THE APPROXIMATION FOR FUNCTIONAL EQUATION ORIGINATING FROM A CUBIC FUNCTION

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Abstract. In this paper, we obtain the general solution of the following cubic type functional equation and establish the stability of this equation

\[ k f \left( \sum_{j=1}^{n-1} x_j + kx_n \right) + k f \left( \sum_{j=1}^{n-1} x_j - kx_n \right) + 2 \sum_{j=1}^{n-1} f(kx_j) \]
\[ + (k^3 - 1)(n - 1)[f(x_1) + f(-x_1)] = 2kf \left( \sum_{j=1}^{n-1} x_j \right) \]
\[ + k^2 \sum_{j=1}^{n-1} [f(x_j + x_n) + f(x_j - x_n)] \]

for any integers \( k \) and \( n \) with \( k \geq 2 \) and \( n \geq 3 \).

1. Introduction

The stability problem of functional equations has originally been formulated by S.M. Ulam [24] in 1940: Under what condition does there exist a homomorphism near an approximate homomorphism? In following year, D.H. Hyers [8] answered the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th.M. Rassias [18]. Since then, a great deal of work has been done by a number of authors and the problems concerned with the generalizations and the applications of the stability to a number of functional equations have been developed as well.

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The cubic function \( f(x) = ax^3 \) satisfies the functional equation
\[
(1.1) \ f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x).
\]
Hence, throughout this paper, we promise that the equation (1.1) is called a cubic functional equation and every solution of the equation (1.1) is said to be a cubic function. The functional equation (1.1) was solved by K.-W. Jun and H.-M. Kim [12]. In addition, they investigated the Hyers-Ulam-Rassias stability for the functional equation (1.1). After then, I.-S. Chang, K.-W. Jun and Y.-S. Jung [6] introduced the cubic type functional equation
\[
(1.2) \ f(x_1 + x_2 + 2x_3) + f(x_1 + x_2 - 2x_3) + f(2x_1) + f(2x_2) \\
+ 7[f(x_1) + f(-x_1)] = 2f(x_1 + x_2) + 4[f(x_1 + x_3) + f(x_1 - x_3)] \\
+ f(x_2 + x_3) + f(x_2 - x_3)].
\]
It is easy to see that the function \( f(x) = ax^3 + b \) is a solution of the functional equations (1.2). Recently, I.-S. Chang and Y.-S. Jung [5] extended the functional equation (1.2) to the n-dimensional cubic type functional equation
\[
(1.3) \ 2f(\sum_{j=1}^{n-1} x_j + 2x_n) + 2f(\sum_{j=1}^{n-1} x_j - 2x_n) + 2 \sum_{j=1}^{n-1} f(2x_j) \\
+ 7(n - 1)[f(x_1) + f(-x_1)] = 4f(\sum_{j=1}^{n-1} x_j) + 8 \sum_{j=1}^{n-1} [f(x_j + x_n)] \\
+ f(x_j - x_n)].
\]
In this paper, we now deal with the cubic type functional equation (0.1), that is to say, we obtain the general solution of the equation (0.1) and offer the stability for the equation (0.1). In 1996, G. Isac and Th.M. Rassias [11] were the first to provide applications of the generalized Hyers-Ulam stability theory of functional equations for the proof of new fixed point theorems.

2. The required results

We now introduce one of fundamental results of fixed point theory.

**Theorem 2.1.** (The alternative of fixed point) [16]. Suppose that we are given a complete generalized metric space \((\Omega, d)\), i.e., one for which \(d\) may assume infinite values, and a strictly contractive mapping \(T : \Omega \to \Omega\) with Lipschitz constant \(L\). Then, for each given \(x \in \Omega\), either
(T_1) \quad d(T^n x, T^{n+1} x) = \infty \text{ for all } n \geq 0,

or

(T_2) \quad \text{There exists a nonnegative integer } n_0 \text{ such that } \\
\quad d(T^n x, T^{n+1} x) < \infty \text{ for all } n \geq n_0; 

Actually, if (T_2) holds, then the followings are true:

- The sequence \((T^n x)\) is convergent to a fixed point \(y^*\) of \(T\);
- \(y^*\) is the unique fixed point of \(T\) in the set \(\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}\);
- \(d(y, y^*) \leq \frac{1}{1-L} d(y, Ty) \text{ for all } y \in \Delta.\)

The reader is referred to the book of D.H. Hyers, G. Isac and Th.M. Rassias [10] for an extensive theory of fixed points with a large variety of applications.

Now we recall some basic facts concerning quasi-Banach spaces and some preliminary results.

**Definition 2.2.** [2, 23] Let \(X\) be a vector space. A quasi-norm \(\| \cdot \|\) is a real-valued function on \(X\) satisfying the following:

1. \(\|x\| \geq 0\) for all \(x \in X\) and \(\|x\| = 0\) if and only if \(x = 0\).
2. \(\|\lambda x\| = |\lambda| \|x\|\) for all \(\lambda \in \mathbb{R}\) and for all \(x \in X\).
3. There is a constant \(M \geq 1\) such that \(\|x+y\| \leq M(\|x\| + \|y\|)\) for all \(x, y \in X\).

The pair \((X, \| \cdot \|)\) is called a quasi-normed space if \(\| \cdot \|\) is a quasi-norm on \(X\). The smallest possible \(M\) is called the modulus concavity of \(\| \cdot \|\). A quasi-Banach space is a complete quasi-normed space. A quasi-norm \(\| \cdot \|\) is called a \(p\)-normed \((0 < p \leq 1)\) if

\(\|x+y\|^p \leq \|x\|^p + \|y\|^p\)

for all \(x, y \in X\). In this case, a quasi-Banach space is called a \(p\)-Banach space. By the Aoki-Rolewicz theorem [23] (see also [2]), each quasi-norm is equivalent to some \(p\)-norm. Since it is much easier to work with \(p\)-norms than quasi-norms, henceforth we restrict our attention mainly to \(p\)-norms.

We will first present the general solution of functional equation (0.1).

**Lemma 2.3.** Let \(X\) and \(Y\) be real vector spaces. A function \(f : X \to Y\) satisfies the functional equation (0.1) for all \(x_1, x_2, \ldots, x_n \in X\) if and only if \(C\) is cubic, where \(C : X \to Y\) is a function defined by \(C(x) = f(x) - f(0)\) for all \(x \in X\).
Proof. (Necessity.) First of all, we have by the assumption

\begin{equation}
(2.1) \quad kC\left(\sum_{j=1}^{n-1} x_j + kx_n\right) + kC\left(\sum_{j=1}^{n-1} x_j - kx_n\right) + 2\sum_{j=1}^{n-1} C(kx_j) \\
+(k^3 - 1)(n - 1)[C(x_1) + C(-x_1)] \\
= 2kC\left(\sum_{j=1}^{n-1} x_j + k^3\sum_{j=1}^{n-1} [C(x_j + x_n) + C(x_j - x_n)]
\right)
\end{equation}

for all \(x_1, x_2, \cdots, x_n \in X\). In particular, it is clear that \(C(0) = 0\). Substituting \(x_j = 0\) (\(j = 1, 2, \cdots, n - 1\)), and \(x_n = x\) in (2.1) yields

\begin{equation}
(2.2) \quad C(kx) + C(-kx) = k^2(n - 1)[C(x) + C(-x)].
\end{equation}

Letting \(x_1 = x\), \(x_2 = -x\), and \(x_j = 0\) (\(j = 3, \cdots, n\)) in (2.1) gives the equation

\begin{equation}
(2.3) \quad C(kx) + C(-kx) = \frac{(3k^3 - 1) - (k^3 - 1)n}{2}[C(x) + C(-x)].
\end{equation}

Now, by combining (2.2) and (2.3), we get \(C(x) + C(-x) = 0\) for all \(x \in X\), that is, \(C\) is an odd function. Hence the relation (2.1) becomes

\begin{equation}
kC\left(\sum_{j=1}^{n-1} x_j + kx_n\right) + kC\left(\sum_{j=1}^{n-1} x_j - kx_n\right) + 2\sum_{j=1}^{n-1} C(kx_j) \\
= 2kC\left(\sum_{j=1}^{n-1} x_j + k^3\sum_{j=1}^{n-1} [C(x_j + x_n) + C(x_j - x_n)]
\right)
\end{equation}

Thus [15, Lemma 2.4] implies that \(C\) is cubic.

(Sufficiency.) Suppose that \(C\) is cubic. Due to [13, Theorem 2.2], we have

\begin{equation}
(2.4) \quad C(kx + y) + C(kx - y) = kC(x + y) + kC(x - y) \\
+ 2k(k^2 - 1)C(x)
\end{equation}

for any integer \(k\) with \(k \neq -1, 0, 1\). Then it is easy to check that

\[C(0) = 0, \ C(x) + C(-x) = 0, \ C(kx) = k^3C(x)\].
On the other hand, by [15, Lemma 2.2], we obtain
\[
kC(\sum_{j=1}^{n-1} x_j + kx_n) + kC(\sum_{j=1}^{n-1} x_j - kx_n) + 2 \sum_{j=1}^{n-1} C(kx_j)
\]
\[
= 2kC(\sum_{j=1}^{n-1} x_j) + k^3 \sum_{j=1}^{n-1} [C(x_j + x_n) + C(x_j - x_n)].
\]
Since \(C\) is an odd function, we see that
\[
kC(\sum_{j=1}^{n-1} x_j + kx_n) + kC(\sum_{j=1}^{n-1} x_j - kx_n) + 2 \sum_{j=1}^{n-1} C(kx_j)
\]
\[
+ (k^3 - 1)(n - 1)[C(x_1) + C(-x_1)]
\]
\[
= 2kC(\sum_{j=1}^{n-1} x_j) + k^3 \sum_{j=1}^{n-1} [C(x_j + x_n) + C(x_j - x_n)],
\]
which gives the functional equation (0.1). This completes the proof of Lemma.

\[\square\]

3. The stability of the functional equation (0.1)

In recent years, L. Cădariu and V. Radu [4] applied the fixed point method to the investigation of the Cauchy additive functional equation. Using such an elegant idea, they could present a short and simple proof for the stability of that equation [3, 17].

From now on, let \(X\) be a real vector space and \(Y\) be a real Banach space. As a matter of convenience, for a given mapping \(f : X \to Y\), we use the following abbreviation
\[
Df(x_1, x_2, \cdots, x_n) := k \sum_{j=1}^{n-1} x_j + kx_n) + k \sum_{j=1}^{n-1} x_j - kx_n)
\]
\[
+ 2 \sum_{j=1}^{n-1} f(kx_j) + (k^3 - 1)(n - 1)[f(x_1) + f(-x_1)] - 2k \sum_{j=1}^{n-1} x_j)
\]
\[
- k^3 \sum_{j=1}^{n-1} [f(x_j + x_n) + f(x_j - x_n)].
\]
for all \( x_1, x_2, \cdots, x_n \in X \), where \( k \geq 2 \) and \( n \geq 3 \) are any integers, and let \( \varphi : X^n \to [0, \infty) \) be a function satisfying

\[
(3.1) \quad \lim_{m \to \infty} \frac{\varphi(\lambda_i^m x_1, \lambda_i^m x_2, \cdots, \lambda_i^m x_n)}{\lambda_i^{3m}} = 0
\]

for all \( x_1, x_2, \cdots, x_n \in X \), where \( \lambda_i = k \) if \( i = 0 \) and \( \lambda_i = \frac{1}{k} \) if \( i = 1 \).

Now we adopt the idea of Cădariu and Radu and prove the stability of the equation (0.1).

**Theorem 3.1.** Suppose that a function \( f : X \to Y \) satisfies the inequality

\[
(3.2) \quad \|Df(x_1, x_2, \cdots, x_n)\| \leq \varphi(x_1, x_2, \cdots, x_n)
\]

for all \( x_1, x_2, \cdots, x_n \in X \). If there exists \( L < 1 \) such that the function

\[
x \mapsto \psi(x) = \varphi(0, \underbrace{\frac{x}{k}, \frac{x}{k}, \cdots, \frac{x}{k}}_{(n-2)-times}, 0)
\]

has the property

\[
(3.3) \quad \psi(x) \leq L \cdot \lambda_i^3 \cdot \psi\left(\frac{x}{\lambda_i}\right)
\]

for all \( x \in X \), then there exists a unique cubic function \( C : X \to Y \) satisfying the inequality

\[
(3.4) \quad \|f(x) - C(x)\| \leq \frac{L^{1-i}}{2(n-2)(1-L)} \psi(x) + \|f(0)\|
\]

for all \( x \in X \).

**Proof.** We consider the set

\[
\Omega := \{ g : g : X \to Y, \ g(0) = 0 \}
\]

and the generalized metric on \( \Omega \):

\[
d(g, h) = d_\psi(g, h) = \inf\{K \in (0, \infty) : \|g(x) - h(x)\| \leq K \psi(x), \ x \in X\}.
\]

One can easily show that \((\Omega, d)\) is complete.

Now we define a function \( T : \Omega \to \Omega \) by

\[
Tg(x) := \frac{1}{\lambda_i^3} \ g(\lambda_i x)
\]
for all $x \in X$. Note that for all $g, h \in \Omega$,

$$d(g, h) < K \implies \|g(x) - h(x)\| \leq K\psi(x), \ x \in X$$

$$\implies \left\| \frac{1}{\lambda_i^3} g(\lambda_i x) - \frac{1}{\lambda_i^3} h(\lambda_i x) \right\| \leq \frac{1}{\lambda_i^3} K\psi(\lambda_i x), \ x \in X$$

$$\implies \|Tg(x) - Th(x)\| \leq LK\psi(x), \ x \in X$$

$$\implies d(Tg, Th) \leq LK.$$ 

Hence we see that for all $g, h \in \Omega$

$$d(Tg, Th) \leq Ld(g, h),$$

that is, $T$ is a strictly contractive on $\Omega$ with the Lipschitz constant $L$.

Here we define a function $F : X \to Y$ by $F(x) = f(x) - f(0)$ for all $x \in X$. Then we have $F(0) = 0$. If we put $x_1 = x_n = 0, x_2 = \cdots = x_{n-1} = y$ in (3.2) and use (3.3) with the case $i = 0,$ then

$$(3.5) \quad \|(n-2)F(ky) - k^3(n-2)F(y)\| \leq \frac{1}{2} \varphi(0, y, y, \cdots, y, 0),$$

which is reduced to

$$\left\| F(y) - \frac{1}{k^3} F(ky) \right\| \leq \frac{1}{2k^3(n-2)} \psi(ky) \leq \frac{L}{2(n-2)} \psi(y)$$

for all $y \in X$, that is,

$$d(F, TF) \leq \frac{L}{2(n-2)} = \frac{L^1}{2(n-2)} \leq \infty.$$ 

If we substitute $y := \frac{y}{k}$ in (3.12) and use (3.3) with the case $i = 1$, then

$$\left\| F(y) - k^3F\left(\frac{y}{k}\right) \right\| \leq \frac{1}{2(n-2)} \psi(y)$$

for all $y \in X$, that is,

$$d(F, TF) \leq \frac{1}{2(n-2)} = \frac{L^0}{2(n-2)} < \infty.$$ 

Thus we conclude that

$$d(F, TF) \leq \frac{L^{1-j}}{2(n-2)} < \infty.$$ 

Now, from the alternative of fixed point in both cases, it follows that there exists a fixed point $C$ of $T$ in $\Omega$ such that

$$(3.6) \quad C(x) = \lim_{m \to \infty} \frac{F(\lambda_i^m x)}{\lambda_i^m}$$
for all \( x \in X \), since \( \lim_{m \to \infty} d(T^m F, C) = 0 \).

Again, using the fixed point alternative, we have

\[
(3.7) \quad d(F, C) \leq \frac{1}{1-L} d(F, TF) \leq \frac{L^{1-i}}{2(n-2)(1-L)},
\]

which yields the inequality (3.4).

To show that the function \( C : X \to Y \) is cubic, let \( x_j := \lambda_i^m x_j \) for \( j = 1, 2, \cdots n \) in (3.2) and divide by \( \lambda_i^{3m} \). Then it follows from (3.1) and (3.6) that

\[
\|DC(x_1, x_2, \cdots, x_n)\| = \lim_{m \to \infty} \frac{\|DF(\lambda_i^m x_1, \lambda_i^m x_2, \cdots, \lambda_i^m x_n)\|}{\lambda_i^{3m}} \leq \lim_{m \to \infty} \frac{\varphi(\lambda_i^m x_1, \lambda_i^m x_2, \cdots, \lambda_i^m x_n)}{\lambda_i^{3m}} = 0
\]

for all \( x_1, x_2, \cdots, x_n \in X \), that is, \( C \) satisfies the functional equation (0.1). Therefore Lemma 2.3 guarantees that \( C \) is cubic, since \( C(0) = 0 \).

To prove the uniqueness of the cubic function, let us assume that there exists another cubic function \( C_1 : X \to Y \) subject to the inequality (3.7). Since \( C_1 \) is a cubic,

\[
C_1(x) = \frac{1}{\lambda_j^3} C_1(\lambda_j x) = (TC_1)(x)
\]

and so \( C_1 \) is a fixed point of \( T \). In view of (3.7) and the definition of \( d \), we deduce that

\[
d(F, C_1) \leq \frac{L^{1-j}}{2(n-2)(1-L)} < \infty,
\]

that is, \( C_1 \in \Delta = \{ g \in X \mid d(F, g) < \infty \} \). By the fixed point alternative, we find that \( C = C_1 \), which proves that \( C \) is unique. This ends the proof of the theorem.

\[ \square \]

From Theorem 3.1, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability [18] of the functional equation (0.1).

**Corollary 3.2.** Let \( X \) and \( Y \) be a normed space and a Banach space, respectively. Let \( p \geq 0 \) be given with \( p \neq 3 \). Assume that \( \varepsilon \geq 0 \) are fixed. Suppose that a function \( f : X \to Y \) satisfies the inequality

\[
\|Df(x_1, x_2, \cdots, x_n)\| \leq \varepsilon (\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p)
\]

for all \( x_1, x_2, \cdots, x_n \in X \). Then there exists a unique cubic function \( C : X \to Y \) such that the inequality

\[
(3.8) \quad \|f(x) - C(x)\| \leq \frac{\varepsilon}{2[k^3 - k^p]} \|x\|^p + \|f(0)\|
\]
for all $x \in X$.

Proof. Let $\varphi : X^n \to [0, \infty)$ be a function defined by

$$
\varphi(x_1, x_2, \cdots, x_n) := \varepsilon(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p)
$$

for all $x_1, x_2, \cdots, x_n \in X$. Then it follows that

$$
\varphi(\lambda^m_{i_1} x_1, \lambda^m_{i_2} x_2, \cdots, \lambda^m_{i_n} x_n)

\lambda^{3m}_{i_1^n}

= (\lambda^m_{i_1^n})^{p-3} \varepsilon(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p) \to 0
$$

as $m \to \infty$, where $p < 3$ if $i = 0$ and $p > 3$ if $i = 1$, that is, (3.1) is seen to be true. Since the inequality

$$
\frac{1}{\lambda^{n}_{i^n}} \psi(\lambda^m_{i^n} x) = \frac{\lambda^{p-3}_{i^n}}{k^p} (n - 2) \varepsilon \|x\|^p \leq \lambda^{p-3}_{i^n} \psi(x)
$$

holds for all $x \in X$, where $p < 3$ if $i = 0$ and $p > 3$ if $i = 1$, we see that the inequality (3.3) holds with either $L = k^{p-3}$ or $L = \frac{1}{k^{p-3}}$. Now the inequality (3.4) yields (3.8), which complete the proof of Corollary. □

The following is the Hyers-Ulam stability [8] of the equation (0.1).

**Corollary 3.3.** Let $X$ and $Y$ be a normed space and a Banach space, respectively. Assume that $\theta \geq 0$ is fixed. Suppose that a function $f : X \to Y$ satisfies the inequality

$$
\|Df(x_1, x_2, \cdots, x_n)\| \leq \theta
$$

for all $x_1, x_2, \cdots, x_n \in X$. Then there exists a unique cubic function $C : X \to Y$ such that the inequality

$$
\|f(x) - C(x)\| \leq \frac{1}{2n(k^3 - 1)} \theta + \|f(0)\|
$$

for all $x \in X$.

Proof. Considering $p := 0$ and $\varepsilon := \frac{\theta}{n}$ in the corollary 3.2, we arrive at the conclusion of the corollary. □

After this, let $X$ be a quasi-normed space and $Y$ be a quasi-Banach space. Also, $M$ will denote the modulus concavity of $\| \cdot \|$.

**Theorem 3.4.** Suppose that $f : X \to Y$ is a function for which there exists a function $\phi : X^n \to [0, \infty)$ such that

$$
\sum_{j=0}^{\infty} \frac{M}{k^3} \phi(k^j x_1, k^j x_2, \cdots, k^j x_n)
$$

(3.9)
converges and
\begin{equation}
\|Df(x_1, x_2, \ldots, x_n)\| \leq \phi(x_1, x_2, \ldots, x_n)
\end{equation}
for all $x_1, x_2, \ldots, x_n \in X$. Then there exists a unique cubic function $C : X \to Y$ satisfying the inequality
\begin{equation}
\|f(x) - C(x)\| \leq \frac{1}{2(n-2)} M \sum_{j=0}^{\infty} \left( \frac{M}{k^3} \right)^j \phi(k^j x) + \|f(0)\|
\end{equation}
holds for all $x \in X$, where $\tilde{\phi}(x) = \phi(0, x, x, \ldots, x, 0)$ for all $x \in X$.

\textit{Proof}. We now define a function $g : X \to Y$ by $g(x) = f(x) - f(0)$ for all $x \in X$. Then we have $g(0) = 0$. Putting $x_1 = x_n = 0$, $x_2 = \cdots = x_{n-1} = x$ in (3.10), we have
\begin{equation}
\|(n-2)g(kx) - k^3(n-2)g(x)\| \leq \frac{1}{2} \tilde{\phi}(x),
\end{equation}
which is written by
\begin{equation}
\left\| g(x) - \frac{g(kx)}{k^3} \right\| \leq \frac{1}{2(n-2)k^3} \tilde{\phi}(x)
\end{equation}
for all $x \in X$. By replacing $x$ by $k^j x$ in (3.13) and dividing by $k^{3j}$, one gets
\begin{equation}
\left\| \frac{g(k^j x)}{k^{3j}} - \frac{g(k^{j+1} x)}{k^{3(j+1)}} \right\| \leq \frac{1}{2(n-2)k^3} \frac{\tilde{\phi}(k^j x)}{k^3}
\end{equation}
for all $x \in X$. Using the induction argument on $l$ and summing up the resulting inequality for $j = 0, 1, \ldots, l-1$ in (3.14), we show that
\begin{equation}
\left\| g(x) - \frac{g(k^l x)}{k^{3l}} \right\| \leq \frac{1}{2(n-2)} M \sum_{j=0}^{l-2} \left( \frac{M}{k^3} \right)^j \tilde{\phi}(k^j x) + \frac{1}{k^3} \left( \frac{M}{k^3} \right)^{l-1} \tilde{\phi}(k^{l-1} x)
\end{equation}
for all $x \in X$ and all $l > 1$, which is considered to be (3.13) for $l = 1$. Now, we figure out by (3.13) and (3.15) for $l+1$
\begin{align*}
\left\| g(x) - \frac{g(k^{l+1} x)}{k^{3(l+1)}} \right\| &\leq M \left\| g(x) - \frac{g(k^l x)}{k^{3l}} \right\| + M \left\| g(k^l x) - \frac{g(k^{l+1} x)}{k^{3l+1}} \right\| \\
\leq &\leq \frac{1}{2(n-2)} M \sum_{j=0}^{l-1} \left( \frac{M}{k^3} \right)^j \tilde{\phi}(k^j x) + \frac{1}{k^3} \left( \frac{M}{k^3} \right)^{l} \tilde{\phi}(k^{l} x)
\end{align*}
for all \( x \in X \), which proves the inequality (3.15) for \( l + 1 \).

In order to prove convergence of the sequence \( \{ \frac{g(k^lx)}{k^3l^m} \} \), we divide inequality (3.15) by \( k^{3m} \) and also replace \( x \) by \( k^mx \) to find that

\[
\left\| \frac{g(k^mx)}{k^{3l}} - \frac{g(k^{l+m}x)}{k^{3(l+m)}} \right\| = \frac{1}{k^{3l}} \left\| g(k^mx) - \frac{g(k^lm^x)}{k^{3l}} \right\|
\leq \frac{1}{2(n-2)} \left[ \frac{M}{M^mk^3} \sum_{j=0}^{l-2} \left( \frac{M}{k^3} \right)^{m+j} \phi(k^{m+j}x) + \frac{1}{M^mk^3} \right] \left( \frac{M}{k^3} \right)^{m+l-1} \phi(k^{m+l-1}x)
\]

for all positive integers \( l \) and \( m \) with \( l > m \) and all \( x \in X \). Since the right-hand side of the inequality tends to 0 as \( m \to \infty \), \( \{ \frac{g(k^lx)}{k^3l^m} \} \) is Cauchy sequence in the quasi-Banach space. Therefore, we may define a function \( C : X \to Y \) by

\[
C(x) := \lim_{l \to \infty} \frac{g(k^lx)}{k^3l}
\]

for all \( x \in X \). By letting \( l \to \infty \) in (3.15), we arrive at (3.11).

Now we show that \( C \) satisfies the functional equation (1.2): Let us replace \( x_j \) by \( k^lx_j \) \((j = 1, 2, \ldots, n) \) in (3.10) and divide by \( k^{3l} \). Then it follows that

\[
DC(x_1, x_2, \ldots, x_n) = \lim_{l \to \infty} \frac{1}{k^{3l}} \| Dg(k^lx_1, k^lx_2, \ldots, k^lx_n) \|
\leq \lim_{l \to \infty} \left( \frac{M}{k^3} \right)^l \phi(k^lx_1, k^lx_2, \ldots, k^lx_n) = 0
\]

for all \( x_1, x_2, \ldots, x_n \in X \). Hence we obtain the desired result. Thus the lemma 2.3 implies that \( C \) is cubic, since \( C(0) = 0 \).

It only remains to claim that \( C \) is unique: Let us assume that there exists a cubic function \( C_1 \) which satisfies (0.1) and (3.11). It is clear that \( C(k^lx) = k^{3l}C(x) \) and \( C_1(k^lx) = k^{3l}C_1(x) \) for all \( x \in X \) and \( l \in \mathbb{N} \). Hence it follows from (3.11) that

\[
\|C(x) - C_1(x)\| = \frac{1}{k^{3l}} \|C(k^lx) - C_1(k^lx)\|
\leq \frac{M}{k^{3l}} \left[ \|C(k^lx) - f(k^lx)\| + \|f(k^lx) - C_1(k^lx)\| \right]
\leq \frac{1}{(n-2)} \frac{M}{k^{3l}} \sum_{j=0}^{\infty} \left( \frac{M}{k^3} \right)^{l+j} \phi(k^{l+j}x) + \frac{2M}{k^{3l}} \|f(0)\|
\]
for all \( x \in X \). Letting \( l \to \infty \), we have \( C(x) = C_1(x) \), which ends the proof of the theorem.

**Corollary 3.5.** Let \( \varepsilon, p \) be nonnegative real numbers. Suppose that a function \( f : X \to Y \) satisfies the inequality
\[
\|Df(x_1, x_2, \cdots, x_n)\| \leq \varepsilon(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p)
\]
for all \( x_1, x_2, \cdots, x_n \in X \). Then there exists a unique cubic function \( C : X \to Y \) such that the inequality
\[
\|f(x) - C(x)\| \leq \frac{M}{2(k^3 - M)^p} \varepsilon \|x\|^p + \|f(0)\|
\]
for all \( x \in X \), where \( p - 3 < -\log_k M \).

**Theorem 3.6.** Suppose that \( f : X \to Y \) is a function for which there exists a function \( \phi : X^n \to [0, \infty) \) such that
\[
\sum_{j=0}^{\infty} (k^3 M)^j \phi\left(\frac{x_1}{k^j}, \frac{x_2}{k^j}, \cdots, \frac{x_n}{k^j}\right)
\]
converges and satisfies the inequality (3.10) for all \( x_1, x_2, \cdots, x_n \in X \). Then there exists a unique cubic function \( C : X \to Y \) satisfying the inequality
\[
\|f(x) - C(x)\| \leq \frac{1}{2(n-2)}\frac{1}{k^3} \sum_{j=0}^{\infty} (k^3 M)^j \overline{\phi}\left(\frac{x}{k^j}\right) + \|f(0)\|
\]
holds for all \( x \in X \), where \( \overline{\phi}(x) = \phi(0, x, x, \cdots, x, 0) \) for all \( x \in X \).

**Proof.** The proof of assertion in Theorem is similarly proved by the following inequality due to (3.13)
\[
\left\|g(x) - k^3 g\left(\frac{x}{k^l}\right)\right\| \leq \frac{1}{2(n-2)}\left[ \frac{1}{k^3} \sum_{j=1}^{l-1} (k^3 M)^j \overline{\phi}\left(\frac{x}{k^j}\right) + \frac{1}{k^3 M (k^3 M)^j} \overline{\phi}\left(\frac{x}{k^j}\right) \right]
\]
for all integers \( l > 1 \) and all \( x \in X \).

**Corollary 3.7.** Let \( \varepsilon, p \) be nonnegative real numbers. Suppose that a function \( f : X \to Y \) satisfies the inequality
\[
\|Df(x_1, x_2, \cdots, x_n)\| \leq \varepsilon(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p)
\]
for all \( x_1, x_2, \cdots, x_n \in X \). Then there exists a unique cubic function \( C : X \to Y \) such that the inequality

\[
\| f(x) - C(x) \| \leq \frac{k^{p-3}}{2(k^p - M k^3)} \varepsilon \| x \| ^p + \| f(0) \|
\]

for all \( x \in X \), where \( p - 3 > \log_k M \).

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