SOME PROPERTIES OF INVARIANT SUBSPACES IN BANACH SPACES OF ANALYTIC FUNCTIONS

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Abstract. Let $B$ be a reflexive Banach space of functions analytic on the open unit disc and $M$ be an invariant subspace of the multiplication operator by the independent variable, $M_z$. Suppose that $\varphi \in H^\infty$ and $M_\varphi : M \to M$, defined by $M_\varphi f = \varphi f$, is the operator of multiplication by $\varphi$. We would like to investigate the spectrum and the essential spectrum of $M_\varphi$ and we are looking for the necessary and sufficient conditions for $M_\varphi$ to be a Fredholm operator. Also we give a sufficient condition for a sequence $\{\omega_n\}$ to be an interpolating sequence for $B$. At last the commutant of $M_\varphi$ under certain conditions on $M$ and $\varphi$ is determined.

1. Introduction

Let $B$ be a reflexive Banach space consisting of analytic functions defined on the open unit disc $D$ and satisfying the following conditions:

1. $1 \in B$ and $\varphi B \subset B$ for every $\varphi \in H^\infty$;
2. for every $\lambda \in D$ the evaluation functional at $\lambda$, $e_\lambda : B \to C$ given by $e_\lambda(f) = f(\lambda)$, is bounded;
3. rang$(M_z - \lambda) = \ker(e_\lambda)$ for every $\lambda \in D$.

Such a space is called a reflexive Banach space of analytic functions.

In what follows we present some examples of such spaces.

(a) The classical Hardy spaces $H^p$ for $1 < p < \infty$.

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(b) The Bergman space of analytic functions defined on \( D \), \( L^p_a(D) \) for \( 1 < p < \infty \).

(c) The spaces \( D_\alpha \) of all functions \( f(z) = \sum \hat{f}(n)z^n \), holomorphic in \( D \), for which \( \|f\|^2 = \sum (n+1)^\alpha |\hat{f}(n)|^2 < \infty \) for every \( \alpha \leq 0 \).

Recall that a bounded linear operator \( T \) on a Banach space is called Fredholm if it is invertible modulo the compact operators. It is known that \( T \) is Fredholm if the kernel of \( T \) is finite dimensional and the range of \( T \) is finite codimensional, i.e.; \( \dim_{\text{range}} \frac{B}{T} \) is infinite. It is worthy of attention to remark that range \( T \) is closed, in fact, if a finite codimensional linear subspace \( Y \) of a Banach space \( \mathcal{B} \) is the range of a continuous linear mapping \( N \) on \( \mathcal{B} \), then it must be closed [5, Lemma 3.3].

The essential spectrum of \( T \), denoted by \( \sigma_e(T) \), is the set of all complex number \( \lambda \) such that \( \lambda - T \) is not Fredholm. Since every invertible operator is Fredholm, we have \( \sigma_e(T) \subset \sigma(T) \), the spectrum of \( T \). The essential norm of \( T \), denoted \( ||T||_e \), is the norm distance from \( T \) to the set of compact operators. Also, \( r(T) \) denotes the spectral radius of \( T \) and by \( r_e(T) \) we mean the essential spectral radius of \( T \); that is, the supremum of the absolute values of numbers in the essential spectrum of \( T \).

Suppose that \( M \) is an invariant subspace of \( M \) on the Bergman space \( L^2_a \). Recently, the necessary and sufficient conditions on the operator \( M_\varphi : M \to M, \varphi \in \mathcal{H}_\infty \), to be a Fredholm operator, besides the essential spectrum and the essential norm of this operator are given by K. Zhu [8]. Also, the special case \( \varphi(z) = z \) is taken into consideration in [7].

2. Spectrum and Essential Spectrum

In all that follows, let \( \mathcal{B} \) be a reflexive Banach space of analytic functions on \( D \) and \( M \) be an invariant subspace of \( M_z \). Also, let \( \mathcal{M}(M) \) denote the set of all multipliers of \( M \); that is, the analytic functions \( \varphi \).
on $D$ such that $\varphi M \subset M$. It is convenient and helpful to introduce the notation $\langle x, x^* \rangle$ stand for $x^*(x)$, where $x \in B$ and $x^* \in B^*$.

**Lemma 2.1** Let $\|M_\varphi\|$ denote the norm of operator $M_\varphi : M \to M$. The following statements are equivalent:

(A1) There is a constant $c > 0$ such that $\|M_p\| \leq c\|p\|_\infty$ for every polynomial $p$.

(A2) $\mathcal{M}(M) = \mathcal{H}^\infty$.

(A3) There is a constant $c > 0$ such that $\|M_\varphi\| \leq c\|\varphi\|_\infty$ for every $\varphi \in \mathcal{H}^\infty$.

**Proof.** (A1) implies (A2): Suppose that $\varphi \in \mathcal{H}^\infty$. Then there is a sequence $\{p_n\}$ of polynomials such that $\sup_n \|p_n\|_\infty < \infty$ and $p_n(z) \to \varphi(z)$ for all $z \in D$. So for every $f \in M$,

$$\sup_n \|p_n f\| = \sup_n \|M_{p_n} f\| \leq c\|f\| \sup_n \|p_n\|_\infty < \infty.$$  

It follows that $p_n f$ converges to $\varphi f$ weakly [1, page 272]. So $\varphi f$ is in the weak closure of $M$ which, in turn, implies that $\varphi f \in M$, because of the convexity of $M$. Hence $\mathcal{H}^\infty \subset \mathcal{M}(M)$. To see the other inclusion, if $\varphi \in \mathcal{M}(M)$, then an application of the closed graph theorem shows that $M_\varphi$ is bounded. Also for every $f \in M$ and $z \in D$, $\langle f, M_\varphi^* e_z \rangle = \langle \varphi f, e_z \rangle = \varphi(z) f(z) = \langle f, \varphi(z) e_z \rangle$. Hence $M_\varphi^* e_z = \varphi(z) e_z$ and $\|\varphi\|_\infty \leq \|M_\varphi^*\| = \|M_\varphi\|$.

(A2) implies (A3): Define the norm $\|\cdot\|$ on $\mathcal{H}^\infty$ by $\|\varphi\|_\infty = \|M_\varphi\|$. If $\{\varphi_n\}$ is a Cauchy sequence in $(\mathcal{H}^\infty, \|\cdot\|)$ then $M_{\varphi_n}$ converges to an operator $A$ in $L(B)$, the set of all bounded operators on $B$. It is easy to see that $\{M_z\}'$, the commutant of $M_z$, is $\{M_\phi \mid \phi \in \mathcal{H}^\infty\}$ (see [3]); so $A = M_\varphi$ for some $\varphi \in \mathcal{H}^\infty$. It follows that $(\mathcal{H}^\infty, \|\cdot\|)$ is a Banach space.

Since $\|\varphi\|_\infty \leq \|\varphi\|$ for every $\varphi \in \mathcal{H}^\infty$, an application of the inverse mapping theorem to the identity map $i : (\mathcal{H}^\infty, \|\cdot\|) \to (\mathcal{H}^\infty, \|\cdot\|_\infty)$ shows that there is a constant $c$ such that $\|M_\varphi\| \leq c\|\varphi\|_\infty$ for every $\varphi \in \mathcal{H}^\infty$.

Clearly, (A3) implies (A1). □
Theorem 2.2 The set of all multipliers of $M$ is $\mathcal{H}^\infty$, and for each $\varphi \in \mathcal{H}^\infty$ the operator $M_\varphi : M \to M$ is a bounded operator. Also, $M_\varphi$ is invertible if and only if $\varphi$ is invertible in $\mathcal{H}^\infty$.

Proof. By Property (1), $\mathcal{H}^\infty \subset \mathcal{M}(\mathcal{B})$, also in the proof of (A1)⇒(A2) in previous Lemma we see that $\mathcal{M}(\mathcal{B}) \subset \mathcal{H}^\infty$. So there is a constant $c > 0$ such that $||M_p|| \leq c||p||_{\infty}$, where $||M_p||$ denotes the norm of $M_p$ on $\mathcal{B}$ which is bigger than the norm of $M_p$ on $M$. By Lemma 2.1, $\mathcal{M}(M) = \mathcal{H}^\infty$. The boundedness of $M_\varphi$ follows from the closed graph theorem. □

In this section the techniques of some proofs are slight refinements of the similar ones, used in [8]. For the reader’s convenience, we include some of the proofs.

Proposition 2.3 Suppose $p$ is a polynomial and

$$q(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$$

where $z_1, z_2, \cdots, z_n$ are the zeros of $p$ in $D$. Then the closure of $pM$ in $\mathcal{B}$ is $qM$.

Proof. Let

$$p(z) = q(z)r(z)s(z),$$

where $r$ is a polynomial with no zero on the closed unit disc, and $s$ is a polynomial with zeros on the unit circle. Since $r(z)$ and $\frac{1}{r(z)}$ are in $\mathcal{H}^\infty$ Theorem 2.2 implies that $rM = M$. Also, since $M_q$ is bounded below, $qM$ is closed in $M$. Hence it is sufficient to show that $sM$ is dense in $M$.

Let

$$s(z) = (z - a_1) \cdots (z - a_m)$$

and for any $n \geq 1$ and $k = 1, 2, \cdots, m$ let $p_{k,n}$ be the $n$-th Taylor polynomial of $(z - a_k)^{-1}$ at $z = 0$; moreover, suppose that

$$p_n = p_{1,n}p_{2,n} \cdots p_{m,n}.$$
If \( f \in M \) then \( s(z)p_n(z)f(z) \to f(z) \) for all \( z \in D \) and
\[
\|sp_nf\| \leq c\|sp_n\|_{\infty}\|f\| \leq c2^n\|f\| < \infty.
\]
Consequently, \( f \) is in the weak closure of \( sM \) which is a convex set, and so \( f \in \overline{sM} \). Hence \( \overline{sM} = M \). \( \square \)

**Proposition 2.4** Suppose \( \varphi \) is a non-vanishing function in \( \mathcal{H}^\infty \) and \( M_\varphi \) is the multiplication operator on \( M \). Then \( M_\varphi \) is Fredholm if and only if \( M_\varphi \) is invertible.

**Proof.** The method of proof is similar to the one used in [8, Lemma 6] and so is omitted.

We recall that the index of \( M \) is \( \dim \frac{M}{zM} \). It is well known that the index of \( M \) can assume any value in \( \{1, 2, 3, \ldots, \infty\} \) also it follows from the Fredholm theory that \( \dim \frac{M}{(z-\lambda)M} \) does not depend on \( \lambda \in D \).

**Theorem 2.5** Suppose \( \varphi \in \mathcal{H}^\infty \) and \( M_\varphi \) is Fredholm on \( M \). If \( \varphi(0) = 0 \), then \( M \) has finite index.

**Proof.** Let \( \varphi = z\psi \) for some \( \psi \in \mathcal{H}^\infty \). If \( x^* \in \ker M_\varphi^* \) and \( f \in M \) then
\[
0 = \langle \psi f, M_\varphi^* x^* \rangle = \langle \varphi f, x^* \rangle = \langle f, M_\varphi^* x^* \rangle
\]
so \( \ker M_\varphi^* \subset \ker M_\varphi^* \). On the other hand,
\[
\dim \frac{M}{zM} = \dim \frac{M}{zM}^* = \dim (zM)^\perp = \dim \ker M_\varphi^* z,
\]
and
\[
\dim \ker M_\varphi^* \leq \dim \ker M_\varphi^* = \dim (\varphi M)^\perp = \dim \frac{M}{\varphi M} = \dim \frac{M}{\varphi M} < \infty.
\]

**Theorem 2.6** Suppose that \( \varphi \in \mathcal{H}^\infty \) and \( M_\varphi \) is the multiplication operator on \( M \). Then
(a) \( \sigma(M_\varphi) = \overline{\varphi(D)} \).

If the index of \( M \) is finite, then
(b) \( M_\varphi \) is Fredholm if and only if \( \varphi \) is invertible in \( \mathcal{H}^\infty \).
\[(c) \sigma_e(M_\varphi) = \overline{\varphi(D)} \text{ and } r(M_\varphi) = r_e(M_\varphi) \leq \|M_\varphi\|_e \leq \|M_\varphi\| \leq c\|\varphi\|_\infty, \text{ for some } c > 0.\]

**Proof.** (a) By Theorem 2.2, \(M_\varphi\) is invertible on \(M\) if and only if \(\varphi\) is invertible in \(H^\infty\). Also \(\lambda - M_\varphi = M_{\lambda - \varphi}\) for every complex number \(\lambda\); so \(\sigma(M_\varphi) = \overline{\varphi(D)}\).

(b) If \(M_\varphi\) is Fredholm, Theorem 2.5 implies that \(\varphi(0) \neq 0\). Since \(\dim \frac{M}{(z - \lambda)M} = \dim \frac{M}{z M}\) for every \(\lambda \in D\), by the same technique as in the proof of the previous theorem one can show that \(\varphi(\lambda) \neq 0\). Then Proposition 2.4 and Theorem 2.2 implies that \(\varphi\) is invertible in \(H^\infty\). The converse follows from Theorem 2.2.

(c) By Part (b) \(\sigma_e(M_\varphi) = \overline{\varphi(D)}\). Since \(\sigma_e(M_\varphi) = \sigma(M_\varphi), r_e(M_\varphi) = r(M_\varphi)\). Using Theorem 2.2 and Lemma 2.1 the proof is complete. \(\square\)

**Lemma 2.7** If \(B(z)\) is a finite Blaschke product then \(M_B\) is bounded below on \(D\).

**Proof.** Let
\[B(z) = z^k \prod_{n=1}^{m} \frac{|a_n|}{a_n} \frac{(a_n - z)}{(1 - a_n z)}\]
and put \(\varphi_n(z) = a_n - z, \psi_n(z) = \frac{1}{1 - a_n z}\) and \(\varphi(z) = z^k\). It is known that \(M_{\varphi_n}\) and \(M_\varphi\) are bounded below [4]. Moreover, by Theorem 2.2, \(M_{\psi_n}, n = 1, 2, \cdots m\) are invertible; hence \(M_B\) is a finite composition of bounded below operators which is certainly bounded below. \(\square\)

**Lemma 2.8** If the index of \(M\) is finite and \(B\) is a finite Blaschke product, then \(M_B\) is Fredholm on \(M\).

**Proof.** Let \(a_1, a_2, \cdots, a_m\) be the zeros of \(B\) according to the multiplicity and put \(p(z) = (z - a_1)(z - a_2) \cdots (z - a_m)\). Proposition 2.3 implies that the closure of \(BM\) is \(pM\), and by the previous lemma, \(BM\) is closed; thus \(BM = pM\). Since \(\dim \frac{M}{(z - \lambda)M}\) is independent of \(\lambda \in D\), if \(M\) is of index \(n\) then \(\dim \frac{B}{BM} = \dim \frac{M}{pM} = nm\). \(\square\)
**Theorem 2.9** Suppose that \( \varphi \in \mathcal{H}^\infty \) and \( M_\varphi \) is the multiplication operator on \( M \). If the index of \( M \) is finite, then the following conditions are equivalent:

(i) The operator \( M_\varphi \) is Fredholm on \( M \).

(ii) There exist constants \( \varepsilon > 0 \) and \( 0 < \delta < 1 \) such that \( |\varphi(z)| \geq \varepsilon \) for all \( z \) with \( \delta \leq |z| < 1 \).

(iii) The function \( \psi \) admits a factorization \( \varphi = B\psi \), where \( B \) is a finite Blaschke product and \( \psi \) is invertible in \( \mathcal{H}^\infty \).

**Proof.** (i) \( \Rightarrow \) (ii). If \( \varphi \) is non-vanishing then by Proposition 2.4, \( M_\varphi \) is invertible. So \( \frac{1}{\varphi} \in \mathcal{H}^\infty \) and (ii) holds. On the other hand, if \( \varphi(a) = 0 \) then there is a function \( S(z) \) in \( \mathcal{H}^\infty \) such that \( \varphi(z) = (z-a)S(z) \). Let \( f \in M \) then \( \varphi f(z) = (z-a)S(z)f(z) \in (z-a)M \) (Theorem 2.2) so \( \varphi M \subset (z-a)M \). Similarly, if \( a_1, a_2, \ldots, a_k \) are zeros of \( \varphi \) in \( \mathbb{D} \) then \( \varphi M \subset pM \) where \( p(z) = (z-a_1)(z-a_2)\cdots(z-a_k) \). Consequently, \( (pM)^\perp \subset (\varphi M)^\perp \) and so \( \dim(pM)^\perp = \dim \frac{M}{pM} = \dim \left( \frac{M}{pM} \right)^* = \dim(pM)^\perp \leq \dim(\varphi M)^\perp = \dim \left( \frac{M}{\varphi M} \right)^* = \dim \frac{M}{\varphi M} \). But \( \dim \frac{M}{\varphi M} = kn \) where \( n \) is the index of \( M \). This implies that \( \varphi \) has only a finite number of zeros in \( \mathbb{D} \); hence (ii) holds.

(ii) \( \Rightarrow \) (i). If \( \varphi \) is non-vanishing then, clearly, \( \frac{1}{\varphi} \in \mathcal{H}^\infty \) and by Theorem 2.2 and Proposition 2.4, \( M_\varphi \) is Fredholm. Otherwise, \( \varphi \) has a finite number of zeros \( a_1, a_2, \ldots, a_m \) in \( \mathbb{D} \) listed according to their multiplicities. Thus \( \varphi(z) = (z-a_1)(z-a_2)\cdots(z-a_m)S(z) \) for some function \( S(z) \) in \( \mathcal{H}^\infty \), with no zeros in \( \mathbb{D} \). It is obvious that \( \frac{1}{S} \in \mathcal{H}^\infty \), and using Theorem 2.2, we see that \( SM = M \). Hence \( \dim \frac{M}{\varphi M} = mn < \infty \).

(ii) \( \Rightarrow \) (iii). Clearly, \( \varphi \) has a finite number of zeros \( a_1, a_2, \ldots, a_m \) in \( \mathbb{D} \). If \( \varphi(z) = (z-a_1)(z-a_2)\cdots(z-a_m)S(z) \) for some nonzero function \( S \) in \( \mathcal{H}^\infty \) then considering

\[
B(z) = \prod_{n=1}^{m} \frac{|a_n|}{a_n} \left( \frac{a_n - z}{1 - a_n z} \right) \quad \text{and} \quad \psi(z) = S(z) \prod_{i=1}^{m} \frac{|a_n|}{a_n} \left( \frac{a_n - z}{1 - a_n z} \right)
\]

(iii) holds.

(iii) \( \Rightarrow \) (ii). It is obvious. \( \square \)
Recall that for $\varphi \in \mathcal{H}^\infty$ cluster($\varphi$) is the set of all complex numbers $\omega$ such that there exists a sequence $\{z_n\}$ in $D$ such that $|z_n| \to 1$ and $\varphi(z_n) \to \omega$ as $n \to \infty$.

**Corollary 2.10** Under the assumptions of the above theorem, we have $\sigma_e(M_\varphi) = \text{cluster}(\varphi)$ and

$$||\varphi||_\infty \leq ||M_\varphi||_e \leq c||\varphi||_\infty$$

for some positive constant $c$.

**Proof.** The formula for the essential spectrum follows from the above theorem and the identity $\lambda - M_\varphi = M_{\lambda - \varphi}$, where $\lambda$ is any complex number. Theorem 2.2 and Lemma 2.1, along with the maximum modulus principle imply that

$$||\varphi||_\infty \leq r_e(M_\varphi) \leq ||M_\varphi||_e \leq ||M_\varphi|| \leq c||\varphi||_\infty$$

for some $c > 0$. $\square$

### 3. Interpolating Sequences

In this section, our aim is to reduce the conditions of the main theorem in [6] but to give more results and a very simpler proof of it. As in [6] we have the following definition.

**Definition 3.1** A sequence $\{\omega_n\}$ in $D$ is said to be an interpolating sequence for $B$ if there exists a positive weight sequence $\{k_n\}$ so that the sequence $\{f(\omega_n)k_n\}$ belongs to $\ell^\infty$ for all $f$ in $B$ and, conversely, every sequence in $\ell^\infty$ can be written in this form.

**Theorem 3.2** If $\{\omega_n\}$ is a sequence in $D$ such that

$$\frac{1 - |\omega_n|}{1 - |\omega_{n-1}|} < \delta < 1 \quad (1)$$

then $\{\omega_n\}$ is an interpolating sequence for $B$. 

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Proof. It is known that if (1) holds then \( \{ \omega_n \} \) is an interpolating
sequence for \( \mathcal{H}^\infty \) (see Page 203 of [2]). But \( \mathcal{H}^\infty \subset \mathcal{B} \), and
\[
|f(\omega_n)| = |\langle f, e_{\omega_n} \rangle| \leq \|f\| \|e_{\omega_n}\| \quad \forall f \in \mathcal{B}.
\]
So the result holds, considering the weights \( k_n = \frac{1}{\|e_{\omega_n}\|} \). \( \square \)

We remark that if \( \{ \omega_n \} \) is a sequence in \( \mathbf{D} \) such that \( \omega_n \to \partial \mathbf{D} \) then
some subsequence of \( \{ \omega_n \} \) satisfies (1), so Theorem 5 of [6] holds.

4. The commutant of \( M_\varphi \)

In this section, we assume that \( M \) is an invariant subspace of \( \mathcal{B} \) with
codimension 1 property, that is, \( \dim(\frac{M}{z^M}) = 1 \), and \( \varphi \) is a certain function
in \( \mathcal{H}^\infty \) or in \( \mathcal{A}(\mathbf{D}) \) the algebra of all continuous functions on \( \overline{\mathbf{D}} \) which
are analytic in \( \mathbf{D} \). Under these conditions we investigate the commutant
of \( M_\varphi \) as an operator from \( M \) to \( M \). Let \( Z_M \) be the intersection of zero
sets of functions in \( M \) and \( e'_\lambda \) be the evaluation functional at \( \lambda \) from \( M \)
to \( \mathbf{C} \).

Theorem 4.1 Let \( \varphi \in \mathcal{H}^\infty \) be a univalent function. Then \( \{ M_\varphi \}' = \{ M_\psi : \psi \in \mathcal{H}^\infty \} \).

Proof. If \( f \in M, \lambda \in \mathbf{D} \) and \( T \in \{ M_\varphi \}' \), then we have
\[
\langle f, M_\varphi T^*(e'_\lambda) \rangle = \langle f, T^*M_\varphi^*(e'_\lambda) \rangle = \langle M_\varphi T(f), e'_\lambda \rangle = \varphi(\lambda)Tf(\lambda) = \varphi(\lambda)\langle T(f), e'_\lambda \rangle
\]
\[
= \varphi(\lambda)(f, T^*(e'_\lambda)) = \langle f, \varphi(\lambda)T^*(e'_\lambda) \rangle.
\]
Consequently, \( M_\varphi T^*(e'_\lambda) = \varphi(\lambda)T^*(e'_\lambda) \) which implies that \( T^*(e'_\lambda) \in \ker(M_\varphi - \varphi(\lambda))' \). Let \( \lambda \in \mathbf{D} - Z_M \). We show that \( \text{ran}(M_\varphi - \varphi(\lambda)) = \ker e'_\lambda \). If \( f \in M \), then
\[
\langle (\varphi - \varphi(\lambda))f, e'_\lambda \rangle = ((\varphi - \varphi(\lambda))f)(\lambda) = 0,
\]
and so \( \text{ran}(M_\varphi - \varphi(\lambda)) \subset \ker e'_\lambda \).
To show the converse, since \( \text{ran}(M_z - \lambda) = \text{kere}_\lambda \), we have \((\varphi - \varphi(\lambda))(z) = (z - \lambda)g(z)\) for some \( g \in B \). Since \( \varphi \) is univalent, \( g \) is bounded below on \( D \). Hence \( \frac{1}{g} \) is in \( \mathcal{H}^\infty \) and, so \( \frac{1}{g} \in \mathcal{M}(M) \) and we have \( z - \lambda = \frac{\varphi(z) - \varphi(\lambda)}{g(z)} \). By Lemma 2.1 in [4], \( \text{ran}((M_z - \lambda)|_M) = \text{kere}'_\lambda \).

So if \( f \in \text{kere}'_\lambda \), then \( f = (z - \lambda)\phi \) for some function \( \phi \in M \). Hence

\[
f = \frac{\varphi - \varphi(\lambda)}{g} \phi = (\varphi - \varphi(\lambda)) \frac{\phi}{g}.
\]

Since \( \phi \in M \) and \( \frac{1}{g} \in \mathcal{H}^\infty \), we conclude that \( \text{kere}'_\lambda \subset \text{ran}(M_\varphi - \varphi(\lambda)) \).

Now, we have \((M_\varphi - \varphi(\lambda))^*(e'_\lambda) = (M_\varphi - \varphi(\lambda))^*T(e'_\lambda) = 0\). Since \( \dim \text{ker}(M_\varphi - \varphi(\lambda))^* = 1 \), we conclude that \( T^*(e'_\lambda) = \psi(\lambda)e'_\lambda \) for some constant \( \psi(\lambda) \). Thus, for every \( \lambda \in D - Z_M \) we have

\[
T(f)(\lambda) = \langle T(f), e'_\lambda \rangle = \langle f, T^*(e'_\lambda) \rangle = \psi(\lambda)\langle f, e_\lambda \rangle = \psi(\lambda)f(\lambda);
\]

in particular, \( \psi(\lambda) = T(1)(\lambda) \). Let \( \psi \) denote \( T(1) \). Since \( D - Z_M \) is dense in \( D \), we conclude that \( T(f)(\lambda) = \psi(\lambda)f(\lambda) \) for every \( \lambda \) in \( D \), and the proof is complete. \( \square \)

The proof of the above theorem is a modification of the proof of Theorem 2.2 in [3].

**Theorem 4.2** Let \( \varphi \in \mathcal{A}(D) \), \( S \) be a subset of \( D - Z_M \) which has a limit point in \( D \). Suppose further that for every \( \lambda \in S \), we have \( \varphi^{-1}(\{\varphi(\lambda)\}) = \{\lambda\} \). Then \( \{M_\varphi\}' = \{M_\psi : \psi \in \mathcal{H}^\infty\} \).

**Proof.** Without loss of generality, we can assume that \( \varphi'(\lambda) \neq 0 \) for every \( \lambda \in S \). As in the proof of Theorem 4.1, for every \( \lambda \in D \) we have \( T^*(e'_\lambda) \subset \ker(M_\varphi - \varphi(\lambda))^* \) and for \( \lambda \in D - Z_M \), \( \text{ran}(M_\varphi - \varphi(\lambda)) \subset \text{kere}'_\lambda \).

Now, let \( \lambda \in S \). Since \( \text{ran}(M_z - \lambda) = \text{kere}_\lambda \), we have \((\varphi - \varphi(\lambda))(z) = (z - \lambda)g(z)\) for some \( g \in \mathcal{A}(D) \) which implies that \( \text{kere}'_\lambda \subset \text{ran}(M_\varphi - \varphi(\lambda)) \). Now, by assumption \( g \) is bounded below on \( D \) and as in the remainder of the proof of Theorem 4.1 we conclude that \( \psi(\lambda) = T(1)(\lambda) \).

Let \( \psi \) denote \( T(1) \) since \( S \) has a limit point in \( D \), we conclude that \( T(f)(\lambda) = \psi(\lambda)f(\lambda) \) for every \( \lambda \) in \( D \) as desired. \( \square \)
References


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