SUPERCYCLICITY OF TWO-ISOMETRIES

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Abstract. A bounded linear operator $T$ on a complex separable Hilbert space $\mathcal{H}$ is called a two-isometry, if $T^*T^2 - 2T^*T + I = 0$. In this paper it is shown that every two-isometry is not supercyclic. This generalizes a result due to Ansari and Bourdon.

1. Introduction

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space over the complex field and use $\mathcal{B}(\mathcal{H})$ to denote the algebra of all bounded linear operators $T: \mathcal{H} \rightarrow \mathcal{H}$. For a vector $f$ in $\mathcal{H}$, the orbit of $f$ under $T$ is defined by

$$orb(T, f) = \{T^n f : n = 0, 1, 2, \cdots\}.$$

We recall that a vector $f$ in $\mathcal{H}$ is cyclic for $T$, if the closed linear span of $orb(T, f)$ is equal to $\mathcal{H}$; it is supercyclic if the set of all scalar multiples of the elements of $orb(T, f)$ is dense in $\mathcal{H}$; also it is said to be hypercyclic if $orb(T, f)$ is dense in $\mathcal{H}$. An operator $T$ is called a cyclic, hypercyclic or supercyclic operator, respectively, if it has a cyclic, hypercyclic, or supercyclic vector. Recently, the cyclicity of operators has attracted much attention (see [2-8]) from operator theorists.

By a two-isometry, we mean an operator $T \in \mathcal{B}(\mathcal{H})$ satisfying $T^*T^2 - 2T^*T + I = 0$. S. I. Ansari and P. S. Bourdon in [2] proved that every isometry is not supercyclic. In the present note, we show that every two-isometry is not supercyclic. Considering the fact that every isometry is a two-isometry this result generalizes the mentioned result of Ansari and Bourdon.

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2. Main Results

For a finite Borel measure $\mu$ on the unit circle $\mathbb{T}$ in the plane, let the Dirichlet type space $D(\mu)$ consist of all analytic functions $f$ on the open unit disc $\mathbb{D}$ such that $\int_0^1 |f'(z)|^2 \varphi_\mu(z) dA(z) < \infty$, where $dA(z) = \frac{1}{\pi} r dr dt$, $z = re^{it}$, and $\varphi_\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} p(z, e^{it}) d\mu(t)$ where $p(z, e^{it}) = \frac{1-|z|^2}{|e^{it}-z|^2}$ is the Poisson kernel. When $\mu$ is the Lebesgue measure on $\mathbb{T}$ the space $D(\mu)$ is the Dirichlet space $D$ and $M_z$ is a two-isometry which is not an isometry; in fact, $\|M_z^n\| = \sqrt{k+1}$ for $k \geq 1$.

In the proof of our main theorem, we use the equivalence of (a) and (c) in Theorem 5. 10 of [9]. For the sake of completeness, we bring it here:

**Theorem 1.** The operator $T \in \mathcal{B}(\mathcal{H})$ is a cyclic two-isometry with $\bigcap_{n>0} T^n \mathcal{H} = \{0\}$, if and only if there is a positive finite Borel measure $\mu$ on $\mathbb{T}$ such that $T$ is unitarily equivalent to the multiplication by $z$, $M_z$ on $D(\mu)$.

**Lemma 1.** An operator $T$ in $\mathcal{B}(\mathcal{H})$ is supercyclic if and only if for every nonzero reducing subspace $M$ of $T$, the restriction of $T$ to $M$, $T|_M$ is supercyclic.

**Proof.** Let $\mathcal{H} = M \oplus M^\perp$ and suppose that $h = g \oplus k$ is a supercyclic vector for $T$. If $g = 0$ then $\mathcal{H} = M^\perp$ which is impossible; so $g \neq 0$. Take $f \in M$, and let $\varepsilon > 0$ be arbitrary. Then there is $n \geq 0$ and $\alpha \in \mathbb{C}$ such that

$$\|\alpha T^n g - f\| \leq \|\alpha T^n (g \oplus k) - f \oplus 0\| < \varepsilon.$$  

Hence $g$ is a supercyclic vector for $T|_M$. The converse is obvious. \qed

Recall that a bounded operator $T$ is pure if it has no nonzero reducing subspace $M$ such that $T|_M$ is normal. Since normal operators are not supercyclic, [3], we have the following corollary.

**Corollary 1.** Every supercyclic operator is pure.

**Theorem 2.** Every two-isometry is not supercyclic.

**Proof.** Suppose that $T$ is a supercyclic two-isometry and let $M = \bigcap_{n>0} T^n \mathcal{H}$. If $M = \{0\}$ then Theorem 1 guarantees the existence of a positive finite Borel measure $\mu$ on $\mathbb{T}$ such that $T$ is unitarily equivalent to $M_z$ on the space $D(\mu)$. By Corollary 3.8 of [9], the point spectrum
of $T^*$ contains more than one point which contradicts the supercyclicity of $T$, [2]. Now suppose that $M \neq \{0\}$. Since $T$ is a two-isometry,

$$\|T^2h\|^2 - 2\|Th\|^2 + \|h\|^2 = 0, \forall h \in \mathcal{H}. \quad (*)$$

Moreover, by Proposition 1.5 of [1], $T^*T - I \geq 0$; thus, $\|Th\| \geq \|h\|$ for all $h \in \mathcal{H}$, which implies that $\ker T = \{0\}$ and $\text{ran } T$ is closed. Define $S : \text{ran } T \rightarrow \mathcal{H}$ by $S(Th) = h$. By the inverse mapping theorem $S$ is bounded. Let $g \in M$ and put $h = S^2g$ in $(*)$, then

$$\|T^2S^2g\|^2 - 2\|TS^2g\|^2 + \|S^2g\|^2 = 0. \quad (**)$$

But since $g = T^2k$ for some $k \in \mathcal{H}$, $(**)$ is, indeed,

$$\|g\|^2 - 2\|Sg\|^2 + \|S^2g\|^2 = 0.$$

Using induction, it is easy to see that

$$\|S^{i+1}g\|^2 - \|S^ig\|^2 = \|Sg\|^2 - \|g\|^2, \quad \forall i \geq 1.$$

Thus,

$$\|S^{m+1}g\|^2 - \|g\|^2 = \sum_{i=0}^{m}(\|S^{i+1}g\|^2 - \|S^ig\|^2) = (m+1)(\|Sg\|^2 - \|g\|^2).$$

This, in turn, implies that

$$0 \leq \|S^{m+1}g\|^2 = (m+1)\|Sg\|^2 - m\|g\|^2.$$

Consequently, $\|Sg\|^2 \geq \frac{m}{m+1}\|g\|^2$. Letting $m \to \infty$, we obtain $\|Sg\| \geq \|g\|$. Put $f = Tg$. Then $f \in M$, and so $\|g\| = \|Sf\| \geq \|f\| = \|Tg\|$. Hence, $\|Tg\| = \|g\|$ for every $g \in M$. It follows that $T^*T = I$ on $M$. Also, the definition of $M$ shows that $TM = M$, and so $T^*M = M$. Since isometries are not supercyclic [2], Lemma 1 complete the proof. □

References


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