SOME TOPOLOGICAL PROPERTIES IN
SUBTRACTION ALGEBRAS

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Abstract. In this paper, we show how certain topologies associate with ideals of subtraction algebras on subtraction algebras. We show subtraction algebras to be topological subtraction algebras with respect to these topologies. Furthermore, we show how certain standard properties may arise. In addition we demonstrate that it is natural for these topologies to have many clopen sets and thus to be highly disconnected via the ideal theory of subtraction algebras.

1. Introduction

B. M. Schein([8]) considered systems of the form \((\Phi; \circ, \setminus)\), where \(\Phi\)
is a set of functions closed under the composition “\(\circ\)” of functions (and hence \((\Phi; \circ)\) is a function semigroup) and the set theoretic subtraction “\(\setminus\)” (and hence \((\Phi; \setminus)\) is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka([10]) discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. H. Kim and H. S. Kim([6]) showed that a subtraction algebra is equivalent to an implicative \(BCK\)-algebra, and a subtraction semigroup is a special case of a \(BCI\)-semigroup which is a generalization of a ring. Y. B. Jun et al.([5]) introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [4], Y. B. Jun and H. S. Kim established the ideal generated by a set and discussed related results. In [3], S. S. Ahn, Y. H. Kim and K. J. Lee introduced the notion of

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a relation on subtraction algebras, called an $SA$-relation, which is a
generalization of a subtraction homomorphism, and then we discussed
some fundamental properties of subtraction algebras. In this paper, we
think about the issue of attaching topologies to subtraction algebras in
a natural manner as possible. It turns out to be useful as the class of
ideals of a subtraction algebra as the underlying structure of a topology.
A certain uniformity and a topology are derived, which provide
natural connection between the notion of a subtraction algebra and a
topology. In the connection, we are able to make a setting under which
a subtraction algebra becomes a topological subtraction algebra. Other
properties are also identified both in the subtraction algebra and in the
topology, such as $\{0\}$ is closed if and only if the topology is Hausdorff,
and $\{0\}$ is open if and only if the topology is discrete among others.

2. Preliminaries

A subtraction algebra is defined as an algebra $(X; -)$ with a binary
operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

$(S1)$ $x - (y - x) = x$;
$(S2)$ $x - (x - y) = y - (y - x)$;
$(S3)$ $(x - y) - z = (x - z) - y$.

The subtraction determines an order relation on $X$: $a \leq b \iff a - b =
0$, where $0 = a - a$ is an element that does not depend on the choice
of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the
sense of [2], that is, it is a meet semilattice with zero $0$ in which every
interval $[0, a]$ is a Boolean algebra with respect to the induced order.
Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$;
and if $b, c \in [0, a]$, then

$$b \lor c = (b' \land c')' = a - (a - b) \land (a - c) = a - ((a - b) - ((a - b) - (a - c))).$$

In a subtraction algebra, the following are true (see [5]):

$(a1)$ $(x - y) - y = x - y$.
$(a2)$ $x - 0 = x$ and $0 - x = 0$.
$(a3)$ $(x - y) - x = 0$.
$(a4)$ $x - (x - y) \leq y$.
$(a5)$ $(x - y) - (y - x) = x - y$.
$(a6)$ $x - (x - (x - y)) = x - y$.
$(a7)$ $(x - y) - (z - y) \leq x - z$. 
(a8) \( x \leq y \iff x = y - w \) for some \( w \in X \).
(a9) \( x \leq y \) implies \( x - z \leq y - z \) and \( z - y \leq z - x \) for all \( z \in X \).
(a10) \( x, y \leq z \) implies \( x - y = x \land (z - y) \).
(a11) \( (x \land y) - (x \land z) \leq x \land (y - z) \).

A non-empty subset \( A \) of a subtraction algebra \( X \) is called a subalgebra of \( X \) if \( x - y \in X \) for all \( x, y \in X \).

**Definition 2.1.** [5] A non-empty subset \( A \) of a subtraction algebra \( X \) is called an *ideal* of \( X \) if it satisfies
(I1) \( 0 \in A \),
(I2) \( (\forall x \in X)(\forall y \in A)(x - y \in A \implies x \in A) \).

**Lemma 2.2.** [5] An ideal \( A \) of a subtraction algebra \( X \) has the following property:
\[ (\forall x \in X)(\forall y \in A)(x \leq y \implies x \in A) \]

**Lemma 2.3.** [3] Let \( (X; -) \) be a subtraction algebra. Then \( (X; -) \) is a poset.

Let \( A \) be an ideal of a subtraction algebra \( X \). For any \( x, y \in X \), define a relation \( \sim_A \) on \( X \) by
\[ x \sim_A y \iff x - y \in A \text{ and } y - x \in A. \]
Then \( \sim_A \) is a congruence relation on \( X \).

### 3. Uniformity in subtraction algebras

Let \( X \) be a non-empty set, and \( U \) and \( V \) any subsets of \( X \times X \). Define
\[ U \circ V := \{(x, y) \in X \times X \mid \text{for some } z \in X, (x, z) \in U \text{ and } (z, y) \in V \} \]
\[ U^{-1} := \{(x, y) \in X \times X \mid (y, x) \in U \} \]
\[ \Delta := \{(x, x) \mid x \in X \}. \]
From now on, \( X \) is a subtraction algebra, unless we specifically state otherwise.

**Definition 3.1.** By a *uniformity* on \( X \), we mean a non-empty collection \( \mathcal{K} \) of subsets of \( X \times X \) which satisfies the following conditions:

\( (U_1) \Delta \subseteq U \) for any \( U \in \mathcal{K} \),
(U_2) if \( U \in \mathcal{K} \), then \( U^{-1} \in \mathcal{K} \),
(U_3) if \( U \in \mathcal{K} \), then there exists \( V \in \mathcal{K} \) such that \( V \circ V \subseteq U \),
(U_4) if \( U, V \in \mathcal{K} \), then \( U \cap V \in \mathcal{K} \),
(U_5) if \( U \in \mathcal{K} \) and \( U \subseteq V \subseteq X \times X \), then \( V \in \mathcal{K} \).

**Theorem 3.2.** Let \( A \) be an ideal of a subtraction algebra \( X \). If we define
\[
U_A := \{(x, y) \in X \times X | x - y \in A \text{ and } y - x \in A\}
\]
and let
\[
\mathcal{K}^* := \{U_A | A \text{ is an ideal of } X\}.
\]
Then \( \mathcal{K}^* \) satisfies the conditions (U_1) \( \sim \) (U_4).

**Proof.** (U_1): If \( (x, x) \in \Delta \), then \( (x, x) \in U_A \) since \( x - x = 0 \in A \). Hence \( \Delta \subseteq U_A \) for any \( U_A \in \mathcal{K}^* \).
(U_2): For any \( U_A \in \mathcal{K}^* \),
\[
(x, y) \in U_A \iff x - y \in A \text{ and } y - x \in A
\]
\[
\iff y \sim_A x
\]
\[
\iff (y, x) \in U_A
\]
\[
\iff (x, y) \in U_A^{-1}.
\]
Hence \( U_A^{-1} = U_A \in \mathcal{K}^* \).
(U_3): For any \( U_A \in \mathcal{K}^* \), the transitivity of \( \sim_A \) implies that \( U_A \circ U_A \subseteq U_A \).
(U_4): For any \( U_I \) and \( U_J \) in \( \mathcal{K}^* \), we claim that \( U_I \cap U_J \in \mathcal{K}^* \).
\[
(x, y) \in U_I \cap U_J \iff (x, y) \in U_I \text{ and } (x, y) \in U_J
\]
\[
\iff x - y, y - x \in I \cap J
\]
\[
\iff x \sim_{I \cap J} y
\]
\[
\iff (x, y) \in U_{I \cap J}.
\]
Since \( I \cap J \) is an ideal of \( X \), \( U_I \cap U_J = U_{I \cap J} \in \mathcal{K}^* \). This proves the theorem. \( \square \)

**Theorem 3.3.** Let \( \mathcal{K} := \{U \subseteq X \times X | U_A \subseteq U \text{ for some } U_A \in \mathcal{K}^*\} \).
Then \( \mathcal{K} \) satisfies the conditions for a uniformity on \( X \) and hence the pair \((X, \mathcal{K})\) is a uniform structure.

**Proof.** By Theorem 3.2, the collection \( \mathcal{K} \) satisfies the conditions (U_1) \( \sim \) (U_4). It suffices to show that \( \mathcal{K} \) satisfies (U_5). Let \( U \in \mathcal{K} \) and \( U \subseteq V \subseteq X \times X \). Then there exists a subset \( U_A \) such that \( U_A \subseteq U \subseteq V \), which means that \( V \in \mathcal{K} \). This proves the theorem. \( \square \)
Let $x \in X$ and $U \in \mathcal{K}$. Define
\[ U[x] := \{ y \in X | (x, y) \in U \}. \]

**Theorem 3.4.** Let $X$ be a subtraction algebra. Then
\[ T := \{ G \subseteq X \mid \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G \} \]
is a topology on $X$.

**Proof.** It is clear that $\emptyset$ and the set $X$ belong to $T$. Also from the definition, it is clear that $T$ is closed under arbitrary union. Finally to show that $T$ is closed under finite intersection, let $G, H \in T$ and suppose $x \in G \cap H$. Then there exist $U$ and $V \in \mathcal{K}$ such that $U[x] \subseteq G$ and $V[x] \subseteq H$. Let $W = U \cap V$. Then $W \in \mathcal{K}$. Also $W[x] \subseteq U[x] \cap V[x]$ and so $W[x] \subseteq G \cap H$. Therefore $G \cap H \in T$. Thus $T$ is a topology on $X$.

Note that for any $x \in X$, $U[x]$ is an open neighborhood of $x$.

**Definition 3.5.** Let $(X, \mathcal{K})$ be a uniform structure. Then the topology $T$ is called the **uniform topology** on $X$ induced by $\mathcal{K}$.

**Proposition 3.6.** Topological space $(X, T)$ is completely regular.

**Proof.** See [9].

**Example 3.7.** Let $X := \{0, a, b, c\}$ be a subtraction algebra ([2]) with the Cayley table as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
<td>0</td>
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<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>a</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $X$, $\{0\}$, $A := \{0, a\}$, $B := \{0, b\}$ are the only ideals of $X$. And then $U_{\{0\}} = \Delta$, $U_A = \Delta \cup \{(a, 0), (0, a), (c, b), (b, c)\}$, $U_B = \Delta \cup \{(b, 0), (0, b), (a, c), (c, a)\}$, and $U_X = X \times X$. Hence $K^* = \{U_{\{0\}}, U_A, U_B, U_X\}$ and $K = \{U \subseteq X \times X | U_A \subseteq U \text{ for some } U_A \in K^*\}$. If we take $U := U_A$, then $U[0] = U[a] = \{0, a\}$ and $U[b] = U[c] = \{b, c\}$. Hence $T = \{G \subseteq X \mid \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G \} \supseteq \{X, \emptyset, \{0, a\}, \{b, c\}\}$. If we take $U := U_B$, then $U[0] = U[b] = \{0, b\}$ and $U[a] = U[c] = \{a, c\}$. Hence $T = \{G \subseteq X \mid \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G \} \supseteq \{X, \emptyset, \{a, c\}, \{0, b\}\}$. If we take $\{0\}$ as an ideal of $X$, then $U_{\{0\}} = \Delta$. If we take $U := U_{\{0\}}$,
$U[x] = \{x\}$ for all $x \in X$, and we have $T = 2^X$, the discrete topology. If we take $X$ as an ideal of $X$, then $U[x] = X$ for all $x \in X$, and we have $T = \{\emptyset, X\}$, the indiscrete topology.

4. Topological properties of the subtraction algebra

Let $X$ be a subtraction algebra and let $C, D$ be subsets of $X$. Then we define $C - D$ as follows:

$$C - D := \{x - y | x \in C, y \in D\}.$$ 

Let $X$ be a subtraction algebra, and let $T$ be a topology on the set $X$. We say that the pair $(X, T)$ is a topological subtraction algebra if the operation "−" is continuous with respect to $T$. The continuity of the operation "−" is equivalent to having the following property satisfied:

(C): Let $O$ be an open set and $a, b \in X$ such that $a - b \in O$. Then there exist open sets $O_1$ and $O_2$ such that $a \in O_1$, $b \in O_2$ and $O_1 - O_2 \subseteq O$.

Theorem 4.1. Let $X$ be a subtraction algebra. Then the pair $(X, T)$ is a topological subtraction algebra, where $T := \{G \subseteq X | \forall x \in G, \exists U \in K, U[x] \subseteq G\}$ as Theorem 3.4.

Proof. By Theorem 3.4, $T$ is a topology on $X$. Enough to show (C). Assume that $x - y \in G$, with $x, y \in X$ and $G$ is an open subset of $X$. Then there exist $U \in K, U[x - y] \subseteq G$ and an ideal $I$ of $X$ such that $U_I \subseteq U$. We claim that the following relation holds:

$$U_I[x] - U_I[y] \subseteq U[x - y].$$

Indeed, for any $h \in U_I[x]$ and $k \in U_I[y]$, we have that $x \sim_I h$ and $y \sim_I k$. Since $\sim_I$ is a congruence relation, it follows that $x - y \sim_I h - k$. From that fact we note $(x - y, h - k) \in U_I \subseteq U$. Hence $h - k \in U_I[x - y] \subseteq U[x - y]$. Then $h - k \in G$. Thus condition (C) is verified. \qed

Theorem 4.2. ([9]) Let $X$ be a set and $B \subseteq \mathcal{P}(X \times X)$ be a family such that for every $U \in B$ the following conditions hold:

(a) $\Delta \subseteq U$,
(b) $U^{-1}$ contains a member of $B$, and
(c) there exists $V \in B$ such that $V \circ V \subseteq U$.

Then there exists a unique uniformity $\mathcal{U}$ for which $B$ is a subbase.

Proof. It's clear from Definition 3.1 and [9]. \qed
**Theorem 4.3.** If we set $B := \{U_I[y] | I$ is an ideal of a subtraction algebra $X\}$, then $B$ is a subbase for a uniformity of $X$.

*Proof.* Since $\sim_I$ is an equivalence relation, it is clear that $B$ satisfies the axiom of Theorem 4.2.  

In Theorem 4.3, we denote the associated topology by $S$. And we say that the topology $\sigma$ is finer than $\tau$ if $\tau \subseteq \sigma$ as subsets of the power set. Then we have:

**Corollary 4.4.** The topology $S$ is finer than $T$.

In Example 3.7, since $\{X, \emptyset, \{0,a\}, \{b,c\}\}$ is a topology on $X$, the topology $T$ induced by the ideal $A := \{0,a\}$ relative to $U_A$ is a finer topology than $\{X, \emptyset, \{0,a\}, \{b,c\}\}$. And since $\{X, \emptyset, \{a,c\}, \{0,b\}\}$ is a topology on $X$, the topology $T$ induced by the ideal $B := \{0,b\}$ relative to $U_B$ is a finer topology than $\{X, \emptyset, \{a,c\}, \{0,b\}\}$.

**Theorem 4.5.** Let $\Lambda$ be an arbitrary family of ideals of a subtraction algebra $X$ which is closed under intersection. Then any ideal is a clopen subset of $X$.

*Proof.* Let $I$ be an ideal of $X$ in $\Lambda$ and $y \in I^c$. Then $y \in U_I[y]$ and hence $I^c \subseteq \bigcup\{U_I[y] | y \in I^c\}$. We claim that $U_I[y] \subseteq I^c$ for all $y \in I^c$. Let $z \in U_I[y]$, then $y \sim_I z$. Hence $y - z \in I$. If $z \in I$, since $I$ is an ideal of $X$, $y \in I$ which is a contradiction. So $z \in I^c$ and we get

$$\bigcup\{U_I[y] | y \in I^c\} \subseteq I^c.$$ 

Hence $I^c = \bigcup\{U_I[y] | y \in I^c\}$. Since $U_I[y]$ is open for all $y \in X$, $I$ is a closed subset. We show that $I = \bigcup\{U_I[y] | y \in I\}$. If $y \in I$ then $y \in U_I[y]$ and we get $I \subseteq \bigcup\{U_I[y] | y \in I\}$. Let $y \in I$, if $z \in U_I[y]$ then $y \sim_I z$ and so $z - y \in I$. Since $y \in I$ and $I$ is an ideal of $X$, we have $z \in I$ and we get that $\bigcup\{U_I[y] | y \in I\} \subseteq I$. So $I$ is also an open subset of $X$.  

**Theorem 4.6.** For any $x \in X$ and $I \in \Lambda$, $U_I[x]$ is a clopen subset of a subtraction algebra $X$.

*Proof.* We show that $(U_I[x])^c$ is open. If $y \in (U_I[x])^c$, then $x - y \in I^c$ or $y - x \in I^c$. Without loss of generality let $y - x \in I^c$. Hence by Theorems 4.1 and 4.5, $(U_I[y] - U_I[x]) \subseteq U_I[y - x] \subseteq I^c$. We claim that $U_I[y] \subseteq (U_I[x])^c$. If $z \in U_I[y]$, then $z - x \in (U_I[y] - U_I[x])$. Hence $z - x \in I^c$ so that $z \in (U_I[x])^c$, proving that $(U_I[x])^c$ is open, i.e., $U_I[x]$ is closed. It is clear that $U_I[x]$ is open. Thus $U_I[x]$ is a clopen subset of $X$.  

$\Box$
A topological space \( X \) is connected if and only if it has only \( X \) and \( \emptyset \) as clopen subsets. Therefore we have:

**Corollary 4.7.** The space \( (X, T) \) is not a connected space.

We denote the uniform topology for an arbitrary family \( \Lambda \) by \( T_\Lambda \) and, in particular, if \( \Lambda = \{I\} \), we denote it by \( T_I \).

**Theorem 4.8.** \( T_\Lambda = T_J \), where \( J = \cap \{I | I \in \Lambda\} \).

*Proof.* Let \( K \) and \( K^* \) be as Theorems 3.2 and 3.3, respectively. Now consider \( \Lambda_0 = \{J\} \). Define \( (K_0)^* = \{U_J\} \) and \( K_0 = \{U \mid U_J \subseteq U\} \). Let \( G \in T_\Lambda \). Given an \( x \in G \), there exists \( U \in K \) such that \( U[x] \subseteq G \). From \( J \subseteq I \), we obtain that \( U_J \subseteq U_I \), for all ideals \( I \) of \( X \). Since \( U \in K \), there exists \( I \in \Lambda \) such that \( U_I \subseteq U \). Hence \( U_J[x] \subseteq U_J[x] \subseteq G \). Since \( U_J \in K_0 \), \( G \in T_J \). So \( T_\Lambda \subseteq T_J \). Conversely, if \( H \in T_J \), then for all \( x \in H \), there exists \( U \in K_0 \) such that \( U[x] \subseteq H \). So \( U_J[x] \subseteq H \) and hence \( \Lambda \) is closed under the intersection, \( J \in \Lambda \). Then we obtain \( U_J \in K \) and so \( H \in T_\Lambda \). Thus \( T_J \subseteq T_\Lambda \). \( \square \)

**Theorem 4.9.** Let \( I \) and \( J \) be ideals of a subtraction algebra \( X \) and \( I \subseteq J \). Then \( J \) is clopen in the topological space \( (X, T_I) \).

*Proof.* Consider \( \Lambda = \{I, J\} \). Then by Theorem 4.8, \( T_\Lambda = T_I \) and thus \( J \) is clopen in the topological space \( (X, T_I) \). \( \square \)

**Theorem 4.10.** Let \( I \) and \( J \) be ideals of a subtraction algebra \( X \). Then \( T_I \subseteq T_J \) if \( J \subseteq I \).

*Proof.* Let \( J \subseteq I \). Consider \( \Lambda_1 = \{J\} \), \( K_1^* = \{U_I\} \), \( K_1 = \{U \mid U_I \subseteq U\} \) and \( \Lambda_2 = \{J\} \), \( K_2^* = \{U_J\} \), \( K_2 = \{U \mid U_J \subseteq U\} \). Let \( G \in T_I \). Then for all \( x \in G \), there exists \( U \in K_1 \) such that \( U[x] \subseteq G \). Since \( J \subseteq I \), \( U_J \subseteq U_I \), so we obtain \( U_J[x] \subseteq U_I[x] \subseteq G \). Then \( G \in T_J \) because of \( U_J \in K_2 \). Thus \( T_I \subseteq T_J \). \( \square \)

Recall that a uniform space \( (X, K) \) is said to be totally bounded if for each \( U \in K \), there exist \( x_1, \cdots, x_n \in X \) such that \( X = \cup_{i=1}^n U[x_i] \), and \( X \) is said to be compact if any open cover of \( X \) has a finite subcover.

**Theorem 4.11.** Let \( I \) be an ideal of a subtraction algebra \( X \). Then the following conditions are equivalent:

1. the topological space \( (X, T_I) \) is compact,
2. the topological space \( (X, T_I) \) is totally bounded,
(3) there exists \( P = \{ x_1, \cdots, x_n \} \subseteq X \) such that for all \( a \in X \) there exist \( x_i \in P(i = 1, \cdots, n) \) with \( a - x_i \in I \) and \( x_i - a \in I \).

Proof. (1)\(\Rightarrow\)(2): It is clear by [9].

(2)\(\Rightarrow\)(3): Let \( U_I \in K \). Since \( (X, T_I) \) is totally bounded, there exist \( x_1, \cdots, x_n \) in \( X \) such that \( X = \bigcup_{i=1}^{n} U_I[x_i] \). If \( a \in X \), then there exists \( x_i \) such that \( a \in U_I[x_i] \), therefore \( a - x_i \in I \) and \( x_i - a \in I \).

(3)\(\Rightarrow\)(1): For any \( a \in X \), by hypothesis, there exists \( x_i \in P \) with \( a - x_i \in I \) and \( x_i - a \in I \). Hence \( a \in U_I[x_i] \). Thus \( X = \bigcup_{i=1}^{n} U_I[x_i] \). Now let \( X = \bigcup_{\alpha \in \Omega} O_\alpha \), where each \( O_\alpha \) is an open set of \( X \). Then for any \( x_i \in X \), there exists \( \alpha_i \in \Omega \) such that \( x_i \in O_{\alpha_i} \). Since \( O_{\alpha_i} \) is an open set, \( U_I[x_i] \subseteq O_{\alpha_i} \), so \( X = \bigcup_{i=1}^{n} U_I[x_i] \subseteq \bigcup_{\alpha \in \Omega} O_{\alpha_i} \), i.e., \( X = \bigcup_{\alpha \in \Omega} O_{\alpha_i} \), which means that \( (X, T_I) \) is compact. \(\square\)

**Theorem 4.12.** If \( I \) is an ideal of a subtraction algebra \( X \), then \( U_I[x] \) is a compact set in the topological space \( (X, T_I) \), for all \( x \in X \).

**Proof.** Let \( U_I[x] \subseteq \bigcup_{\alpha \in \Omega} O_\alpha \), where each \( O_\alpha \) is an open set of \( X \). Since \( x \in U_I[x] \), there exists \( \alpha \in \Omega \) such that \( x \in O_{\alpha} \). Then \( U_I[x] \subseteq O_{\alpha} \). Hence \( U_I[x] \) is compact. \(\square\)

**Definition 4.13.** A topological subtraction algebra \( X \) is said to be **discrete** if every element admits a neighborhood consisting of that element only.

**Proposition 4.14.** If \( \{0\} \) is an open set in a topological subtraction algebra \( X \), then \( X \) is discrete.

**Proof.** Since \( x - x = 0 \in \{0\} \) for all \( x \in X \) and \( \{0\} \) is open, there exist neighborhoods \( U \) and \( V \) of \( x \) such that \( U - V = \{0\} \). Let \( W := U \cap V \). Then \( W - W \subseteq U - V = \{0\} \) and so \( W - W = \{0\} \). It follows from \( x \in W \) that \( W = \{x\} \), which means that \( X \) is discrete. \(\square\)

**Proposition 4.15.** Let \( X \) be a topological subtraction algebra. Then \( \{0\} \) is closed in \( X \) if and only if \( X \) is Hausdorff.

**Proof.** Assume that \( \{0\} \) is closed and let \( x, y \in X \) with \( x \neq y \). Then \( x - y \neq 0 \) or \( y - x \neq 0 \). We may assume that \( x - y \neq 0 \) without loss of generality. Then there exist neighborhoods \( U \) and \( V \) of \( x \) and \( y \) respectively such that \( U - V \subseteq X \setminus \{0\} \). It follows that \( U \cap V = \emptyset \) and hence \( X \) is Hausdorff. Conversely, let \( X \) be Hausdorff. We claim that \( X \setminus \{0\} \) is open. If \( x \in X \setminus \{0\} \), then \( x \neq 0 \) and so there exist disjoint
neighborhoods $U$ and $V$ of $x$ and 0, respectively. Therefore $0 \notin U$ and hence $U \subseteq X \setminus \{0\}$, which implies that $X \setminus \{0\}$ is open. This completes the proof. 

**Proposition 4.16.** Let $A$ be an ideal of a topological subtraction algebra $X$. If 0 is an interior point of $A$, then $A$ is open.

**Proof.** Let $x \in A$. Since $x - x = 0 \in A$ and 0 is an interior point of $A$, there exists a neighborhood $U$ of 0 which is contained in $A$. Then there exist neighborhoods $G$ and $H$ of $x$ such that $G - H \subseteq U \subseteq A$. On the other hand, for any $y \in G$, $y - x \in G - H \subseteq A$. Since $A$ is an ideal and $x \in A$, it follows that $y \in A$, so that $x \in G \subseteq A$. Hence $A$ is open. 

**Proposition 4.17.** Let $X$ be a topological subtraction algebra. If $A$ is an open set in $X$ which is an ideal of $X$, then it is a closed set in $X$.

**Proof.** Let $A$ be an ideal which is an open set in $X$ and let $x \in X \setminus A$. Then there exists a neighborhood $U$ of $x$ such that $U - U \subseteq A$ because of $x - x = 0 \in A$ and $A$ is open. We claim that $U \subseteq X \setminus A$. If $U \not\subseteq X \setminus A$, then there exists $y \in U \cap A$. Note that $z - y \in U - U \subseteq A$ for all $z \in U$. Since $y \in A$ and $A$ is an ideal, it follows that $z \in A$, which shows that $U \subseteq A$. This is a contradiction and we are done.

**Proposition 4.18.** Let $X$ be a topological subtraction algebra and $\{0\}$ be closed. Then $\cap \mathcal{N}_0 = \{0\}$, where $\mathcal{N}_0$ is a neighborhood system of 0.

**Proof.** Since $\{0\}$ is closed, $X$ is Hausdorff by Proposition 4.15. Then 0 has a neighborhood $U$ such that $x \notin U$ for any $x \neq 0$, and so $x \notin \cap \mathcal{N}_0$. Hence $\cap \mathcal{N}_0 = \{0\}$. 

Let $X$ be a subtraction algebra. For an arbitrary element $a \in X$ and any non-empty subset $V$ of $X$, denote

$$V(a) := \{x \in X | x - a \in V \text{ and } a - x \in V\}.$$ 

Note that $V(a) \subseteq U(a)$ whenever $V \subseteq U \subseteq X$. In a uniform space $(X, \mathcal{T})$, a **filter base** is a family $\Omega = \{V_\alpha | \alpha \in \Lambda\}$ of non-empty subsets of $X$ which satisfies $\forall \alpha, \beta \in \Lambda, \exists \gamma \in \Lambda$ such that $V_\gamma \subseteq V_\alpha \cap V_\beta$.

**Theorem 4.19.** Let $\Omega$ be a filter base on a subtraction algebra $X$ such that for every $p, q \in V$, and $V \in \Omega$
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(i) $0 - p \in V$,
(ii) $(x - p) - q = 0$ implies $x \in V$.

Then $\mathcal{T} := \{ O \subseteq X \mid \forall a \in O \, \exists V \in \Omega : V(a) \subseteq O \}$ is a topology on $X$ and $\Omega$ is a local base at $0$.

Proof. Let $\mathcal{T} := \{ O \subseteq X \mid \forall a \in O \, \exists V \in \Omega : V(a) \subseteq O \}$. Clearly $\emptyset, X \in \mathcal{T}$. Let $\{ O_\alpha \}$ be a family of members of $\mathcal{T}$ and let $a \in \cup O_\alpha$. Then $a \in O_\alpha$ for some $\alpha$. It follows that there exists $V \in \Omega$ such that $V(a) \subseteq O_\alpha \subseteq \cup O_\alpha$, so that $\cup O_\alpha \in \mathcal{T}$. Assume that $O_\alpha$ and $O_\beta$ belong to $\mathcal{T}$ and let $a \in O_\alpha \cap O_\beta$. Then there exist $V_\alpha \in \Omega$ and $V_\beta \in \Omega$ such that $V_\alpha(a) \subseteq O_\alpha$ and $V_\beta(a) \subseteq O_\beta$, respectively. Since $\Omega$ is a filter base, there exists $V \in \Omega$ such that $V \subseteq V_\alpha \cap V_\beta$. Thus we have

$V(a) \subseteq (V_\alpha \cap V_\beta)(a) \subseteq V_\alpha(a) \cap V_\beta(a) \subseteq O_\alpha \cap O_\beta$,

and so $O_\alpha \cap O_\beta \in \mathcal{T}$. This proves that $\mathcal{T}$ is a topology on $X$. (In this case, we call this the topology induced by $\Omega$, and is denoted by $\mathcal{T}_\Omega$).

Now we will show that $\Omega$ is the filter base of a neighborhood of 0 with respect to the topology $\mathcal{T}$. Let $p \in V$ and $V \in \Omega$. Then $0 - p \in V$ by (i). And it follows from (ii) that $0 \in V$ because $(0 - p) - (0 - p) = 0$, i.e., every element $V \in \Omega$ contains 0. If $x \in V(p)$, then $x - p \in V$ and $p - x \in V$. And so $x - p = v$ for some $v \in V$. Hence $(x - p) - v = 0$, which implies that $x \in V$. Therefore $V(p) \subseteq V$ and $V \in \mathcal{T}$. Thus $V$ is a neighborhood of 0. Now we can let $V$ be a neighborhood of 0. Then there is a $U \in \Omega$ such that $U(0) \subseteq V$. Note that $0 - a \in U$ and $a - 0 \in U$ for some $a \in U$; so $a \in U(0)$ and $0 \in U \subseteq U(0) \subseteq V$. Hence $\Omega$ is a local base at 0 with respect to the topology $\mathcal{T}$. \qed

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