \text{ and } \rho(\mathcal{U}(L)) \subset M_m(F), \] the full matrix algebra of degree \( m \) [4]. Now let \( \xi_i \) for \( i = 1, 2, \cdots , 78, 78+1 \) be values in \( F \) corresponding to \( x_{\alpha_1}^p, x_{-\alpha_1}, h_{\alpha_1}, \cdots , x_{-\alpha_6}^p, x_{\alpha_7}^p, x_{-\alpha_7}, \cdots , x_{-\alpha_{36}}^p, x_{\alpha_{36}}^p, s \) respectively which must satisfy \( \text{Irr}(s, \mathcal{O}(L)) \) for the universal Casimir element \( s \) of \( L \) and for the \( p \)-center \( \mathcal{O}(L) \) of \( L \), where \( H \) is a standard Cartan subalgebra of \( L \) and \( \{ x_\alpha, \alpha \in \Phi ; h_\alpha, \alpha \in H, 1 \leq i \leq 6 \} \) is the standard Chevalley basis of \( L \) with a root system \( \Phi \). Recollect here that \( s \) has the form \( s = \sum_{\alpha \in \Phi^+} a_\alpha x_\alpha x_{-\alpha} + \sum_{i=1}^6 b_i h_{\alpha_i} (b_i \neq 0 \in F) + \) other terms of degree \( 2 \) in \( \mathcal{U}(H) \). Suppose now that \( \bar{p} \) is any left maximal ideal of \( \mathcal{U}(L) \); then the annihilator \( A(\mathcal{U}(L)/\bar{p}) := \{ x \in \mathcal{U}(L) \mid x \cdot \mathcal{U}(L) \subset \bar{p} \} \) becomes the largest two sided ideal contained in \( \bar{p} \), and hence it is just a maximal ideal \( \mathfrak{m}_\chi \) of \( \mathcal{U}(L) \) with an associated \( p \)-character \( \chi \) to \( \bar{p} \). It is easy to see that \( A(\mathcal{U}(L)/\bar{p}) \cap 3 = 3(x_{\alpha_1}^p - \xi_1) + \cdots + 3(x_{-\alpha_{36}}^p - \xi_{78}) + 3(s - \xi_{79}) \) is equal to a maximal ideal \( \mathfrak{m}_\chi \) of \( 3 := 3(\mathcal{U}(L)) \) for its associated \( p \)-character \( \chi \in L^* \) and that \( A(\mathcal{U}(L)/\bar{p}) \) is the kernel of some irreducible representation \( \rho \) of \( L \). On the other hand any maximal ideal \( \mathfrak{m}_\chi \) of \( \mathcal{U}(L) \) must contain some ideal \( \{ \sum_{i=1}^{78} \mathcal{U}(L)(x_i^p - x_i) + \mathcal{U}(L)(s - \xi_{79}) \} \), where \( x_i \) represents \( x_{\alpha_1}, x_{-\alpha_1}, h_{\alpha_1}, \cdots , x_{\alpha_6}, x_{-\alpha_6}, h_{\alpha_6}, \) \( x_{\alpha_7}, x_{-\alpha_7}, \cdots , x_{-\alpha_{36}}, x_{\alpha_{36}} \) in order. In the next section 2, we shall deal with the dimensions of \( L \)-irreducible modules in connection with this. In section 3, we shall be concerned with the byproducts of Kac-Weisfeiler conjecture which is now clearly confirmed by A.Premet [5]. Finally in §4, some errata to the previous paper [3] shall be presented. By way of Jacobson’s density theorem, we will argue about the dimension of irreducible modules over the \( E_6 \)-type Lie algebra.

**Proposition (2.1).** In the notations as above, an arbitrary nonrestricted simple \( L \)-module has its dimension \( p^{36} \) over \( F \).

**Proof.** Noting that \( E_6 \) has a sub-root system of \( E_8 \)’s, we shall imitate the proof for \( E_8 \) in chapter 7 in [4]. We shall do in several steps.

(I) Let \( \alpha_1 = \varepsilon_1 - \varepsilon_2 \) and let \( \xi_1 \neq 0 \). For \( 1 \leq i \leq 72 \), we set \( B_i := b_{i1}(c_{i1} + h_{\varepsilon_1 - \varepsilon_2}) + b_{i2}(c_{i2} + h_{\varepsilon_2 - \varepsilon_3}) + b_{i3}(c_{i3} + h_{\varepsilon_3 - \varepsilon_4}) + b_{i4}(c_{i4} + h_{\varepsilon_4 - \varepsilon_5}) + b_{i5}(c_{i5} + h_{\varepsilon_5 - \varepsilon_6}) + b_{i6}(c_{i6} + h_{\frac{1}{2}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7)}) \), where we have chosen \( c_{ij} \)’s \( \in F \) so that parentheses are invertible mod \( \mathfrak{m}_\chi \), and we have chosen \( (b_{i1}, \cdots , b_{i6}) \in F^6 \) so that any 7 \( B_i \)’s are linearly independent in \( F^6 \), the \( B_i \cdot x_{\varepsilon_1 - \varepsilon_2} \neq x_{\varepsilon_1 - \varepsilon_2} \cdot B_i \).
Irreducible modules over the $E_6$-type Lie algebra

(mod $\mathfrak{m}_X$), and the $B$ below becomes an $F$-linearly independent set in $\mathcal{U}(L)$ if necessary. Here we select a basis candidate like $B := \{(B_1 + A_{\varepsilon_1 - \varepsilon_2})^{i_1} \otimes (B_2 + A_{-(\varepsilon_1 - \varepsilon_2)})^{i_2} \otimes \cdots \otimes (B_{12} + A_{-(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \sqrt{2} \varepsilon_7)}^{i_{12}} \otimes (\otimes_{j=13}^{72} (B_j + A_{\alpha_j})^{i_j})\}$ for $0 \leq i_j \leq p - 1$, where we set

\begin{align*}
A_{\varepsilon_1 - \varepsilon_2} &= x_{\alpha_1} = x_{\varepsilon_1 - \varepsilon_2}, \\
A_{-(\varepsilon_1 - \varepsilon_2)} &= \{c_{\varepsilon_2 - \varepsilon_1} + (h_{\varepsilon_1 - \varepsilon_2} + 1)^2 + 4x_{-(\varepsilon_1 - \varepsilon_2)} \cdot x_{\varepsilon_1 - \varepsilon_2}\} \\
\text{(with $c_{\varepsilon_2 - \varepsilon_1}$ a constant),} \\
A_{-(\varepsilon_1 - \varepsilon_k)} &= x_{\varepsilon_3 - \varepsilon_k} (c_{\varepsilon_k - \varepsilon_1} + x_{\varepsilon_1 - \varepsilon_k} \cdot x_{-(\varepsilon_1 - \varepsilon_k)} \pm x_{-(\varepsilon_1 - \varepsilon_k)} \cdot x_{\varepsilon_k - \varepsilon_2}) (k \neq 1, 2, 3), \\
A_{-(\varepsilon_1 - \varepsilon_3)} &= x_{-\varepsilon_2 + \varepsilon_3} (c_{\varepsilon_3 - \varepsilon_1} + x_{\varepsilon_1 - \varepsilon_3} \cdot x_{-(\varepsilon_1 - \varepsilon_3)} \pm x_{-(\varepsilon_1 - \varepsilon_3)} \cdot x_{\varepsilon_3 - \varepsilon_2}), \\
A_{(\varepsilon_2 - \varepsilon_k)} &= x_{-\varepsilon_3 + \varepsilon_k} (c_{\varepsilon_k - \varepsilon_2} + x_{\varepsilon_2 - \varepsilon_k} \cdot x_{-(\varepsilon_1 - \varepsilon_k)} \pm x_{-(\varepsilon_1 - \varepsilon_k)} \cdot x_{\varepsilon_k - \varepsilon_2}) (k \neq 1, 2, 3), \\
A_{\varepsilon_2 - \varepsilon_3} &= x_{-\varepsilon_3 + \varepsilon_2} (c_{\varepsilon_2 - \varepsilon_3} + x_{\varepsilon_3 - \varepsilon_2} \cdot x_{\varepsilon_2 - \varepsilon_3} \pm x_{-(\varepsilon_1 - \varepsilon_k)} \cdot x_{\varepsilon_1 - \varepsilon_3}), \\
A_{\frac{1}{2}(\sum_{i=1}^{6} (-1)^{k(i)} \varepsilon_i + \sqrt{2} \varepsilon_7) &= x_{\frac{1}{2}(\sum_{i=1}^{6} (-1)^{k(i)} \varepsilon_i \pm \sqrt{2} \varepsilon_7})}
\end{align*}

except that, for $\beta_k(i) = -\frac{1}{2} \varepsilon_1 + \frac{1}{2} \varepsilon_2 + \frac{1}{2} \sum_{i=3}^{6} (-1)^{k(i)} \varepsilon_i \pm \sqrt{2} \varepsilon_7$, we put

\begin{align*}
A_{\alpha_k(i)} &= x_{\frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \frac{1}{2} \sum_{i=3}^{6} (-1)^{k(i)} \varepsilon_i \pm \sqrt{2} \varepsilon_7)} (c_{\beta_k(i)} + x_{\beta_k(i)} \cdot x_{\beta_k(i)} \pm x_{\gamma_k(i)} \cdot x_{\gamma_k(i)}) \text{ with} \\
\gamma_k(i) &= \frac{1}{2} \varepsilon_1 - \frac{1}{2} \varepsilon_2 - \frac{1}{2} \sum_{i=3}^{6} (-1)^{k(i)} \varepsilon_i \pm \frac{\sqrt{2}}{2} \varepsilon_7, \text{ and for other roots $\alpha \in \Phi$ we set $A_{\alpha} = x_{\alpha}^2$ or $x_{\alpha}^3$ accordingly as in proposition 5.2.1 in [4] so that each factor in the basis element of $B$ may have a different weight from others with respect to $H$. Here $c_{\alpha}$'s are chosen so that parentheses containing them are invertible and the signs $\pm$ are chosen so that they commute with $x_{\alpha_1}$ (and so with $x_{-\alpha_1}$). $B$ will be shown to be a basis in $\mathcal{U}(L)/\mathfrak{m}_X$. Due to P-B-W theorem, $B$ is a linearly independent set over $F$ in $\mathcal{U}(L)$. Furthermore $\forall \alpha \in \Phi, A_{\alpha} \notin \mathfrak{m}_X$ (see below for detail).

Suppose next that there is a dependence equation which is of least degree with respect to $h_{\alpha_j} \in H$ and the number of whose highest degree terms is also least. If it has an $h$-component, then conjugation of it by $x_{\alpha_1}$ gives a nontrivial dependence equation of lower degree than the given one, a contradiction. If it has not, it reduces to the following forms:

\begin{enumerate}
\item[(i)] $x_{\varepsilon_i - \varepsilon_j} \cdot C + C' \in \mathfrak{m}_X; \\
\item[(ii)] $x_{\varepsilon_1 - \varepsilon_j} \cdot C + C' \in \mathfrak{m}_X; \\
\item[(iii)] $x_{\varepsilon_i - \varepsilon_k} \cdot C + C' \in \mathfrak{m}_X; \text{ if } i, k \neq 1, 2, \\
\item[(iv)] $x_{\frac{1}{2}(\sum_{i=1}^{6} (-1)^{k(i)} \varepsilon_i \pm \sqrt{2} \varepsilon_7} \cdot C + C' \in \mathfrak{m}_X.
\end{enumerate}
where $C$ and $C'$ commute with $x_{a_1}$.

For the case (i), consider $x_{e_{2} - e_{j}} \cdot x_{e_{j} - e_{2}} \cdot C + x_{e_{2} - e_{j}} \cdot C' \in \mathfrak{m}_x$, from which we get $(h_{e_{2} - e_{j}} + x_{e_{j} - e_{2}} \cdot x_{e_{2} - e_{j}}) \cdot C + x_{e_{2} - e_{j}} \cdot C' \in \mathfrak{m}_x$. Applying $adx_{e_{1} - e_{2}}$, we get $x_{e_{1} - e_{2}} \cdot C \in \mathfrak{m}_x$ implying $C \in \mathfrak{m}_x$. So the first assumed dependence equation reduces to that of lower degree.

For the case (ii), considering $x_{e_{j} - e_{1}}(x_{e_{1} - e_{j}} \cdot C + C') \in \mathfrak{m}_x$ gives $C \in \mathfrak{m}_x$ similarly to (i). For (iii), we consider $x_{e_{2} - e_{k}}(x_{e_{1} - e_{k}} \cdot C + C') \in \mathfrak{m}_x$; so we get $(x_{e_{2} - e_{k}} + x_{e_{1} - e_{k}} \cdot x_{e_{2} - e_{k}}) \cdot C + x_{e_{2} - e_{k}} \cdot C' \in \mathfrak{m}_x$. Next applying $adx_{e_{1} - e_{2}}$ to the last relation yields $(x_{e_{1} - e_{k}} + x_{e_{1} - e_{k}} \cdot x_{e_{1} - e_{k}}) \cdot C + x_{e_{1} - e_{k}} \cdot C' \in \mathfrak{m}_x$, which again yields $x_{e_{1} - e_{k}} \cdot C \in \mathfrak{m}_x$. This gives $C \in \mathfrak{m}_x$ by the argument as above. Similarly we may show that (iv) gives $C \in \mathfrak{m}_x$. Repeating such processes gives rise to a contradiction $x_{a} \in \mathfrak{m}_x$ for some $a \in \Phi$.

(II) Suppose next that $\xi_1 = 0$ but $\xi_2 \neq 0$; we then see without difficulty that we have $\dim_F(U(L)/\mathfrak{m}_x) = p^{2m} = p^{72}$ in view of a Lie algebra automorphism sending $x_{-a_1}$ to $x_{a_1}$.

(III) Suppose that $\xi_1 = \xi_2 = 0$ but $\xi_3 \neq 0$. In this case we set $B := \{(B_1 + A_{e_{1} - e_{2}})^{12} \otimes (B_2 + A_{e_{3} - e_{4}})^{12} \otimes \cdots \otimes (B_{12} + A_{\frac{1}{2}(-e_{1} - e_{2} + e_{3} + e_{4} + e_{5} + e_{6} + \sqrt{2} \varepsilon_{7})})^{12} \otimes (\otimes_{j=13}^{72}(B_j + A_{a_j})^{12})\}$ for $0 \leq i_j \leq p - 1$, where we designate

$$A_{e_{1} - e_{2}} = x_{e_{1} - e_{2}} - x_{-(e_{1} - e_{2})} = g_{e_{1} - e_{2}},$$

$$A_{-(1 - e_{1} - e_{2})} = \{c_{e_{2} - e_{1}} + (h_{e_{1} - e_{2}} + 1)^{2} + 4x_{(e_{1} - e_{2})} \cdot x_{e_{1} - e_{2}}\}$$

(with $c_{e_{2} - e_{1}}$ a constant),

$$A_{-(e_{1} - e_{k})} = x_{e_{2} - e_{k}}(c_{e_{2} - e_{1}} + x_{e_{1} - e_{k}} \cdot x_{-(e_{1} - e_{k})} \pm x_{-(e_{2} - e_{k})} \cdot x_{e_{k} - e_{2}})(k \neq 1, 2, 3),$$

$$A_{e_{1} - e_{k}} = x_{e_{1} - e_{k}}(c_{e_{1} - e_{k}} + x_{e_{1} - e_{k}} \cdot x_{e_{k} - e_{2}} \pm x_{e_{2} - e_{k}} \cdot x_{e_{k} - e_{2}})(k \neq 1, 2, 3),$$

$$A_{-(e_{1} - e_{3})} = x_{e_{4} - e_{3}}^{2} - x_{-(e_{1} - e_{3})} \pm x_{-(e_{2} - e_{3})} \cdot x_{e_{2} - e_{3}}(k \neq 1, 2, 3),$$

$$A_{e_{1} - e_{3}} = x_{e_{2} - e_{3}}(c_{e_{1} - e_{3}} + x_{e_{1} - e_{3}} \cdot x_{e_{3} - e_{2}} \pm x_{e_{2} - e_{3}} \cdot x_{e_{3} - e_{2}}),$$

$$A_{e_{2} - e_{k}} = x_{e_{2} - e_{k}}(c_{e_{2} - e_{k}} + x_{-(e_{2} - e_{k})} \cdot x_{e_{k} - e_{2}} \pm x_{-(e_{2} - e_{k})} \cdot x_{e_{k} - e_{2}})(k \neq 1, 2, 3),$$

$$A_{e_{1} - e_{2}} = x_{e_{2} - e_{1}}(c_{e_{2} - e_{1}} + x_{e_{2} - e_{1}} \cdot x_{e_{1} - e_{2}} \pm x_{e_{1} - e_{2}} \cdot x_{e_{1} - e_{2}})(k \neq 1, 2, 3),$$

$$A_{e_{1} - e_{3}} = x_{e_{3} - e_{3}}^{2} - x_{-(e_{1} - e_{3})} \pm x_{-(e_{2} - e_{3})} \cdot x_{e_{2} - e_{3}}(k \neq 1, 2, 3),$$

$$A_{e_{2} - e_{3}} = x_{e_{3} - e_{3}}^{2} - x_{-(e_{2} - e_{3})} \pm x_{-(e_{2} - e_{3})} \cdot x_{e_{2} - e_{3}}(k \neq 1, 2, 3),$$

$$A_{e_{1} - e_{3}} = x_{e_{3} - e_{3}}^{2} - x_{-(e_{1} - e_{3})} \pm x_{-(e_{1} - e_{3})} \cdot x_{e_{1} - e_{3}}(k \neq 1, 2, 3),$$

$$A_{\frac{1}{2} \sum_{i=1}^{6} (-1)^{k(i)\varepsilon_{i} \pm \sqrt{2} \varepsilon_{7}}} = x_{\frac{1}{2} \sum_{i=1}^{6} (-1)^{k(i)\varepsilon_{i} \pm \sqrt{2} \varepsilon_{7}}}$$
except that for \( \beta_{k(i)} = -\frac{1}{2} \varepsilon_1 + \frac{1}{2} \varepsilon_2 + \frac{1}{2} \sum_{i=3}^6 (-1)^{k(i)} \varepsilon_i \pm \frac{v}{2} \varepsilon_7 \) and \( \beta'_{k(i)} = \frac{1}{2} \varepsilon_1 - \frac{1}{2} \varepsilon_2 + \frac{1}{2} \sum_{i=3}^6 (-1)^{k(i)} \varepsilon_i \pm \frac{v}{2} \varepsilon_7 \), we put

\[ A_{\beta_{k(i)}} = x_1^{1+2} 
\frac{1}{2} \varepsilon_1 + \frac{1}{2} \varepsilon_2 + \frac{1}{2} \sum_{i=3}^6 (-1)^{k(i)} \varepsilon_i \pm \frac{v}{2} \varepsilon_7 \cdot (c \beta_{k(i)} + x \beta_{k(i)} \cdot x - \beta_{k(i)} \pm x \gamma_{k(i)} \cdot x - \gamma_{k(i)}) \]

and

\[ A_{\beta'_{k(i)}} = x_1^{1+2} 
\frac{1}{2} \varepsilon_1 - \frac{1}{2} \varepsilon_2 + \frac{1}{2} \sum_{i=3}^6 (-1)^{k'(i)} \varepsilon_i \pm \frac{v}{2} \varepsilon_7 \cdot (c \beta'_{k(i)} + x \beta'_{k(i)} \cdot x - \beta'_{k(i)} \pm x \gamma'_{k(i)} \cdot x - \gamma'_{k(i)}) \]

respectively appropriately as in proposition 2.1, and for other roots \( \alpha \in \Phi \) we set \( A_\alpha = x_\alpha^2 \) or \( x_\alpha^3 \) suitably as in (I). We may verify without difficulty that \( \mathcal{L}(L)/\mathfrak{M}_\chi \) has \( \mathcal{B} \) as its \( F \)-basis. Its proof almost goes in parallel with that of (I).

(IV) Since we have no short root and since any root is sent to any other root by the Weyl group, the remaining cases are also verified automatically.

(V) Since \( \dim \mathfrak{U}(L)/\mathfrak{M}_\chi = p^{72} \), we get our conclusion by virtue of Jacobson's density theorem.

\[ \square \]

§3. Kac-Weisfeiler conjecture

The well-known Kac-Weisfeiler conjecture was proved by A.Premet around 1995, but the dimensions of the coadjoint orbits are still unknown. In the present situation we can get the following fact.

**Proposition (3.1).** For \( p \geq 7 \), \( E_6 \) has no subregular point. In other words, we have \( S(L, p, \chi) = \emptyset \) for \( p \geq 7 \), \( \chi \neq 0 \), and \( L = E_6 \)-type.

**Proof.** Proposition 2.1 ensures our assertion in view of §3-3 in [4].

**Lemma (3.2).** We can obtain \( \text{Irr}(s, \mathcal{O}(L)) \) by expanding out both sides of the relation:

\[ N_{Q(3)}^{Q(3)}(h_{\alpha_1}, \ldots, h_{\alpha_6}) \left( \sum_{\alpha \in \Phi^+} a_\alpha x_\alpha x_{-\alpha} \right) = N_{Q(3)}^{Q(3)}(h_{\alpha_1}, \ldots, h_{\alpha_6}) \{ s - (b_1 h_{\alpha_1} + \cdots + b_6 h_{\alpha_6} + \sum_{i,j} a_{ij} h_{\alpha_i} h_{\alpha_j}) \} \]

where \( \sum_{i,j} a_{ij} h_{\alpha_i} h_{\alpha_j} \) indicates the part of other terms of degree 2 of \( s \) in \( \mathcal{U}(H) \).
Proof. See proposition 3.2.2 in [4].

COROLLARY (3.3). For any $\chi \in L^*$, where have at most $p^6$-nonisomorphic irreducible $L$-modules.

Proof. By virtue of [1], we have exactly $p^6$-nonisomorphic irreducible restricted $L$-modules. Since $|Q(U(L)) : Q(3)| = p^{72}$, $U(L) / M_\chi$ for $\chi \in L^* \setminus \{0\}$ can be expressed as $\sum_{i=1}^{72} 3 \cdot (x_i + M_\chi) / M_\chi$ for basis elements $x_i \in B$. So theorem 3.1 gives a unique maximal ideal $\sum U(L)(x_{\beta_i} - \xi_i) + U(L)(s - (s_\chi)_j)$ for each root $(s_\chi)_j$ of $\text{Irr}(s, \mathcal{O}(L))$ associated with $\chi$. Hence we obtain the conclusion by virtue of the above lemma 3.2.

REMARK (3.4). Using various facts up to now, we can conclude that the dimension of the coadjoint orbit of any nonzero $\chi \in L^* \setminus \{0\}$ becomes $2 \times 36$ if $p$ is good. V.G.Kac confirms that we have an irreducible representation $\rho_\chi$ of dimension $p^{2m'}$ if the dimension of the coadjoint orbit of $\chi$ is $2m' \leq 2m = 72$. So we can assert that the dimension of the coadjoint orbits is $2m = 72$ in view of Premet's proof of Kac-Weisfeiler conjecture.

§4. Errata to a previous paper

In [3], we presented the root system $\Phi$ of $E_7$ incorrectly, thereby gave an incorrect proof of its proposition 2.1. So we correct some part of [3] and give the result in this section.

PROPOSITION (4.1). Let $\Phi$ be the root system of $E_7$-type. Then $\Phi$ has a base $\Delta = \{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \cdots, \varepsilon_7 - \varepsilon_8, \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8)\}$ and $\Phi = \{\varepsilon_i - \varepsilon_j, 1 \leq i \neq j \leq 8; \frac{1}{2}(\pm \varepsilon_1 \pm \cdots \pm \varepsilon_8)\}$ (four pluses) and any nonrestricted simple module for $E_7$ has its dimension $p^{63}$ for $p \geq 7$.

Proof. For $\Phi$ and $\Delta$, refer to §6.7 in [2]. Since $E_7$ has a sub-root system of $E_8$, we have only to imitate the proof for $E_8$ in chapter 7 in [4]. But we may also imitate proposition (2.1)'s proof because of similar reason for $E_6$. Several steps are in order.

1. If $\alpha_1 = \varepsilon_2 - \varepsilon_3$ and let $\xi_1 \neq 0$, then we set $B_i := b_{i1}(c_{i1} + h_{\varepsilon_2 - \varepsilon_3}) + b_{i2}(c_{i2} + h_{\varepsilon_3 - \varepsilon_4}) + \cdots + b_{i6}(c_{i6} + h_{\varepsilon_7 - \varepsilon_8}) + b_{i7}(c_{i7} + h_{\frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8)})$,
where we have chosen \( c'_{ij} s \in F \) so that parentheses are invertible \( \text{mod} \mathfrak{M}_\lambda \), and we have chosen \( (b_1, \ldots, b_7) \in F^7 \) so that any 8 \( B_i \)'s are linearly independent in \( \mathbb{P}^7(F) \), the \( B_i \cdot x_{\varepsilon_2-\varepsilon_3} \neq x_{\varepsilon_2-\varepsilon_3} \cdot B_i \) (mod\( \mathfrak{M}_\lambda \)), and the \( \mathcal{B} \) below becomes an \( F \)-linearly independent set in \( \mathcal{U}(L) \) if necessary. Here we pick out a basis candidate like \( \mathcal{B} := \{ (B_1 + A_{\varepsilon_2-\varepsilon_3})^{i_1} \otimes (B_2 + A_{-(\varepsilon_2-\varepsilon_3)})^{i_2} \otimes \cdots \otimes (B_{14} + A_{-\frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-4+\varepsilon_5+\varepsilon_6+\varepsilon_7+\varepsilon_8)} \otimes (\otimes_{j=15}^{126} (B_j + A_{\alpha_j})^{i_j}) \} \) for \( 0 \leq i_j \leq p - 1 \), where we designate

\[
A_{\varepsilon_2-\varepsilon_3} = x_{\alpha_1} = x_{\varepsilon_2-\varepsilon_3}, \\
A_{-(\varepsilon_2-\varepsilon_3)} = \{ c_{\varepsilon_2-\varepsilon_3} + (h_{\varepsilon_2-\varepsilon_3} + 1)^2 + 4x_{-(\varepsilon_2-\varepsilon_3)} \cdot x_{\varepsilon_2-\varepsilon_3} \} \\
\text{with } c_{\varepsilon_2-\varepsilon_3} \text{ a constant}, \\
A_{-(\varepsilon_2-\varepsilon_k)} = x_{\varepsilon_2-\varepsilon_k} (c_{\varepsilon_2-\varepsilon_k} + x_{\varepsilon_2-\varepsilon_k} \cdot x_{-(\varepsilon_2-\varepsilon_k)} \pm x_{-(\varepsilon_2-\varepsilon_k)} \cdot x_{\varepsilon_2-\varepsilon_k})(k \neq 1, 2, 3), \\
A_{-(\varepsilon_2-\varepsilon_1)} = x_{\varepsilon_2-\varepsilon_1} (c_{\varepsilon_2-\varepsilon_1} + x_{\varepsilon_2-\varepsilon_1} \cdot x_{-(\varepsilon_2-\varepsilon_1)} \pm x_{-(\varepsilon_2-\varepsilon_1)} \cdot x_{\varepsilon_2-\varepsilon_1}), \\
A_{\varepsilon_3-\varepsilon_1} = x_{\varepsilon_3-\varepsilon_1} (c_{\varepsilon_3-\varepsilon_1} + x_{\varepsilon_3-\varepsilon_1} \cdot x_{\varepsilon_3-\varepsilon_1} \pm x_{\varepsilon_3-\varepsilon_1} \cdot x_{\varepsilon_3-\varepsilon_1}), \\
A_{\varepsilon_3-\varepsilon_k} = x_{\varepsilon_3-\varepsilon_k} (c_{\varepsilon_3-\varepsilon_k} + x_{\varepsilon_3-\varepsilon_k} \cdot x_{\varepsilon_3-\varepsilon_k} \pm x_{\varepsilon_3-\varepsilon_k} \cdot x_{\varepsilon_3-\varepsilon_k})(k \neq 1, 2, 3), \\
A_{\beta_{k(i)}} = x_{\frac{1}{2} \varepsilon_2 - \frac{1}{2} \varepsilon_3 + \frac{1}{2} \sum_{i=4}^{8} (1)^{k(i)} \varepsilon_i \pm \frac{1}{2} \varepsilon_1} (c_{\beta_{k(i)}} + x_{\beta_{k(i)}} \cdot x_{-\beta_{k(i)}} \pm x_{\gamma_{k(i)}} \\
\text{for } \beta_{k(i)} = -\frac{1}{2} \varepsilon_2 + \frac{1}{2} \varepsilon_3 + \frac{1}{2} \sum_{i=4}^{8} (1)^{k(i)} \varepsilon_i \pm \frac{1}{2} \varepsilon_1 \text{ and } \gamma_{k(i)} = \frac{1}{2} \varepsilon_2 - \frac{1}{2} \varepsilon_3 + \frac{1}{2} \sum_{i=4}^{8} (1)^{k(i)} \varepsilon_i \pm \frac{1}{2} \varepsilon_1, \text{ and for other roots } \alpha \in \Phi \text{ we designate } A_{\alpha} = x_{\alpha}^2 \text{ or } x_{\alpha}^3 \text{ or even } x_{\alpha} \text{ accordingly as in proposition 2.1. Similarly we may show } \mathcal{B} \text{ is an } F\text{-basis of } \mathcal{U}(L)/\mathfrak{M}_\lambda \text{ as before.}

(II) Next if \( \xi_1 = 0 \) but \( \xi_2 \neq 0 \), then we get \( \dim_F(\mathcal{U}(L)/\mathfrak{M}_\lambda) = p^{2m} = p^{126} \) by a Lie algebra automorphism sending \( x_{-\alpha_1} \) to \( x_{\alpha_1} \).

(III) Suppose that \( \xi_1 = \xi_2 = 0 \) but \( \xi_3 \neq 0 \). In this case we put \( \mathcal{B} := \{ (B_1 + A_{\varepsilon_2-\varepsilon_3})^{i_1} \otimes (B_2 + A_{-(\varepsilon_2-\varepsilon_3)})^{i_2} \otimes \cdots \otimes (B_{14} + A_{-\frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-4+\varepsilon_5+\varepsilon_6+\varepsilon_7+\varepsilon_8)} \otimes \)
\((\otimes_{j=1}^{126} (B_j + A_\alpha_j)^{i_j})\) for \(0 \leq i_j \leq p - 1\), where we designate

\[
A_{e_2 - e_3} = x_{e_2 - e_3}^{p-1} - x_{-(e_2 - e_3)} = g_{e_2 - e_3},
\]

\[
A_{-(e_2 - e_3)} = \{c_{e_2 - e_3} + (h_{e_2 - e_3} + 1)^2 + 4x_{-(e_2 - e_3)} \cdot x_{e_2 - e_3}\}
\]

(with \(c_{e_2 - e_3}\) a constant),

\[
A_{-(e_2 - e_k)} = x_{e_1 - e_k} \quad (c_{e_2 - e_k} + x_{e_2 - e_k} - x_{-(e_2 - e_k)} \pm x_{-(e_1 - e_k)} \cdot x_{e_1 - e_k})(k \neq 1, 2, 3),
\]

\[
A_{e_2 - e_k} = x_{e_1 - e_k} \quad (c_{e_2 - e_k} + x_{e_2 - e_k} \pm x_{e_1 - e_k} \cdot x_{e_1 - e_k})(k \neq 1, 2, 3),
\]

\[
A_{e_1 - e_2} = x_{e_1 - e_2} \quad (c_{e_1 - e_2} + x_{e_2 - e_1} \cdot x_{e_2 - e_1} \pm x_{e_1 - e_3} \cdot x_{e_3 - e_1}),
\]

\[
A_{e_2 - e_1} = x_{e_2 - e_1} \quad (c_{e_2 - e_1} + x_{e_2 - e_1} \cdot x_{e_1 - e_2} \pm x_{e_3 - e_1} \cdot x_{e_1 - e_1}),
\]

\[
A_{e_3 - e_1} = x_{e_2 - e_1} \quad (c_{e_3 - e_1} + x_{e_3 - e_2} \cdot x_{e_3 - e_3} \pm x_{e_2 - e_2} \cdot x_{e_2 - e_2})(k \neq 2, 3, 5),
\]

\[
A_{e_2 - e_3} = x_{e_2 - e_3} \quad (c_{e_3 - e_1} + x_{e_3 - e_3} \cdot x_{e_3 - e_3} \pm x_{e_2 - e_3} \cdot x_{e_2 - e_3})(k \neq 2, 3, 5),
\]

\[
A_{e_1 - e_3} = x_{e_1 - e_3} \quad (c_{e_1 - e_3} + x_{e_3 - e_3} \cdot x_{e_3 - e_3} \pm x_{e_2 - e_3} \cdot x_{e_2 - e_3})(k \neq 2, 3, 5),
\]

and for other roots \(\alpha\) we put \(A_{\alpha}\) similarly as in (III) of the proof in proposition

2.1. Similarly we can show \(B\) is an \(F\)-basis of \(U(L)/\mathfrak{M}_X\) as before.

(IV) Since there is only one root length, the remaining cases are easily verified automatically.

(V) Since \(\text{dim}_F(U(L)/\mathfrak{M}_X) = p^{126}\), we have our conclusion by dint of Jacobson's density theorem.

\[\square\]

References


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