A GENERALIZATION OF THE SCHUR-COHN TEST

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ABSTRACT. In this note we provide a generalization of the Schur-Cohn test. If $p_n$ be a polynomial of degree $n$ then $p_n$ is a reduced-stable polynomial whose reduced degree is equal to $r$ if and only if $D_n := P_n Q_n^{-1}$ is a contraction such that $\text{rank}(I - D_n D_n^*) = r$.

Consider first the following interpolation problem, called the Carathéodory–Schur interpolation problem (CSIP). Given $c_0, c_1, \ldots, c_{N-1}$ in $\mathbb{C}$, find an analytic function $k$ on the open unit disc $\mathbb{D}$ such that

(i) $\widehat{k}(j) = c_j, \quad j = 0, \ldots, N-1$ (where $\widehat{k}(j)$ denotes the $j$-th Fourier coefficient of $k$) and

(ii) $\sup_{z \in \mathbb{D}} |k(z)| \leq 1$.

CSIP can be analyzed by a matricial argument (cf. [4]): CSIP is solvable if and only if the Toeplitz matrix

$$
C := \begin{pmatrix}
c_0 & 0 & 0 & \cdots & 0 \\
c_1 & c_0 & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
c_{N-2} & \ddots & \ddots & c_0 & 0 \\
c_{N-1} & c_{N-2} & \cdots & c_1 & c_0
\end{pmatrix}
$$

is a contraction, that is, $||C|| \leq 1$, or equivalently, $I - CC^* \geq 0$. Today this result is also called the Carathéodory–Fejér Theorem. A proof of this result can be obtained by means of the Commutant Lifting Theorem (cf. [3, Proposition XXVII.7.2]). In particular, by Pick's Theorem (cf. [2, Corollary I.2.3]) CSIP has a unique solution $k$ if and only if $\det(I - CC^*) = 0$. Moreover, in the cases


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where \( \det (I - CC^*) = 0 \), the unique solution is a finite Blaschke product \( k \) of the form

\[
k(z) = e^{i\theta} \prod_{j=1}^{n} \frac{z - \zeta_j}{1 - \overline{\zeta}_j z} \quad (|\zeta_j| < 1 \text{ for } j = 1, \ldots, n; \ \theta \in [0, 2\pi]),
\]

such that \( \deg (k) = \text{rank} (I - CC^*) \), where \( \deg (k) \) denotes the \textit{degree} of \( k \), that is, the number of zeros of \( k \) in the open unit disc \( \mathbb{D} \) (see also [5, Theorem]).

As an application of the preceding result, we will derive a generalization of the Schur-Cohn test.

A polynomial \( p \) is called \textit{stable} if all the zeros of \( p \) are in the open unit disk \( \mathbb{D} \), and is called \textit{reduced-stable} if for every zero \( \zeta \) of \( p \) such that \( |\zeta| > 1 \), the number \( 1/\zeta \) is a zero of \( p \) in the open unit disk \( \mathbb{D} \) of multiplicity greater than or equal to the multiplicity of \( \zeta \). If \( p \) is reduced-stable, we define the \textit{reduced degree} of \( p \) as the integer \( Z_D - Z_{C \setminus D} \), where \( Z_D \) and \( Z_{C \setminus D} \) are the number of zeros of \( p \) in \( \mathbb{D} \) and in \( C \setminus \overline{D} \) counting multiplicity. Thus if \( p \) is a stable polynomial of degree \( n \) then evidently the reduced degree of \( p \) is \( n \). Let \( p_n \) denote the polynomial

\[
p_n(z) = c_0 + c_1 z + \cdots + c_n z^n.
\]

The \( n \times n \) Toeplitz matrices \( P_n \) and \( Q_n \) are defined by

\[
P_n = \begin{pmatrix}
c_0 & 0 & 0 & \cdots & 0 \\
c_1 & c_0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
c_{n-2} & \cdots & 0 & c_0 & c_1 \\
c_{n-1} & c_{n-2} & \cdots & c_1 & c_0
\end{pmatrix}
\quad \text{and} \quad
Q_n = \begin{pmatrix}
\overline{c_0} & 0 & 0 & \cdots & 0 \\
\overline{c_1} & c_0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\overline{c_{n-2}} & \cdots & 0 & c_0 & c_1 \\
\overline{c_{n-1}} & \overline{c_{n-2}} & \cdots & c_1 & c_0
\end{pmatrix}.
\]

Then the \textit{Schur-Cohn test} for the stability of a polynomial states that if \( p_n \) is a polynomial of degree \( n \) then the following statements are equivalent ([1]).

1. \( p_n \) is a stable polynomial.
2. \( P_n Q_n^{-1} \) is a strict contraction.

The following theorem is a generalization of the Schur-Cohn test.

\textbf{Theorem 1.} Let \( p_n \) be a polynomial of degree \( n \). Then the following statements are equivalent.

1. \( p_n \) is a reduced-stable polynomial whose reduced degree is equal to \( r \).
2. \( \frac{p_n}{z^n} \) is a finite Blaschke product of degree \( r \).
3. \( D_n := P_n Q_n^{-1} \) is a contraction such that \( \text{rank} (I - D_n D_n^*) = r \).
Proof. (1) $\Leftrightarrow$ (2): If $p_n = \alpha \prod_{j=1}^{n}(z - \zeta_j)$ then $z^n \frac{p_n}{\overline{p_n}} = \overline{\alpha} \prod_{j=1}^{n}(1 - \overline{\zeta_j}z)$. Then $\frac{p_n}{z^n \overline{p_n}}$ is a finite Blaschke product of degree $r$ if and only if for every zero $\zeta$ of $p_n$ such that $|\zeta| > 1$, the number $1/\overline{\zeta}$ is a zero of $p_n$ in the open unit disk $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of $\zeta$ and $r = Z_{\mathbb{D}} - Z_{\mathbb{C}\setminus\mathbb{D}}$ if and only if $p_n$ is a reduced-stable polynomial of reduced degree $r$.

(2) $\Leftrightarrow$ (3): Let $p_n = c_0 + c_1 z + \cdots + c_n z^n$ ($c_n \neq 0$). Then $z^n \frac{p_n}{\overline{p_n}} = \overline{c_n} + \overline{c_{n-1}} z + \cdots + \overline{c_0} z^n$ and so $\frac{p_n}{z^n \overline{p_n}}$ is analytic in a neighborhood of 0. Let $A$ be the $n \times n$ matrix given by

\[
\begin{pmatrix}
0 & \ldots & \ldots & 0 \\
1 & 0 & \cdots & \cdots \\
0 & 1 & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 1 & 0
\end{pmatrix}
\]

Using the usual functional calculus, we have that $p_n(A) = P_n$ and $(z^n \overline{p_n})(A) = Q_n$. In particular, since $A$ is nilpotent, the expression $\left( \frac{p_n}{z^n \overline{p_n}} \right)(A)$ makes sense. Note that if $\frac{p_n}{z^n \overline{p_n}} = \sum_{j=0}^{\infty} d_j z^j$ then

\[
\left( \frac{p_n}{z^n \overline{p_n}} \right)(A) = \begin{pmatrix}
d_0 & 0 & 0 & \ldots & 0 \\
d_1 & d_0 & \ddots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \cdots \\
d_{n-2} & \cdots & \cdots & 0 & d_1 \\
d_{n-1} & d_{n-2} & \cdots & d_1 & d_0
\end{pmatrix}
\]

Thus by CSIP, $D_n := \left( \frac{p_n}{z^n \overline{p_n}} \right)(A)$ is a contraction if and only if $\frac{p_n}{z^n \overline{p_n}}$ is analytic on $\mathbb{D}$ if and only if $\frac{p_n}{z^n \overline{p_n}}$ is a finite Blaschke product. By the functional calculus we see that $\left( \frac{p_n}{z^n \overline{p_n}} \right)(A) = P_n Q_n^{-1}$. The argument for degree comes from an argument of S. Takahashi [5, Theorem], which states that if $I - CC^* \geq 0$ then there exists a finite Blaschke product whose degree is equal to the rank of $I - CC^*$.

\[\square\]

References


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