OPTIMAL POLYNOMIAL LOWER BOUNDS FOR THE EXPONENTIAL FUNCTION

JAEGUG BAE

ABSTRACT. In this paper, for each natural number \( n \), we construct a polynomial \( p_n(x) \) of degree \( n \) so that \( p_n(x) \leq p_{n+1}(x) \leq e^x \) for \( x \geq -1 \). These polynomials are optimal in the sense that if \( p(x) \) is a polynomial of degree \( n \) with \( p_{n-1}(x) \leq p(x) \leq e^x \), then \( p(x) \leq p_n(x) \).

1. Introduction

A typical bound of the exponential function is

\[
1 + x \leq e^x \quad \text{or} \quad e^x \leq \frac{1}{1 - x}.
\]

Here the first inequality holds on the whole real line but the second holds for \( x < 1 \). Pólya proved the inequality between the arithmetic and geometric means using this simple inequality (see [2, §4.2], [7]). In fact, almost the same proof can be applied to show the inequality between the generalized arithmetic and geometric means:

\[
\sum_{j=1}^{n} \lambda_j a_j \geq \prod_{j=1}^{n} a_j^{\lambda_j}
\]

where \( \lambda_j \)'s and \( a_j \)'s are nonnegative real numbers with \( \sum_{j=1}^{n} \lambda_j = 1 \).

As a part of an effort to approximate \( e^x \) by other (algebraic) functions, various bounds for the exponential function have been developed (see [3], [5, pp. 266-270], [6]). Recently, Kim established the following densely algebraic bounds in [1], [4]

\[
e^x \leq 1 - \frac{1}{n} + \frac{1}{n} \left( \frac{1 + \left( 1 - \frac{1}{n} \right) x}{1 - \frac{x}{n}} \right)^n
\]


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for real \( n \geq 1 \) and \(-\frac{n}{n-1} < x < n\). In this paper, we present the following simple polynomial bounds of \( e^x \) extending (1.1).

**Theorem 1.1.** For \( n \geq 1 \), let

\[
p_n(x) = 1 + \alpha_1 x + \sum_{j=2}^{n} \alpha_j x^j (x + 1)^{j-2}
\]

where \( \alpha_1 = 1 \) and

\[
\alpha_n = \frac{n-2}{e} \left( \sum_{j=0}^{n-2} \frac{1}{j!} - e \right) + \frac{1}{(n-2)!} e, \quad n \geq 2.
\]

Then \( p_n(x) \leq p_{n+1}(x) \leq e^x \) for \( x \geq -1 \). Furthermore, these are optimal in the sense that if \( p(x) \) is a polynomial of degree \( n \) with \( p_{n-1}(x) \leq p(x) \leq e^x \) for \( x \geq -1 \), then \( p(x) \leq p_n(x) \) for \( x \geq -1 \).

This theorem gives lower bounds of \( e^x \) but by writing \(-x\) for \( x \), we have the following rational upper bound version.

**Theorem 1.2.** For all natural numbers \( n \), we have

\[
e^x \leq \frac{1}{p_{n+1}(-x)} \leq \frac{1}{p_n(-x)}, \quad x < 1.
\]

If \( p(x) \) is a polynomial of degree \( n \) with \( e^x \leq \frac{1}{p(x)} \leq \frac{1}{p_{n-1}(-x)}, \) \( x < 1 \), then

\[
\frac{1}{p_n(-x)} \leq \frac{1}{p(-x)}, \quad x < 1.
\]

With some preliminaries in the next section, we will prove the theorem. We assume that all functions in this paper are real valued on \( \mathbb{R} \) and analytic on \( \mathbb{C} \). Also, note that when we say the number of zeros of a function, it includes all the multiplicities.

### 2. Preliminaries

At first, we give a couple of observations on calculus.
Proposition 2.1. Let $a$ be a fixed real number. Suppose $f^{(k-1)}(a) = 0$ and $f^{(k)}(a) > 0$ for some positive integer $k$.

(i) If $k$ is even, then there exists an interval around $a$ on which $f(x)$ achieves the unique minimum $f(a) = 0$.

(ii) If $k$ is odd, then there exists an interval around $a$ on which $f(x)$ is strictly increasing.

Proof. Let $h(x)$ be a function defined on an open interval $I$ containing $a$. Let’s say that $h(x)$ has unique minimum (UM) property when $h(a) = 0$ and it is the unique minimum on $I$, and $h(x)$ has strictly increasing (SI) property when $h(a) = 0$ and $h(x)$ is strictly increasing on $I$. It’s easy to see that if $h'(x)$ has UM-property and $h(a) = 0$ then $h(x)$ has SI-property. On the other hand, if $h'(x)$ has SI-property and $h(a) = 0$ then $h(x)$ has UM-property. Since $f^{(k)}(a) > 0$, by the continuity of $f^{(k)}(x)$, there exists an open interval $I$ containing $a$ on which $f^{(k)}(x) > 0$. Hence, on the interval $I$, $f^{(k-1)}(x)$ has SI-property, $f^{(k-2)}(x)$ has UM-property, $f^{(k-3)}(x)$ has SI-property, \ldots, and so on. Pursuing these alternations $k$ times, we have the conclusions of the proposition. \hfill \Box

Proposition 2.2. Let $p(x)$ be a polynomial of degree $n$. Then $e^x - p(x)$ has at most $n + 1$ zeros.

Proof. Suppose a function $f(x)$ has $k$ zeros. Applying Roll’s Theorem, we know that its derivative $f'(x)$ has at least $k - 1$ zeros. In other words, if $f'(x)$ has $k - 1$ zeros, then $f(x)$ has at most $k$ zeros. Now the proof is straightforward by using induction on $n$, the degree of $p(x)$. \hfill \Box

We introduce one more proposition. We need the following lemma for the proof of it.

Lemma 2.3. Suppose there exists an interval $[a, a + \delta]$ on which $f(x) \geq 0$ and $f(a) = 0$. Then there exists $\lambda$ ($0 < \lambda < \delta$) such that $f'(x) \geq 0$ on $[a, a + \lambda]$. 
Proof. We may assume \( f(x) \) is not constant. Let \( \beta = \inf \{ x \in [a, a + \delta] \mid f'(x) < 0 \} \). If \( \beta > a \), then we may take any real number between \( a \) and \( \beta \) as our \( \lambda \). In fact, \( \beta = a \) is impossible because (as an analytic function) \( f(x) \) has only finitely many zeros on \( [a, a + \delta] \) and for any \( \epsilon > 0 \), there is no open interval \( (a, a + \epsilon) \) on which \( f(x) < 0 \). \( \square \)

**Definition 2.4.** We say that \( a \) is an \( m \)-multiple zero of \( f(x) \) when \( f^{(j)}(a) = 0 \) for \( j = 0, 1, 2, \cdots, m - 1 \) and \( f^{(m)}(a) \neq 0 \). In this case, \( Z(a, f) = m \) is called the multiplicity of the zero \( a \) of \( f \).

**Proposition 2.5.** Suppose \( 0 \leq f(x) \leq g(x) \) for \( x \geq a \) and \( Z(a, g) = m \). Then \( Z(a, f) \geq m \) and \( f^{(m)}(a) \leq g^{(m)}(a) \).

**Proof.** By the definition of a multiple zero, \( g^{(j)}(a) = 0 \) for \( j = 0, 1, 2, \cdots, m - 1 \). Hence, for each \( j \in \{0, 1, 2, \cdots, m\} \), it suffices to show that \( 0 \leq f^{(j)}(x) \leq g^{(j)}(x) \) on the interval \( [a, a + \lambda_j] \) for some \( \lambda_j > 0 \). We use an induction on \( j \). Assume \( 0 \leq f^{(j-1)}(x) \leq g^{(j-1)}(x) \) on the interval \( [a, a + \lambda_{j-1}] \). Then note \( f^{(j-1)}(a) = g^{(j-1)}(a) = 0 \). Now applying Lemma 2.3 twice, one for \( f^{(j-1)}(x) \) and another for \( g^{(j-1)}(x) - f^{(j-1)}(x) \), we can find an interval \( [a, a + \lambda_j] \) on which \( 0 \leq f^{(j)}(x) \leq g^{(j)}(x) \). \( \square \)

### 3. Proof of the main theorem

In this section, we prove Theorem 1.1. At first, we establish some properties of the sequence \( \alpha_n \).

**Lemma 3.1.** For \( n \geq 1 \), we have

(i) \( \alpha_n > 0 \)

(ii) \( \alpha_{n+2} = 2\alpha_{n+1} - \alpha_n + \frac{1}{n\epsilon} \)

(iii) \( \sum_{n=1}^{\infty} \alpha_n = \frac{3}{2} \).
Proof. In this proof, we will use the following inequality frequently.

\begin{equation}
(3.1) \quad e - \sum_{j=0}^{n-1} \frac{1}{j!} = \frac{1}{n!} \left( \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots \right) < \frac{1}{n!} \left( \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots \right) = \frac{n+1}{n!} \frac{1}{n}.
\end{equation}

By this inequality, we have

\[
\alpha_n = \frac{n-2}{e} \left( \sum_{j=0}^{n-2} \frac{1}{j!} - e \right) + \frac{1}{(n-2)!} e > \frac{n-2}{e} \frac{n}{(n-1)!} \frac{1}{(n-1)} + \frac{1}{(n-2)!} e = \frac{1}{(n-2)!} \left( 1 - \frac{n(n-2)}{(n-1)^2} \right) > 0.
\]

For (ii), the direct calculation shows that

\[
2\alpha_{n+1} - \alpha_n + \frac{1}{n! e} = \frac{2n-2}{e} \left( \sum_{j=0}^{n-1} \frac{1}{j!} - e \right) + \frac{2}{(n-1)!} e - \frac{n-2}{e} \left( \sum_{j=0}^{n-2} \frac{1}{j!} - e \right) - \frac{1}{(n-2)!} e + \frac{1}{n! e} = \sum_{j=0}^{n} \frac{1}{j!} - e + \frac{1}{n! e} = \alpha_{n+2}.
\]

For (iii), it’s enough to show that \( \sum_{n=1}^{\infty} n \left( e - \sum_{j=0}^{n} \frac{1}{j!} \right) = \frac{1}{2} e \) since \( \alpha_1 = 1 \) and \( \sum_{n=2}^{\infty} \frac{1}{(n-2)! e} = 1 \). Let \( \sigma_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \) with \( \sigma_0 = 0 \).

Then

\[
\sum_{k=1}^{n} k \left( e - \sum_{j=0}^{k} \frac{1}{j!} \right) = \sigma_n e - \sigma_n \frac{1}{0!} - \sum_{j=1}^{n} (\sigma_n - \sigma_{j-1}) \frac{1}{j!} = \sigma_n \left( e - \sum_{j=0}^{n} \frac{1}{j!} \right) + \sum_{j=2}^{n} \frac{\sigma_{j-1}}{j!}.
\]

Note that, by (3.1), the first term of the last expression goes to zero as \( n \) goes to infinity. Thus we obtain

\[
\sum_{n=1}^{\infty} n \left( e - \sum_{j=0}^{n} \frac{1}{j!} \right) = \sum_{j=2}^{\infty} \frac{\sigma_{j-1}}{j!} = \sum_{j=1}^{\infty} \frac{\sigma_{j}}{(j+1)!} = \sum_{j=1}^{\infty} \frac{1}{2} j(j+1) \frac{1}{(j+1)!} = \frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{j!} = \frac{1}{2} e.
\]

\( \square \)
Lemma 3.2. For $n \geq 1$, let $g_n(x) = e^x - p_n(x)$. Then

(i) $g_n(0) = g'_n(0) = 0$, $g''_n(0) = 1 - 2\sum_{j=2}^n \alpha_j > 0$ (with $g''_n(0) = 1$).

(ii) For $n \geq 2$, $g_n(x)$ has $n-1$ multiple zeros at $-1$ and $g_n^{(n-1)}(-1) > 0$.

Proof. Note that

$$p'_n(x) = 1 + 2\sum_{j=2}^n \alpha_j x(x+1)^{j-2} + \sum_{j=3}^n (j-2)\alpha_j x^2(x+1)^{j-3},$$

$$p''_n(x) = 2\sum_{j=2}^n \alpha_j (x+1)^{j-2}$$

$$+ 4\sum_{j=3}^n (j-2)\alpha_j x(x+1)^{j-3} + \sum_{j=4}^n (j-2)(j-3)\alpha_j x^2(x+1)^{j-4}.$$ 

Invoking Lemma 3.1, (i) is immediate now. For $k \geq 2$, let us define $f_k(x) = \alpha_k x^2(x+1)^{k-2}$. Then we observe that

$$f_k^{(j)}(-1) = 0 \text{ if } 0 \leq j < k - 2 \text{ or } j > k$$

$$f_k^{(k)}(x) = k!\alpha_k$$

(3.2) $$f_k^{(k-1)}(x) = (k-1)!2\alpha_k x + (k-1)!(k-2)\alpha_k (x+1)$$

$$f_k^{(k-2)}(x) = (k-2)!\alpha_k x^2 + (k-2)!2(k-2)\alpha_k x(x+1)$$

$$+ \frac{1}{2}(k-2)!(k-2)(k-3)\alpha_k (x+1)^2.$$ 

To prove (ii), let $n \geq 2$. Since $g_n(x) = e^x - 1 - \alpha_1 x - \sum_{k=2}^n f_k(x)$, we have

$$g_n^{(n-1)}(-1) = \frac{1}{e} - f_{n-1}^{(n-1)}(-1) - f_n^{(n-1)}(-1)$$

$$= \frac{1}{e} - (n-1)!\alpha_{n-1} + (n-1)!2\alpha_n$$

$$= (n-1)! \left( 2\alpha_n - \alpha_{n-1} + \frac{1}{(n-1)!} \right) = (n-1)!\alpha_{n+1} > 0$$

by (3.2) and Lemma 3.1 (ii). Finally, it remains to show $g_n^{(j)}(-1) = 0$, for $0 \leq j \leq n - 2$. Clearly, $g_n^{(0)}(-1) = g_n(-1) = \frac{1}{e} - 1 + \alpha_1 - \alpha_2 = 0$ and $g_n^{(1)}(-1) = \frac{1}{e} - \alpha_1 + 2\alpha_2 - \alpha_3 = 0$. Using (3.2) and Lemma 3.1 (ii) again for
\[2 \leq j \leq n - 2, \quad \text{we obtain}\]
\[g_n^{(j)}(-1) = \frac{1}{e} - \sum_{k=2}^{n} f_k^{(j)}(-1) = \frac{1}{e} - f_j^{(j)}(-1) - f_{j+1}^{(j)}(-1) - f_{j+2}^{(j)}(-1)\]
\[= \frac{1}{e} - j! \alpha_j + j! 2 \alpha_{j+1} - j! \alpha_{j+2}\]
\[= j! \left(-\alpha_{j+2} + 2\alpha_{j+1} - \alpha_j + \frac{1}{j!e}\right) = 0.\]

Now we are ready to prove the main theorem.

**Proof of Theorem 1.1.**

Since \(\alpha_n\) is a positive sequence by Lemma 3.1 and \(x^2(x+1) \geq 0\) for \(x \geq -1\), it's clear that \(p_n(x) \leq p_{n+1}(x)\) for \(x \geq -1\). We are to show \(g_n(x) \geq 0\) for \(x \geq -1\). Lemma 3.2 shows that \(g_n(x)\) has double zeros at \(x = 0\) and \(n - 1\)-multiple zeros at \(x = -1\). Also, Proposition 2.2 says that these are all the zeros of \(g_n(x)\). Lemma 3.2 (i) implies that \(g_n(0) = 0\) is a local minimum. Moreover, combining Lemma 3.2 (ii) with Proposition 2.1, we realize that \(g_n(-1) = 0\) is a local minimum for odd \(n(>1)\) and \(g_n(x)\) is increasing in some neighborhood of \(x = -1\) for even \(n\). Based on these analyses, we may conclude \(g_n(x) \geq 0\) for \(x \geq -1\). Finally, suppose \(p(x)\) is a polynomial of degree \(n\) with \(p_{n-1}(x) \leq p(x) \leq e^x\) for \(x \geq -1\). This means \(0 \leq p(x) - p_{n-1}(x) \leq g_{n-1}(x)\) for \(x \geq -1\). Let \(f(x) = p(x) - p_{n-1}(x)\). Then \(f(x)\) is a polynomial of degree \(n\) and by Lemma 3.2 and Proposition 2.5, it has double zeros at 0 and \(n - 2\)-multiple zeros at \(-1\). Hence
\[f(x) = \alpha x^2(x+1)^{n-2}\]
for some \(\alpha \in \mathbb{R}\). Proposition 2.5 also says that
\[(n-2)! \alpha = f^{(n-2)}(-1) \leq g_{n-1}^{(n-2)}(-1) = (n-2)! \alpha_n\]
which implies \(\alpha \leq \alpha_n\). Therefore we have
\[p_n(x) - p(x) = p_n(x) - p_{n-1}(x) - f(x) = (\alpha_n - \alpha)x^2(x+1)^{n-2} \geq 0\]
for \(x \geq -1\). \(\square\)
References


Jaegug Bae
Department of Applied Mathematics,
Korea Maritime University,
# 1 Dongsam-Dong, Yeongdo-Gu, Pusan 606-791, S. Korea
E-mail: jgbae@hhu.ac.kr