AN APPLICATION OF TILINGS IN THE HYPERBOLIC PLANE

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Abstract. We will construct several types of semi-regular tilings of a hyperbolic unit disk model by defining geometric features of the definition of distance in a hyperbolic plane, area of triangle, and isometry of inversions. We researched the method of regular tilings and semi-regular tilings of hyperbolic unit disk model and wrote an semi-regular tiling construction algorithm using Cabri2 program and Cinderella program. Lastly, We want to make a product related to traditional heritage cultural patterns using Photoshop, so we’ll model the advertising photos of cites; Seoul, Gwangju.

1. Introduction

The theory and method of Tilings in non-Euclidean geometry is a new research area. We remind the general concepts of tilings of Euclidean plane, sphere and hyperbolic plane.

1.1. Regular Tiling

Regular tiling with only one regular polygon.

(1) Tiling is possible only with regular triangle, square and regular hexagon on a plane.

(2) There are only 5 regular tilings on a sphere because it is the same as the regular polyhedron.

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(3) The condition of regular tiling where the $k$ regular $n$-gons gather at one vertex is $\frac{1}{n} + \frac{1}{k} < \frac{1}{2}$. \{$n, k\}$ means that there are $k$ regular $n$-gons at one vertex.

1.2. Semi-regular Tiling

Semi-regular tiling means the tiling to arrange the array of regular polygons with two or more regular polygons at one vertex.

(1) Plane: Only 8.
(2) Sphere: There are only 13 semi-regular tiling except 5 regular polyhedron, regular $n$-prism, and alternate $n$-prism.
(3) Hyperbolic plane: It should satisfy the followings in order to satisfy the semi-regular tiling in a hyperbolic disk. $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_q} < \frac{q-2}{2}$, (regular $p_1$-gon, regular $p_2$-gon, \ldots, regular $p_q$-gon)
(4) Semi-regular tiling on a hyperbolic disk: $P_q$ means the regular tiling where there are $q$ regular $p$-gons at one vertex.

(1) $p^q \rightarrow (2p, 2p, q)$
(8, 8, 5) Tiling is constructed at $4^5$. 
(2) $p^q \rightarrow (4p, 2p, 2q)$
(4, 10, 8) Tiling is constructed at $5^4$.

(3) $p^q \rightarrow (p, q, p, q)$
(5, 4, 5, 4) Tiling is constructed at $5^4$. 
2. Semi-regular tiling on a hyperbolic disk model

2.1. Tiling theory on a hyperbolic disk model

In a hyperbolic trigonometry, we have
\[ \cosh a = \frac{\cos \beta \cos \gamma + \cos \alpha}{\sin \beta \sin \gamma}. \]

According to the radius, although \( \overline{OA} \) changes, since \( \frac{e^n - 1}{e^n + 1} \) is determined by \( \{n, k\} \), we have

<table>
<thead>
<tr>
<th>{n, k}</th>
<th>( \frac{e^n - 1}{e^n + 1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>{3, 7}</td>
<td>0.141</td>
</tr>
<tr>
<td>{4, 5}</td>
<td>0.259</td>
</tr>
<tr>
<td>{5, 4}</td>
<td>0.304</td>
</tr>
<tr>
<td>{6, 5}</td>
<td>0.486</td>
</tr>
<tr>
<td>{7, 4}</td>
<td>0.489</td>
</tr>
</tbody>
</table>

2.2. Construction and application of a tiling on a hyperbolic disk model using Cabri Geometry

First, we make the regular tiling \( \{6, 5\} \) on a hyperbolic disk model:
Let \( \alpha, \beta, \gamma \) be the angles at \( A, B, C \) respectively and \( a, b, c \) be the opposite sides of \( A, B, C \) on a hyperbolic triangle \( ABC \). Then the following is satisfied
\[ \cosh a = \frac{\cos \beta \cos \gamma + \cos \alpha}{\sin \beta \sin \gamma}. \]

Using the above equation, find the hyperbolic length of \( \overline{OA} \).

Since
\[ \cosh^{-1} \left[ \frac{\cos \frac{\pi}{6} \cos \frac{\pi}{2} + \cos \frac{\pi}{3}}{\sin \frac{\pi}{6} \sin \frac{\pi}{2}} \right] \approx \ln(1.618 + \sqrt{1.618^2 - 1}) \approx \ln 2.890, \]
however, when the hyperbolic length from \( O \) to \( Z \) on a unit disk is \( h \), and \( |Z| = \frac{e^h - 1}{e^h + 1} \), we have \( |\overline{OA}| = \frac{2.890 - 1}{2.890 + 1} \approx 0.486 \) on a unit disk. Thus, the length of \( \overline{OA} \) on an arbitrary disk \( D \) whose center is \( O \) and radius is \( R \) is \( R \times 0.486 \).
If the ray that forms the angle $\frac{\pi}{6}$ from $\overline{OA}$ is constructed, there is an intersection point between this ray and the circle $D'$, and the intersection point is $B$ and the $\triangle OAB$ is constructed, whose three interior angles are $\frac{\pi}{2}$, $\frac{\pi}{6}$, $\frac{\pi}{3}$.

And we can have the regular hexagon which is located in the very middle on a $D$ by reflecting $\triangle OAB$ 12 times as you can see in the figure 2.1 below.

![Figure 2.1](image)

Now we can have the tiling with regular hexagons by doing inversion infinite times as you can see in the figure 2.2.

![Figure 2.2](image)
3. Construction and application of Semi-regular tiling on a hyperbolic disk model using Cinderella

We are going to construct semi-regular tiling using Cinderella, which is the special program for mathematics.

Let \( AB = s, \ AC = \frac{1}{s}, \ EB = R. \)

\[
R \cos B = \frac{\frac{1}{s} + s}{2} \tan A, \quad R \sin B = \frac{\frac{1}{s} - s}{2}, \quad \tan A \tan B = \frac{\frac{1}{s} - s}{\frac{1}{s} + s} = \frac{1 - s^2}{1 + s^2}
\]

\[
s^2 = \frac{1 - \tan A \tan B}{1 + \tan A \tan B} = \frac{\cos(A + B)}{\cos(A - B)}
\]

We may obtain the length from the center of the polygon to the vertices when the angles \( \angle A \) and \( \angle B \) are determined. \( \{m, n\} \) semi-regular tiling can be constructed by using \( \angle A = \frac{180}{m}, \angle B = \frac{180}{n}. \)
Now, we may construct \{4, 6\} semi-regular tiling on a hyperbolic disk model using Cinderella by the following order.

(a) Construct a circle and draw a straight line going through the center of this circle and another straight line that is perpendicular to the former straight line. We call them $x$-axis and $y$-axis.

(b) Draw two angle bisectors of two angles formed by two straight lines.
(c) Draw a straight line forming 60 degrees with the left transverse.

(d) Draw a straight line forming 15 degrees with the right transverse that passes through the intersection point $C$ of the straight line 1 and the right transverse.

(e) Draw a circle whose center is the intersection point $C$ and radius is $CD$. 
(f) Do inversion of the circle for $x$-axis and $y$-axis.

(g) Draw the circle passing through points $Q$, $W$, and $E$.

(h) Do inversion of the circle just like (g). Now the quadrilateral that should be in the center was completed.
(i) Do inversion of two circles that are located in the opposite side of the big circle respectively.

(j) Draw the geodesic line to two small circles after the inversion in order to make a hexagon.

(k) Semi-regular tiling is completed once the geodesic lines are constructed to the other circles.
(1) The figure below is done with the painting program.

The following is painted after \( \{5, 6\}, \{6, 4\} \) semi-regular tiling is finished.
4. To make an art using Semi-regular tilings on a hyperbolic disk model

We put the Korean traditional patterns onto the semi-regular tiling on a hyperbolic disk model, and made a beautiful art introducing famous cities of Korea.
References


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