MIXED VECTOR $F$-IMPLICIT COMPLEMENTARITY PROBLEMS AND CORRESPONDING VECTOR VARIATIONAL INEQUALITY PROBLEMS

Byung-Soo Lee

Abstract. We consider existence theorems for a new class of mixed vector $F$-implicit complementarity problems and the corresponding mixed vector $F$-implicit variational inequality problems.

1. Introduction and Preliminaries

The following $F$-complementarity problem: Find $x \in K$ such that

$$\langle Tx, x \rangle + F(x) = 0 \text{ and } \langle Tx, y \rangle + F(y) \geq 0 \text{ for all } y \in K,$$

and the corresponding $F$-variational inequality problem:

Find $x \in K$ such that

$$\langle Tx, y - x \rangle + F(y) - F(x) \geq 0 \text{ for all } y \in K$$

were introduced and considered in [6], where $K$ is a nonempty closed convex cone of a real Banach space $X$, $T : K \to X^*$ (the dual space) is a mapping and $F : K \to (-\infty, +\infty)$ is a positively homogeneous and convex function.

In 2003, Fang and Huang [2] studied a new class of vector $F$-complementarity problems with demi-pseudomonotone mappings in Banach spaces by considering the solvability of the problems. Huang and Li [3]...

Received July 2, 2007. Accepted August 13, 2007.
2000 Mathematics Subject Classification: 49J40, 90C33.
Key words and phrases: Mixed vector $F$-implicit complementarity problems, mixed vector $F$-implicit variational inequality problems, KKM-mapping, Fan-KKM Theorem.
considered a class of scalar $F$-implicit complementarity problems and another class of $F$-implicit variational inequality problems in Banach spaces, in 2004. And then, Li and Huang [5] extended and generalized the scalar case to some vector case. The equivalence between the $F$-implicit complementarity problem and $F$-implicit variational inequality problem was presented and some new existence theorems of solutions for $F$-implicit variational inequality problems were also proved.

The main objective of this work is to generalize some results of [4, 5] to more generalized vector case. We introduce a new class of generalized mixed vector $F$-implicit complementarity problems and corresponding new class of generalized mixed vector $F$-implicit variational inequality problems in Banach spaces and prove the equivalence between them under certain assumptions. Furthermore, we derive some new existence theorems of solutions for the generalized mixed vector $F$-implicit complementarity problems and the generalized mixed vector $F$-implicit variational inequality problems by using Fan-KKM Theorem [1] under some suitable assumptions without any monotonicity.

An ordered Banach space $(Y, P)$ is a real Banach space $Y$ with an ordering defined by a closed cone $P \subseteq Y$ with an apex at the origin in the form of

$$x \geq y \Leftrightarrow x - y \in P$$

and

$$x \not\geq y \Leftrightarrow x - y \not\in P$$

A mapping $F : K \to Y$ is said to be positively homogeneous if $F(\alpha x) = \alpha F(x)$ for all $x \in K$ and $\alpha \geq 0$, where $X$ and $Y$ are vector spaces and $K$ a subspace of $X$. 
2. Main Results

**Definition 2.1.** Let $K$ be a nonempty subset of a topological vector space $X$. A mapping $G : K \to 2^X$ is called a KKM-mapping if for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of $K$

$$\text{conv}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i),$$

where $\text{conv}$ denotes the convex hull.

**Fan-KKM Theorem [1].** Let $K$ be a nonempty subset of Hausdorff topological vector space $X$. Let $G : K \to 2^X$ be a KKM-mapping such that for any $y \in K$ $G(y)$ is closed and $G(y^*)$ is compact for some $y^* \in K$, then there exists $x^* \in K$ such that $x^* \in G(y)$ for all $y \in K$, i.e.,

$$\bigcap_{y \in K} G(y) \neq \emptyset.$$  

**Lemma 2.1 [5].** Let $(Y, P)$ be an ordered Banach space by a pointed closed convex cone $P$. If $x \geq 0$ and $y \geq 0$, then $x+y \geq 0$, for all $x, y \in Y$.

Let $X$ be a real Banach space, $K \subseteq X$ be a nonempty closed convex cone and $(Y, P)$ be an ordered Banach space. Denote $L(X, Y)$ the space of all continuous linear mappings from $X$ into $Y$ and $\langle t, x \rangle$ the value of a linear continuous mapping $t \in L(X, Y)$ at $x$. Let $A, T : K \to L(X, Y)$, $g, h : K \to K$, $F : K \to Y$ and $N : L(X, Y) \times L(X, Y) \to L(X, Y)$ be mappings.

In this section, we consider the following mixed vector $F$-implicit complementarity problem (MVF-ICP): Find $x^* \in K$ such that

$$\langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*)) = 0$$

and

$$\langle N(Ax^*, Tx^*), h(y) \rangle + F(h(y)) \geq 0,$$

for all $y \in K$. 

We also introduce the following mixed vector \( F \)-implicit variational inequality problem (MVF-IVIP): Find \( x^* \in K \) such that

\[
\langle N(Ax^*, Tx^*), h(y) - g(x^*) \rangle + F(h(y)) - F(g(x^*)) \geq 0, \text{ for all } y \in K.
\]

**Remark 2.1.** (i) By putting \( g = h \) in (MVF-ICP) and (MVF-IVIP), we obtain (GVF-ICP) and (GVF-IVIP) in [4].

(ii) The following vector \( F \)-implicit complementarity problem (VF-ICP) of finding \( x^* \in K \) such that

\[
\langle f(x^*), g(x^*) \rangle + F(g(x^*)) = 0
\]

and

\[
\langle f(x^*), y \rangle + F(y) \geq 0, \text{ for all } y \in K,
\]

is a particular form of (MVF-ICP) and the corresponding vector \( F \)-implicit variational inequality problem (VF-IVIP) of finding such that

\[
\langle f(x^*), y - g(x^*) \rangle + F(y) - F(g(x^*)) \geq 0, \text{ for all } y \in K,
\]

is a particular form of (MVF-IVIP) for the identities \( A \) and \( T \) and a mapping \( f : K \to L(X, Y) \) defined by \( f(x) = N(x, x) \) for \( x \in K \), and for the identity \( h \), which were considered and studied by Li and Huang [5].

We first establish the equivalence between (MVF-ICP) and (MVF-IVIP).

**Theorem 2.1.**

(i) If \( x^* \) solves (MVF-ICP), then it solves (MVF-IVIP).

(ii) Let \( F : K \to Y \) be a positively homogeneous mapping and \( h \) be a surjective mapping.

If \( x^* \) solves (MVF-IVIP), then it solves (MVF-ICP).

**Proof.** (i) If \( x^* \in K \) is a solution of (GVF-ICP), then

\[
\langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*)) = 0
\]
and
\[ \langle N(Ax^*, Tx^*), h(y) \rangle + F(h(y)) \geq 0 \text{ for all } y \in K. \]

Hence
\[
\langle N(Ax^*, Tx^*), h(y) - g(x^*) \rangle + F(h(y)) - F(g(x^*)) \\
= \langle N(Ax^*, Tx^*), h(y) \rangle + F(h(y)) \rangle - \langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*)) \\
= \langle N(Ax^*, Tx^*), h(y) \rangle + F(h(y)) \\
\geq 0,
\]

for all \( y \in K. \)

(ii) If \( x^* \in K \) is a solution of (MVF-IVIP), then
\[ \langle N(Ax^*, Tx^*), h(y) - g(x^*) \rangle + F(h(y)) - F(g(x^*)) \geq 0 \text{ for all } y \in K. \]

Since \( F : K \to Y \) is positively homogeneous, \( h \) is sujective and \( K \) is a convex cone, we can take \( y \in K \) such that \( h(y) = 2g(x^*) \) in (MVF-IVIP), thus we have
\[ \langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*)) \geq 0. \]

Similarly we take \( y \in K \) such that \( h(y) = \frac{1}{2}g(x^*) \), then we have
\[ \langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*)) \leq 0. \]

Hence,
\[ \langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*)) \in P \cap \{-P\}. \]

Since \( P \) is a pointed cone
\[ \langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*)) = 0. \]
Thus, we obtain
\[
\langle N(Ax^*, Tx^*), h(y) \rangle + F(h(y)) \\
= \langle N(Ax^*, Tx^*), h(y) - g(x^*) \rangle + F(h(y)) - F(g(x^*)) \\
+ \langle N(Ax^*, Tx^*), g(x^*) \rangle + F(g(x^*)) \\
= \langle N(Ax^*, Tx^*), h(y) - g(x^*) \rangle + F(h(y)) - F(g(x^*)) \\
\geq 0
\]
for all \( y \in K \).

\[ \square \]

If \( A, T \) and \( h \) are identity mappings on \( K \), then we have the following result.

**Corollary 2.1** [5].

(i) If \( x^* \) solves (VF-ICP), then it solves (VF-IVIP).

(ii) Let \( F : K \to Y \) be positively homogeneous.

If \( x^* \) solves (VF-IVIP), then it solves (VF-ICP).

Now we consider the existence of solutions to (MVF-IVIP) and the solution sets.

**Theorem 2.2.** Assume that

(a) mappings \( N : L(X,Y) \times L(X,Y) \to L(X,Y) \), \( A, T : K \to L(X,Y) \) and \( F : K \to Y \) are continuous, and \( g, h : K \to K \) are continuous and \( h \) is surjective;

(b) there exists a mapping \( i : K \times K \to Y \) such that

(i) \( i(x, x) \geq 0 \) for all \( x \in K \);

(ii) \( \langle N(Ax, Tx), h(y) - g(x) \rangle + F(h(y)) - F(g(x)) - i(x, y) \geq 0 \) for all \( x, y \in K \);

(iii) the set \( \{ y \in K : i(x, y) \geq 0 \} \) is convex for all \( x \in K \);
(c) there exists a nonempty compact convex subset $C$ of $K$ such that for all $x \in K \setminus C$ there exists $y \in C$ such that
\[
\langle N(Ax, Tx), h(y) - g(x) \rangle + F(h(y)) - F(g(x)) \ngeq 0.
\]

Then (MVF-IVIP) has a solution. Furthermore, the solution set is closed.

Proof. First we define a mapping $G : K \to 2^C$ by
\[
G(y) = \{x \in C : \langle N(Ax, Tx), h(y) - g(x) \rangle + F(h(y)) - F(g(x)) \geq 0\},
\]
for all $y \in K$.

By the assumption (a), for any $y \in K$, $G(y)$ is closed in $C$. Since every element $x^* \in \bigcap\limits_{y \in K} G(y)$ is a solution of (MVF-IVIP), we have to show that $\bigcap\limits_{y \in K} G(y) \neq \emptyset$. Since $C$ is compact it is sufficient to prove that the family $\{G(y)\}_{y \in K}$ has the finite intersection property. Let $\{y_1, y_2, \ldots, y_n\}$ be a finite subset of $K$ and set $B := \text{conv}(C \cup \{y_1, y_2, \ldots, y_n\})$. Then $B$ is a compact and convex subset of $K$.

Define mappings $F_1, F_2 : B \to 2^B$ as follows:
\[
F_1(y) = \{x \in B : \langle N(Ax, Tx), h(y) - g(x) \rangle + F(h(y)) - F(g(x)) \geq 0\}
\]
for all $y \in B$,

and
\[
F_2(y) = \{x \in B : i(x, y) \geq 0\} \text{ for all } y \in B.
\]

From the conditions (i) and (ii), we have
\[
i(y, y) \geq 0
\]
and
\[
\langle N(Ay, Ty), h(y) - g(y) \rangle + F(h(y)) - F(g(y)) - i(y, y) \geq 0.
\]

Now Lemma 2.1 implies
\[
\langle N(Ay, Ty), h(y) - g(y) \rangle + F(h(y)) - F(g(y)) \geq 0
\]
and so \( F_1(y) \) is nonempty. Similarly, we can prove that for any \( y \in K, F_1(y) \) is closed. Since \( F_1(y) \) is a closed subset of a compact set \( B \), we know that \( F_1(y) \) is compact. Next, we prove that \( F_2 \) is a KKM-mapping. Suppose that there exists a finite subset \( \{u_1, u_2, \ldots, u_n\} \) of \( B \) and \( \lambda_i \geq 0 \) \( (i = 1, 2, \ldots, n) \) with \( \sum_{i=1}^{n} \lambda_i = 1 \) such that

\[
   u = \sum_{i=1}^{n} \lambda_i u_i \notin \bigcup_{j=1}^{n} F_2(u_j).
\]

Then

\[
   i(u, u_j) \geq 0, \ j = 1, 2, \ldots, n.
\]

From the condition (iii), we have

\[
   i(u, u) \geq 0,
\]

which contradicts the condition (i). Hence \( F_2 \) is a KKM-mapping. On the other hand, from the condition (ii), we have

\[
   F_2(y) \subseteq F_1(y) \quad \text{for all } y \in B.
\]

Now \( x \in F_2(y) \) implies that \( i(x, y) \geq 0 \) and by the condition (ii), we have

\[
   \langle N(Ax, Tx), h(y) - g(x) \rangle + F(h(y)) - F(g(x)) - i(x, y) \geq 0.
\]

It follows from Lemma 2.1, that

\[
   \langle N(Ax, Tx), h(y) - g(x) \rangle + F(h(y)) - F(g(x)) \geq 0,
\]

i.e., \( x \in F_1(y) \). Thus \( F_1 \) is a KKM mapping. From the Fan-KKM Theorem, there exists \( x^* \in B \), such that \( x^* \in F_1(y) \) for all \( y \in B \). Hence

\[
   \langle N(Ax^*, Tx^*), h(y) - g(x^*) \rangle + F(h(y)) - F(g(x^*)) \geq 0 \quad \text{for all } y \in B.
\]

By assumption (c), we get \( x^* \in C \) and moreover \( x^* \in G(y_i), \ i = 1, 2, \ldots, n \). Hence \( \{G(y)\}_{y \in K} \) has the finite intersection property.

Since \( A, T : K \to L(X, Y), \ g, h : K \to K, \ F : K \to Y \) and \( N : L(X, Y) \times L(X, Y) \to L(X, Y) \) are continuous, the solution set of (MVF-IVIP) is obviously closed.
Theorem 2.3. Assume that $A, T : K \rightarrow L(X, Y), N : L(X, Y) \times L(X, Y) \rightarrow L(X, Y)$ and $g, h : K \rightarrow K$ are continuous and $h$ is surjective, and $F : K \rightarrow Y$ is positively homogeneous and continuous. If assumptions (b) and (c) in Theorem 2.2 hold, then (MVF-ICP) has a solution. Furthermore, the solution set of (MVF-ICP) is closed.

Proof. The conclusion follows directly from Theorems 2.1 and 2.2.

Corollary 2.2 [5]. Assume that

(a) $f : K \rightarrow L(X, Y), g : K \rightarrow K$ and $F : K \rightarrow Y$ are continuous,

(b) there exists a mapping $i : K \times K \rightarrow Y$ such that
   
   (i) $i(x, x) \geq 0,$ for all $x \in K$;
   
   (ii) $\langle f(x), y - g(x) \rangle + F(y) - F(g(x)) - i(x, y) \geq 0$ for all $x, y \in K$;
   
   (iii) the set $\{y \in K : i(x, y) \geq 0\}$ is convex, for all $x \in K$;

(c) there exists a nonempty, compact, convex subset $C$ of $K$ such that for all $x \in K \setminus C$, there exists $y \in C$ such that

$$\langle f(x), y - g(x) \rangle + F(y) - F(g(x)) \geq 0.$$ 

Then (VF-IVIP) has a solution. Furthermore, the solution set is closed.

Remark 2.2. Putting $g = h$ in Theorems 2.1, 2.2 and 2.3, we obtain results in [4] as corollaries.

References


Department of Mathematics,
Kyungsung University,
Busan 608-736, Korea
E-mail: bslee@ks.ac.kr