THE REPRESENTABILITY OF MODULAR FORMS BY A CERTAIN THETA SERIES AND ITS APPLICATIONS

SUNGTAE JUN AND INSUK KIM

Abstract. The primitive orders in a quaternion algebra play a central role of the theory of Hecke operators. In this paper, we study theta series generated by Brandt matrices and its applications to almost Ramanujan graphs.

1. Introduction

An order $M$ of a quaternion algebra $A$ over a local field $k$ is called primitive if it satisfies one of following conditions. If $A$ is a division algebra, $M$ contains the full ring of integers of a quadratic extension field of $k$. If $A$ is isomorphic to $\text{Mat}_{2\times 2}(k)$, then $M$ contains a subring which is isomorphic either to $\mathcal{O} \oplus \mathcal{O}$ where $\mathcal{O}$ is the ring of integers in $k$ or to the full ring of integers in a quadratic extension field of $k$. A special kind of primitive orders, called special orders, were studied in [10]. Generally, primitive orders in quaternion algebra were studied in [3] and in [14] using different methods.

In [15], the representability of modular forms by theta series was studied on the primitive orders over a nondyadic local field. The modular forms studied in [15] were defined without characters unlike the special order case. In this paper, to complete [15] results, we will study the dyadic local field case.

Brandt matrices we construct will give the representations of the Hecke operators on a certain space generated by theta series. From this, we obtain several results on the representability of modular forms by theta series. Finally, as an application of Brandt matrices, we found some “almost Ramanujan graphs”.

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2. Orders and Brandt Matrices

Let $A$ be a rational quaternion algebra ramified precisely at an odd prime $q$ and $\infty$. That is, $A_q = A \otimes k_q$ and $A_\infty = A \otimes \mathbb{R}$ are division algebras where $k_q$ is a local field. On the other hand, $A_p = A \otimes k_p$ is isomorphic to $M_{2 \times 2}(k_p)$ for a finite prime $p \neq q$.

Let $L$ be a quadratic extension field of $k_p$. It is well known that

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \beta^* & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in L \right\}$$

is a quaternion algebra over $k_p$ which is isomorphic to $M_{2 \times 2}(k_p)$. Let $A_p = L + \xi L = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta^* & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in L \right\}$ where

$$\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\alpha \in L$ is identical with $\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}$ in $A_p$. Hence, we can define the norm and the trace of an element in $A$ as its determinant and trace.

Let $P_L$ be the prime ideal of $\mathcal{O}_L$, the ring of integers in $L$. In [15], we have proved that the possibilities of an order, $\Lambda$, of $A_p$ containing $\mathcal{O}_L$ are as follows:

$$\Lambda = \begin{cases} \Lambda_\nu(L) = \mathcal{O}_L + \xi P_L^{\nu} & \text{if } L \text{ is unramified, or} \\ \Lambda_\nu(L) = \mathcal{O}_L + (1 + \xi) P_L^{\nu-t} & \text{if } L \text{ is ramified,} \end{cases}$$

for some nonnegative integer $\nu$.

**Remark.** In the unramified case, $\nu$ is always even number.

**Definition 1.** Let $A$ be a quaternion algebra ramified precisely at one finite prime $q$ and $\infty$. An order $M$ has level $2^{2\nu}q$ if

(i) $M_p$ is the maximal order of $A_p$ for an odd prime $p$,

(ii) there exists a quadratic unramified extension field $L(2)$ of $k_2$ such that $M_2 = R_{2^{2\nu}(2)}(L(2))$.

For the notational convenience, we assume that $N = 2^{2\nu}q$. Let $M$ be an order of level $N$ in $A$. A left $M$ ideal $I$ is a lattice on $A$ such that $I_p = M_p a_p$ (for some $a_p \in A_p^\times$) for all $p < \infty$. Two left $M$ ideals $I$ and $J$ are said to belong to the class if $I = J a$ for some $a \in A^\times$. Analogously, right $M$ ideals can be defined. The class number of left ideals for any order $M$ of level $N$ is the number of distinct classes of such ideals as usual sense.

The norm of an ideal, denoted by $N(I)$, is the positive rational number which generates the fractional ideal of $k$ generated by $\{N(a)\mid a \in I\}$. The conjugate of an ideal $I$, denoted by $\overline{I}$, is given by $I = \{\overline{a} \mid a \in I\}$. The inverse of an ideal, denoted by $I^{-1}$, is given by $I^{-1} = \{a \in A \mid IaI \subset I\}$. 
Proposition 2.1. Let $M$ be an order of level $2^{2^\nu}q$ in $A$. Let $I_1, I_2, I_3, \ldots, I_H$ be the complete set of representatives of all the distinct left $M$ ideal classes. Let $M_j$ be the right order of $I_j$, $j = 1, 2, \ldots, H$. Then $I_j^{-1}I_1, I_j^{-1}I_2, \ldots, I_j^{-1}I_H$ is a complete set of representatives of all the distinct left $M$ ideal classes (for $j = 1, 2, \ldots, H$).

Proof. See Proposition 2.13 and 2.15 of [16].

There are several literatures dealt with the definition of Brandt matrices with an order of quaternion algebra. Here, we define the generalized Brandt matrices $B(n) = B(n;N)$ associated with an order $M$, in the exactly same manner as Eichler's (See [7]), equation 15 and 15a on the page 105). Here $n$ is a nonnegative integer. For the details, see [7], [16] or [18]. Let $M$ be an order of $A$ with level $N$ and let $I_1, I_2, \cdots, I_H$ be a representatives of (left) ideal classes of $M$ with $H$, the class number. Then let $e_j$ be the number of unit elements in $M_j$, the right order of $I_j$. The entries of Brandt matrices are defined as $b_{ij}(n) = \frac{1}{e_j}$ the number of elements in $I_j^{-1}I_i$ with norm $nN(I_i)/N(I_j)$ for $n \geq 1$. If $n = 0$, $b_{ij}(0) = \frac{1}{e_j}$. Then $B(n,N) = (b_{ij}(n))$ is called a Brandt matrix, which is a $H \times H$ matrix.

Proposition 2.2. The entries of the matrix series

$$\Theta(\tau,N) = \left(\theta_{ij}(\tau)\right) = \sum_{n=0}^{\infty} B(n;N) \exp(2\pi in\tau)$$

are modular forms of weight 2 on $\Gamma_0(N)$.

Proof. See Theorem 6.16 [16].

Lemma 2.3. Let $M$ be an order of level $N$ and let $I, J$ be left $M$ ideals. Let $\theta_I = \sum_{\alpha} \exp(\tau N(\alpha)/N(I))$ be the theta series attached to $I$ and similarly for $J$. Then $\theta_I - \theta_J$ is a cusp form of weight 2 on $\Gamma_0(N)$.

Proof. For a finite prime $l$, let $I_l = M_l a$ and $J_l = M_l b$ for some $a, b \in A_l^\times$. There exists $u \in \mathbb{Z}_l$ such that

$$u N(a)/N(I) = N(b)/N(J).$$

Since $M_l$ is an order containing the ring of integers of unramified quadratic extension field of $k_l$, there exists a unit $v \in M_l$ such that $N(v) = u$ (see [17] page 188). So quadratic forms $N(x)/N(I)$ for $x \in I$ and $N(x)/N(J)$ for $x \in J$ are locally equivalent for a finite prime $l$. If $l = \infty$, then it is clear. Thus $\theta_I$ and $\theta_J$ have same genus. It is classical result that the difference, $\theta_I - \theta_J$ is a cusp form (See [16] page 376).
Lemma 2.4. Let $B(n; N) = (b_{ij}(n))$ where $1 \leq i \leq H$ and $1 \leq j \leq H$. Then we have
(a) $e_j b_{ij}(n) = e_i b_{ij}(n)$ for all $i, j, 1 \leq i, j \leq H$ and all $n \geq 0$,
(b) $\sum_{j=1}^{H} b_{ij}(n) = b(n)$ are independent of $i$.

Proof. See the proof of Lemma 2.18 in [16].

Lemma 2.5. Let the notation be as above.
Let $B(n; N) = B(n, L(p), 2\nu) = (b_{ij}(n))$.

Consider the matrix $A = \begin{pmatrix} 1 & e_1 e_2^{-1} & \cdots & e_1 e_H^{-1} \\ 1 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & -1 \end{pmatrix}$, that is, $A = (a_{ij})$ where $a_{i1} = 1$ for $i = 1, \ldots, H$; $a_{1j} = e_1 e_j^{-1}$ for $j = 1, \ldots, H$, $a_{ii} = -1$ for $i = 2, \ldots, H$ and all other $a_{ij} = 0$ if $i \neq j, j \neq 1, i \neq j$. Then $AB(n)A^{-1} = B'(n)$ for all $n \geq 0$ where $B'(n) = (b_{ij}'(n))$ and $b_{11}'(n) = b(n) = \sum_{j=1}^{H} b_{ij}(n)$ (independent of $i$ by Lemma 2.4); $b_{ii}'(n) = b_{ii}'(n) = 0$ for $i = 2, \ldots, H$ and $b_{ij}'(n) = b_{ij}(n) - b_{ij}(n)$ for $2 \leq i, j \leq H$.

Proof. See the proof of Lemma 2.19 in [16].

We are now able to consider the case of cusp forms of weight 2.

(2.1) $AB(n; N')A^{-1} = \begin{pmatrix} b(n) \\ 0 \\ 0 \end{pmatrix}$

where $B'(n; N)$ is a $H - 1 \times H - 1$ matrix. From now we will denote $B'(n; N)$ for the $H - 1 \times H - 1$ matrix appearing in $AB(n; N')A^{-1}$.

Theorem 2.6. Let $B'(n; N)$ be as above. Then the entries of modified Brandt matrices

$$\Theta'(\tau, N) = \sum_{n=0}^{\infty} B'(n; N) \exp(2\pi i n \tau)$$

are cusp forms of weight 2 on $\Gamma_0(N)$ where $N = qp^{2\nu}$.

Proof. Each entry of $\Theta'(\tau, N)$ is $\sum_{n=0}^{\infty} (b_{ij}(n) - b_{ij}(n))e^{2\pi i n \tau}$. By Lemma 2.3, this is a cusp form of weight 2 on $\Gamma_0(N)$.

Proposition 2.7. Fix $q$. Then the $B(n; N)$ with $(n, 2q) = 1$ generate a commutative semi simple ring. Similarly, the $B'(n; N)$ with $(n, 2q) = 1$ generate a commutative semi simple ring.

Proof. By the Theorem 2 on page 106 of Eichler[7], $B(n; N')$ with $(n, 2q) = 1$ generates a commutative ring.
By Proposition 2.7, there exists a $H \times H$ matrix $E$ such that $EB(n; N)E^{-1}$ is simultaneously diagonal matrix for all $n$ with $(n, 2q) = 1$. Similarly, there exists a $H - 1 \times H - 1$ matrix $E'$ such that $E'B'(n, N)E'^{-1}$ is simultaneously diagonal matrix for all $n$ with $(n, 2q) = 1$.

3. Trace Formula

As we mention at the introduction, we will compare the trace of Brandt matrix $B(n; N)$ with $(n, 2q) = 1$ with the trace of Hecke operator $T(n)$ acting on the space of cusp forms $S_2(2^{2q}q)$. For the convenience, let $tr_NT_2(n)$ be the trace of Hecke operator. Hijikata has computed $tr_NT_2(n)$, which is given by [9].

**Theorem 3.1.**

$$tr_NT_2(n) = -\sum_s a_k(s) \sum_f b(s, f) \prod_{p|N} c'(s, f, p)$$

$$+ \delta(\sqrt{n}) k - 1 \cdot 12 \cdot N \cdot \prod_{p|N} (1 + \frac{1}{p}) + \delta(k) \text{deg}T_2(n)$$

where the notation is exactly the same as in [9].

**Remark.** Let $s$ run over all integers such that $s^2 - 4n$ has one of followings.

$$s^2 - 4n = \begin{cases} 
0 \\
\frac{t^2}{4} & 0 < m \equiv 1 \mod 4 \\
\frac{t^2}{4}m & 0 < m \equiv 2, 3 \mod 4
\end{cases}$$

Let $\phi(X) = X^2 - sX + n$ and let $x, y$ be roots of $\phi(X)$ in $C$. Then

$$a(s) = \begin{cases} |x| \cdot (4N)^{-1} & \text{if } s^2 - 4n = 0 \\
\min(|x|, |y|)/(|x - y|) & \text{if } s^2 - 4n = t^2 \\
\frac{1}{2} & \text{otherwise}
\end{cases}$$

For each fixed $s$, $a(s)$ is corresponding to its classification.

Let $f$ run over the following set.

$$f = \begin{cases} 
1, 2, \cdots, N & \text{if } s^2 - 4n = 0 \\
\text{all positive divisors of } t & \text{otherwise}
\end{cases}$$
and

\[ b(s, f) = \begin{cases} 
1 & \text{if } s^2 - 4n = 0 \\
\frac{1}{2} \varphi(\sqrt{s^2 - 4n}/f) & \text{if } s^2 - 4n = t^2 \\
h((s^2 - 4n)/f^2)/\omega((s^2 - 4n)/f^2) & \text{otherwise}
\end{cases} \]

where \( \varphi \) is Euler \( \varphi \)-function, \( h(d) \) (resp. \( \omega(d) \)) denotes the class number of locally principal ideals (resp. \( \frac{1}{2} \) the cardinality of the unit group) of the order \( Q(\sqrt{d}) \) with discriminant \( d \).

Proof. See [9]. \( \square \)

**Theorem 3.2.** The trace of the Brandt matrix is given by

\[ \text{tr}B(n; 2^{2\nu}q) = \sum_s \sum_f b(s, f)c(s, f, 2)c(s, f, q) + \delta(\sqrt{n}) \frac{1}{12}(q - 1)2^{2\nu - 1} \]

where \( \nu \geq 1 \).

**Remark.** The meaning of \( s, f, b(s, f) \) and \( c(s, f, l) \) are as follows. Let \( s \) run over all integers such that \( s^2 - 4n \) is negative. Hence with some positive integer \( t \) and square free integer \( m \), we can classify \( s^2 - 4n \) by

\[ s^2 - 4n = \begin{cases} 
t^2m & m \equiv 1 \mod 4 \\
t^24m & m \equiv 2, 3 \mod 4
\end{cases} \]

For each \( s \), let \( f \) run over all positive divisors of \( t \). Let \( L = Q[x]/(\Psi_s(x)) \) where \( \Psi_s(x) = x^2 - sx + n \) and \( \xi \) is the canonical image of \( x \) in \( L \). Then \( L \) is an imaginary quadratic number field and \( \xi \) generates the order \( Z + Z\xi \) of \( L \). For each \( f \), there is uniquely determined order \( \mathcal{O}_f \) containing \( Z + Z\xi \) as a submodule of index \( f \). Let \( \Delta(\mathcal{O}_f) = s^2 - 4n/f^2 \). Let \( h(\Delta(\mathcal{O}_f))/(\omega(\Delta(\mathcal{O}_f))) \) denote the number of locally principal \( \mathcal{O}_f \) ideals (resp. \( \frac{1}{2}[U(\mathcal{O}_f)] \)). Then \( b(s, f) = \frac{h(\Delta(\mathcal{O}_f))}{\omega(\Delta(\mathcal{O}_f))} \).

Let \( M \) be an order of level \( N \) of \( A \). Then \( c(s, f, l) \) is the number of \( M_t^x = (M \otimes Z_l)^x \) equivalence classes of optimal embeddings of \( \mathcal{O}_f \otimes Z_l \) into \( M \otimes Z_l \). In other words, let \( Z + Z\alpha \) be the maximal order of \( L \), then \( \mathcal{O}_f \otimes Z_l = Z_l + Z_l\alpha \) and \( (s^2 - 4n)/f^2 \equiv t^{2m}\Delta(\alpha) \mod (Z_l^x)^2 \). So it is easy to find \( c(s, f, l) \), the number of \( M_t^x = R_{\nu(l)}^x(L(l)) \) (See Definition 2.1) equivalent classes of optimal embeddings of \( t^{m\alpha} (= Z_l + Z_l\alpha) \) into \( M_l = R_{\nu(l)}(L(l)) \), in Table 5.28 in [15], if \( s, f \) and \( n \) are given.

Proof. See the proof of Theorem 6.22 [15]. \( \square \)
**Theorem 3.3.** For all positive integer \(n\) with \((n, 2q) = 1\) and for all \(\nu \geq 2\) we have

\[
(3.1) \quad trB(n; 2^{2\nu}q) - trB(n; 2^{2\nu-2}q) \\
(3.2) \quad = tr_{2^{2\nu-2}q}T(n) - 2tr_{2^{2\nu}}T(n) - 2(tr_{2^{2\nu-1}q}T(n) - 2tr_{2^{2\nu-1}}T(n)) \\
+ tr_{2^{2\nu-2}q}T(n) - 2tr_{2^{2\nu-2}}T(n) \\
= trT(n) \text{ on } S^0_2(2^{2\nu}q)
\]

**Proof.** To prove this identity we will compare with term by term. First, the degree term of (3.2) is

\[
\deg T(n) - 2\deg T(n) - 2(\deg T(n) - 2\deg T(n)) + \deg T(n) - 2\deg T(n) = 0
\]

and the degree terms of (3.1) do not occur.

Second, we consider mass terms. By Theorem 6.18 in [15] or [3], mass term of (3.1) is

\[
(q - 1)2^{2\nu-1} - (q - 1)2^{2\nu-3} = (q - 1)(2 + 1)2^{2\nu-3}.
\]

On the other hand, mass term of (3.2) is

\[
(q + 1)(2 + 1)2^{2\nu-1} - 2(2 + 1)2^{2\nu-1} - 2((q + 1)(2 + 1)2^{2\nu-2} - 2(2 + 1)2^{2\nu-2}) \\
+ (q + 1)(2 + 1)2^{2\nu-3} - 2(2 + 1)2^{2\nu-3} = (q - 1)(2 - 1)(2 + 1)2^{2\nu-3}
\]

Finally we will check the main part of (3.1) and (3.2).

Suppose \(s^2 - 4n = 0\) or \(t^2\) for some integer \(t\). Then \(M_q\) is the maximal order of \(A_q\) which is a division ring and \(c'(s, f, q)\) is the number of \(M_q^\times = (M \otimes Z_q)^\times\) equivalence classes of optimal embeddings of \(O \otimes Z_q\) with discriminant 0 or \(t^2\) into \(M \otimes Z_q\) which is independent from \(s, f\) (See Table following Theorem 2.1 of [15]). Since there are no optimal embeddings into the division algebra, (3.1) is 0. On the other hand, (3.2) is split into two parts,

\[
(3.3) \quad tr_{2^{2\nu}}T(n) - 2(tr_{2^{2\nu-1}}T(n)) + tr_{2^{2\nu-2}}T(n)
\]

and

\[
(3.4) \quad 2(tr_{2^{2\nu}}T(n) - 2tr_{2^{2\nu-1}}T(n) - tr_{2^{2\nu-2}}T(n)).
\]

If \(s^2 - 4n = 0\), then (3.3) is
\[
\frac{s}{2} \left( \frac{1}{4} c'(s, f, q) \cdot \left( \frac{1}{2^{2\nu} q} \sum_{1}^{2^2\nu} c'(s, f, 2)_{2^{2\nu}} \right) - 2 \frac{1}{2^{2\nu-1} q} \sum_{1}^{q2^{2\nu-1}} c'(s, f, 2)_{2^{2\nu-1}} + \frac{1}{2^{2\nu-2} q} \sum_{1}^{2^2\nu-2q} c'(s, f, 2)_{2^{2\nu-2}} \right) \\
= \frac{s}{2} \left( \frac{1}{4} c'(s, f, q) (2 - 4 + 2) = 0. \right)
\]

Similarly (3.4) is 0.

Next, if \( s^2 - 4n = t^2 \), then by the Remark following Theorem 3.1, \( a(s) \) and \( b(s, f) \) are independent from level \( 2^{2\nu} q \), \( 2^{2\nu-1} q \) and \( 2^{2\nu-2} q \).

So it suffices to compare with \( c'(s, f, 2). \) Now, (3.3) is \( c'(s, f, 2)_{2^{2\nu}} - 2c'(s, f, 2)_{2^{2\nu-1}} + c'(s, f, 2)_{2^{2\nu-2}} = 0. \) Similarly, (3.4)=0. Hence (3.2) is 0.

We now have

\[
\Delta = \frac{(s^2 - 4n)}{f^2} = 2^a q^b d
\]

where \((2q, d) = 1\).

We now need the tables of the number of inequivalent optimal embeddings.

Here \( u \in \mathbb{Z}_2^\times \).

\[
\Delta = 2^{2m}
\]

<table>
<thead>
<tr>
<th>( c'(s, f, 2)_{2^{2\nu+1}} )</th>
<th>( \nu &lt; m )</th>
<th>( \nu = m )</th>
<th>( \nu &gt; m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2 \cdot 2^\nu )</td>
<td>( 2 \cdot 2^m + 2 \cdot 2^m-1 )</td>
<td>( 2 \cdot 2^m + 2 \cdot 2^m-1 )</td>
<td></td>
</tr>
</tbody>
</table>

\[
\Delta = 2^{2m} u
\]

<table>
<thead>
<tr>
<th>( c'(s, f, 2)_{2^{2\nu+1}} )</th>
<th>( \nu &lt; m )</th>
<th>( \nu = m )</th>
<th>( \nu &gt; m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2 \cdot 2^\nu )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

\[
\Delta = 2^{2m+1} a \text{ where } a = 1 \text{ or } a = u.
\]

<table>
<thead>
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<th>( c'(s, f, 2)_{2^{2\nu+1}} )</th>
<th>( \nu &lt; m )</th>
<th>( \nu = m )</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( 2 \cdot 2^\nu )</td>
<td>( 2^m )</td>
<td>( 0 )</td>
<td>( 0 )</td>
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\[
\Delta = 2^{2m+1} b
\]

<table>
<thead>
<tr>
<th>( c'(s, f, 2)_{2^{2\nu+1}} )</th>
<th>( \nu &lt; m )</th>
<th>( \nu = m )</th>
<th>( \nu &gt; m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2 \cdot 2^\nu )</td>
<td>( 2^m )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>
Suppose that \( b \geq 2 \). Then \( c'(s, f, q)_q = 2 \) and \( c(s, f, q)_q = 0 \). Since \( b(s, f) \) is fixed if \( s \) and \( f \) is given,

\[
(3.1) \quad \sum_{s} \sum_{f} b(s, f) c(s, f, q) c(s, f, 2)_{2^{2\nu}} - \sum_{s} \sum_{f} b(s, f) c(s, f, q) c(s, f, 2)_{2^{2\nu-2}} \\
= \sum_{s} \sum_{f} b(s, f) \cdot 0 \cdot c(s, f, 2)_{2^{2\nu}} - \sum_{s} \sum_{f} b(s, f) \cdot 0 \cdot c(s, f, 2)_{2^{2\nu-2}} \\
= 0
\]

\[
(3.2) \quad \sum_{s} \sum_{f} b(s, f) c'(s, f, q) c'(s, f, 2)_{2^{2\nu}} - 2 \sum_{s} \sum_{f} b(s, f) c'(s, f, 2)_{2^{2\nu}} \\
- 2(\sum_{s} \sum_{f} b(s, f) c'(s, f, q) c'(s, f, 2)_{2^{2\nu-1}} - 2 \sum_{s} \sum_{f} b(s, f) c'(s, f, 2)_{2^{2\nu-1}}) \\
+ \sum_{s} \sum_{f} b(s, f) c'(s, f, q) c'(s, f, 2)_{2^{2\nu-2}} - 2 \sum_{s} \sum_{f} b(s, f) c'(s, f, 2)_{2^{2\nu-2}} \\
= \sum_{s} \sum_{f} b(s, f) \cdot 2 \cdot c'(s, f, 2)_{2^{2\nu}} - 2 \sum_{s} \sum_{f} b(s, f) c'(s, f, 2)_{2^{2\nu}} \\
- 2(\sum_{s} \sum_{f} b(s, f) \cdot 2 \cdot c'(s, f, 2)_{2^{2\nu-1}} - 2 \sum_{s} \sum_{f} b(s, f) c'(s, f, 2)_{2^{2\nu-1}}) \\
+ \sum_{s} \sum_{f} b(s, f) \cdot 2 \cdot c'(s, f, 2)_{2^{2\nu-2}} - 2 \sum_{s} \sum_{f} b(s, f) c'(s, f, 2)_{2^{2\nu-2}} \\
= 0
\]

Hence, we need to check cases that \( \frac{s^2 - 4m}{j^2} \) is \( 2^a d \) or \( 2^a qd \).

Case A. \( 2^a d \).

If \( a = 2m \) and \( (\frac{d}{q}) = 1 \), then \( c(s, f, q)_q = 0 \) and \( c'(s, f, q)_q = 2 \). So it is clear.

If \( a = 2m \) and \( (\frac{d}{q}) = -1 \), then \( c(s, f, q)_q = 2 \) and \( c'(s, f, q)_q = 0 \). So (3.1) becomes
\[
2 \left( \sum_{s,f} b(s,f)c(s,f,2)_{2^{2\nu}} - \sum_{s,f} b(s,f)c(s,f,2)_{2^{2\nu-2}} \right) \\
= -2 \sum_{s,f} b(s,f)c'(s,f,2)_{2^{2\nu}} + 4 \sum_{s,f} b(s,f)c'(s,f,2)_{2^{2\nu-1}} \\
- 2 \sum_{s,f} b(s,f)c'(s,f,2)_{2^{2\nu-2}}
\]

Hence, to check (3.1), it suffices to compare

\[
c(s,f,2)_{2^{2\nu}} - c(s,f,2)_{2^{2\nu-2}} \\
= c'(s,f,2)_{2^{2\nu}} - 2c'(s,f,2)_{2^{2\nu-1}} + c(s,f,2)_{2^{2\nu-2}}.
\]

In the following table, we will read \(2^{-1} = 2^{-2} = 0\). For the (3.1), let

\[
g(\nu) = c(s,f,2)_{2^{2\nu}} - c(s,f,2)_{2^{2\nu-2}}.
\]

<table>
<thead>
<tr>
<th>(\nu)</th>
<th>(c(s,f,2)_{2^{2\nu}})</th>
<th>(c(s,f,2)_{2^{2\nu-2}})</th>
<th>(g(\nu))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\nu &gt; m + 1)</td>
<td>(2 \cdot 2^{m} - 2 \cdot 2^{m-1})</td>
<td>(2 \cdot 2^{m} - 2 \cdot 2^{m-1})</td>
<td>(0)</td>
</tr>
<tr>
<td>(\nu = m + 1)</td>
<td>(2 \cdot 2^{m} - 2 \cdot 2^{m-1})</td>
<td>(2^{m} - 2 \cdot 2^{m-1})</td>
<td>(2^{m})</td>
</tr>
<tr>
<td>(\nu = m)</td>
<td>(2^{m} - 2^{m-1})</td>
<td>(2^{m-1} - 2^{m-2})</td>
<td>(2^{m} - 3 \cdot 2^{m-1} + 2^{m-2})</td>
</tr>
<tr>
<td>(\nu &lt; m)</td>
<td>(2^{\nu} - 2^{\nu-1})</td>
<td>(2^{\nu-1} - 2 \cdot 2^{\nu-2})</td>
<td>(2^{\nu} - 2 \cdot 2^{\nu-1} + 2^{\nu-2})</td>
</tr>
</tbody>
</table>

For the (3.2), let \(t(\nu) = c'(s,f,2)_{2^{2\nu}} - 2c'(s,f,2)_{2^{2\nu-1}} + c'(s,f,2)_{2^{2\nu-2}}\).

<table>
<thead>
<tr>
<th>(\nu)</th>
<th>(c'(s,f,2)_{2^{2\nu}})</th>
<th>(c'(s,f,2)_{2^{2\nu-1}})</th>
<th>(c'(s,f,2)_{2^{2\nu-2}})</th>
<th>(t(\nu))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\nu &gt; m + 1)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
</tr>
<tr>
<td>(\nu = m + 1)</td>
<td>(0)</td>
<td>(0)</td>
<td>(2^{m})</td>
<td>(2^{m})</td>
</tr>
<tr>
<td>(\nu = m)</td>
<td>(p^{m})</td>
<td>(2^{m-1})</td>
<td>(2^{m-1} + 2^{m-2})</td>
<td>(2^{m} - 3 \cdot 2^{m-1} + 2^{m-2})</td>
</tr>
<tr>
<td>(\nu &lt; m)</td>
<td>(2^{\nu} - 2^{\nu-1})</td>
<td>(2^{\nu-1})</td>
<td>(2^{\nu-1} - 2^{\nu-2})</td>
<td>(2^{\nu} - 2 \cdot 2^{\nu-1} + 2^{\nu-2})</td>
</tr>
</tbody>
</table>

If \(a = 2m + 1\), as in the above case it suffices to check \(\frac{2d}{q} = -1\).

<table>
<thead>
<tr>
<th>(\nu)</th>
<th>(c(s,f,2)_{2^{2\nu}})</th>
<th>(c(s,f,2)_{2^{2\nu-2}})</th>
<th>(g(\nu))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\nu &gt; m + 1)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
</tr>
<tr>
<td>(\nu = m + 1)</td>
<td>(0)</td>
<td>(2^{m} - 2^{m-1})</td>
<td>(-2^{m} + 2^{m-1})</td>
</tr>
<tr>
<td>(\nu = m)</td>
<td>(2^{m} - 2^{m-1})</td>
<td>(2^{m-1} - 2^{m-2})</td>
<td>(2^{m} - 2 \cdot 2^{m-1} + 2^{m-2})</td>
</tr>
<tr>
<td>(\nu &lt; m)</td>
<td>(2^{\nu} - 2^{\nu-1})</td>
<td>(2^{\nu-1} - 2 \cdot 2^{\nu-2})</td>
<td>(2^{\nu} - 2 \cdot 2^{\nu-1} + 2^{\nu-2})</td>
</tr>
</tbody>
</table>
The Representability of Modular forms by Theta series and its applications 353

For the R.H.S of (3-2), let \( t(\nu) = c'(s, f, 2)_{2^\nu} - 2c'(s, f, 2)_{2^{\nu-1}} + c'(s, f, 2)_{2^{\nu-2}}. \)

<table>
<thead>
<tr>
<th>( \nu &gt; m + 1 )</th>
<th>( c'(s, f, 2)_{2^\nu} )</th>
<th>( c'(s, f, 2)_{2^{\nu-1}} )</th>
<th>( c'(s, f, 2)_{2^{\nu-2}} )</th>
<th>( t(\nu) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu = m + 1 )</td>
<td>0</td>
<td>( 2^m )</td>
<td>( 2^{m-1} + 2^{m-2} )</td>
<td>( 2^m - 2 \cdot 2^{m-1} + 2^{m-2} )</td>
</tr>
<tr>
<td>( \nu &lt; m )</td>
<td>( 2^\nu - 2^{\nu-1} )</td>
<td>( 2 \cdot 2^{\nu-1} )</td>
<td>( 2^{\nu-1} - 2^{\nu-2} )</td>
<td>( 2^\nu - 2 \cdot 2^{\nu-1} + 2^{\nu-2} )</td>
</tr>
</tbody>
</table>

Case B. \( 2^aqd. \)

If \( a = 2m \) and \( (\frac{d}{q}) = 1 \), then \( c(s, f, q) = 1 \) and \( c'(s, f, q) = 1 \). This case becomes Case A. \( a = 2m \) case.

If \( a = 2m + 1 \) and \( (\frac{2d}{q}) = 1 \), then \( c(s, f, q) = 0 \) and \( c'(s, f, q) = 2 \). This is clear from Case A. \( a = 2m + 1 \).

Finally, we like to prove the second equality. By Lemma 15 and Theorem 5 of [1],

\[
\text{tr} T(n) \text{ on } S_2(q2^{2\nu}) - 2\text{tr} T(n) \text{ on } S_2(2^{2\nu}) \\
-2(\text{tr} T(n) \text{ on } S_2(q2^{2\nu-1}) - 2\text{tr} T(n) \text{ on } S_2(2^{2\nu-1})) \\
+\text{tr} T(n) \text{ on } S_2(q2^{2\nu-2}) - 2\text{tr} T(n) \text{ on } S_2(2^{2\nu-2}) \\
=\text{tr} T(n) \text{ on } S_0^0(q2^{2\nu})
\]

**Lemma 3.4.** Let the notations be as above. Then

\[
\text{tr} B(n; q) = \text{tr} T(n) - 2\text{tr} T(n) + \deg T(n).
\]

Note that the Brandt matrix \( B(n; q) \) is independent of which quadratic extension \( L(2) \) of \( k_2 \) we use, since in all cases the order \( R(L(p)) \) is simply a maximal order of \( A_p \) for a finite odd prime \( p \).

**Proof.** See Lemma 6.5 in [15].

In the previous section, trace identity between Brandt matrix and Hecke operators is proved. We are now in position to determine the subspace generated by the set of entries in the Brandt matrix series \( \sum_{n=0}^{\infty} B'(n; N)e^{2\pi in\tau} \). For the next theorem, we need following facts. Let \( S_0^0(N) \) be the set of all new forms in \( S_2(N) \) (See [1]). An important result from the theory of new forms is the decomposition,

\[
S_2(N) \simeq \sum_{a|N} \sum_{d|(N/a)} S_0^0(a).
\]
Thus

\[ S_2(2^{2\nu}q) \simeq \sum_{k=0}^{2\nu} (2\nu - k + 1)S_2^0(2^kq) \oplus 2 \sum_{k=0}^{2\nu} (2\nu - k + 1)S_2^0(2^k) \quad \text{and} \]

\[ S_2(2^{2\nu}) \simeq \sum_{k=0}^{2\nu} (2\nu - k + 1)S_2^0(2^k) \]

and \( \text{tr}T(n) \) on \( S_2(2^{2\nu}q) - \text{tr}T(n) \) on \( S_2(2^{2\nu}) = \text{tr}T(n) \) on \( \sum_{k=0}^{2\nu} (2\nu - k + 1)S_2^0(q2^k) \) where the sum means direct sum.

By 2.5, there exists a matrix \( E' \) such that \( E'B'(n; N)E'^{-1} \) is a diagonal matrix for all \( (n, 2q) = 1 \). Let \( \Phi(n; q, L(2), 2\nu) \) be the subspace of cusp forms generated by \( \{\theta_1(\tau), \ldots, \theta_d(\tau)\} \) which is appearing on the diagonal of the diagonalized matrix series \( \sum_{n=0}^\infty E'B'(n; N)E'^{-1}\exp(n\tau) \).

**Theorem 3.5.** Let \( \Phi(n; N) \) and \( \{\theta_1(\tau), \ldots, \theta_d(\tau)\} \) be as above and let \( <\theta_1> \) denote the 1-dimensional (complex) vector space generated by \( \theta_1(\tau) \). Then

\[
<\theta_1> \oplus <\theta_2> \oplus \cdots \oplus <\theta_d>
\]

\[ \simeq S_2^0(2^{2k}q) \oplus S_2^0(2^{2k-2}q) \oplus \cdots \oplus S_2^0(2^2q) \oplus S_2^0(q) \]

where the isomorphism is a module for the Hecke algebra \( H \) generated by \( T(n) \) with \( (n, 2q) = 1 \).

**Proof.** By theorem 3.2,

\[
\Phi(n; q, L(2), 2\nu) \simeq \Phi(n; 2 \cdot 2\nu - 2) \oplus S_2^0(2^{2\nu}q)
\]

\[
\simeq \Phi(n; q, L(2), 0) \oplus S_2^0(2^{2\nu}q)
\]

\[ \oplus S_2^0(2^{2\nu-2}q) \oplus \cdots \oplus S_2^0(2^2q). \]

\[ \text{tr}B'(n; 2^{2\nu}q) = \text{tr}B(n, 2^{2\nu}) - b(n) = \text{tr}B(n, 2^{2\nu}) - \text{deg}T(n) \text{ for } (n, 2q) = 1, \text{ where } b(n) = \text{deg}T(n) \text{ for } (n, 2q) = 1 \text{ (See p94 [11]).} \]

By Lemma 3.4 \( \text{tr}B(n; q) = \text{tr}T(n) \text{ on } S_2(q) - \text{tr}T(n) \text{ on } 2S_2(1) + \text{deg}T(n). \) Hence \( \text{tr}B'(n; q) = \text{tr}T(n) \text{ on } S_2(q) - \text{tr}T(n) \text{ on } 2S_2(1) = \text{tr}T(n) \text{ on } S_2^0(q) \text{ by (3-1), which means } \Phi(n; q) \simeq S_2^0(q). \) we conclude that \( <\theta_1> \oplus <\theta_2> \oplus \cdots <\theta_d> \simeq \sum_{k=0}^{2\nu} S_2^0(2^{2k}q). \)

\[ \square \]
4. Application to Ramanujan Graphs

Let $G$ be a multi-graph (i.e., we allow loops and multiple edges) with $n$ vertices $v_i$ and edges $e_j$. A walk $W$ on $G$ is an alternating sequence of vertices and edges $v_0e_1v_1e_2\cdots e_rv_r$ where each edge $e_j$ has endpoints $v_{j-1}$ and $v_j$. A walk is said to be without backtracking if a point can traverse the walk without stopping and backtracking. Let $a_{ij}^{(r)}$ denote the number of without backtracking walks of length $r$ from $v_i$ to $v_j$ in $G$ and put $A_r = (a_{ij}^{(r)})$. The $A_r$ are symmetric $n \times n$ matrices with nonnegative integer entries and even diagonal entries. $A_1$ is called the adjacency matrix of $G$. $G$ is determined by $A_1$ and every symmetric $n \times n$ matrices with nonnegative integer entries and even diagonal entries determines a multigraph.

Let the notation be as above. Then it is easy to see that

$$A_1A_1 = A_2 + D A_rA_1 = A_{r+1} + A_{r-1}(D - I)$$

for $r \geq 2$.

We define $n \times n$ matrices $B_r$ recursively as follows:

$$B_{-1} = 0, \quad B_0 = I, \quad B_1 = A_1$$

$$B_rB_1 = B_{r+1} + B_{r-1}(D - I)$$

for $r \geq 0$. Then the relation between $A_r$ and $B_r$ is given by $A_r = B_r - B_{r-2}$ for all $r \geq 1$.

Ramanujan means that all eigenvalues $\lambda$ of the adjacency matrix not equal to $\pm(l+1)$ satisfy $|\lambda| \leq 2\sqrt{l}$ which asymptotically best possible. As an application, we try to construct so called "almost Ramanujan graphs" using the similar method that Pizer has done in [17]. Since the following graphs have loops and multiple edges, we call almost Ramanujan graphs rather than Ramanujan graphs.

**Theorem 4.1.** Let the notation be as above and assume $m, m'$, and $l$ are relatively prime to $N = q \prod_{i=1}^d 2^{\nu_i}$. Then the ideals $I_1, \cdots, I_H$ can be ordered so that, simultaneously for all $m, B(m; N) = (b_{ij}(m))$.

Further:

1. The $B(N; m)$ form a commuting family of diagonalizable matrices which satisfy the following relations:

   $$B(m; N)B(m'; N) = B(mm'; N), \quad \text{if } m \text{ and } m' \text{ are relatively prime.}$$

   and

   $$B(l^r; N)B(l^s; N) = \sum_{k=0}^{\min(r, s)} l^k B(l^{r+s-2k}; N), \quad \text{if } l \text{ is prime.}$$
2. The $\lambda_0 = \sigma_1(m)$ is an eigenvalue of $B(m; N)$ and the other eigenvalues $\lambda_i$, $1 \leq i \leq H - 1$ satisfy $|\lambda_i| \leq \sigma_0(m)\sqrt{m}$, where $\sigma_r(m) = \sum_{d|m} d^r$.

3. $\sum_{j=1}^H b_{ij}(m) = \sigma_1(m)$.

**Proof.** In Theorem 3.5, we have proved that $B(m; N)$ is a representative of Hecke operators acting on the certain spaces of theta series of weight 2. The eigenvalues of $\lambda$ of $B(m; N)$ not equal to $\sigma_1(m)$ are eigenvalues for the actions of $T_m$ on a space of cusp forms of weight 2 on $\Gamma_0(N)$ and so $|\lambda| \leq \sigma_0(m)\sqrt{m}$ by the Petersson Ramanujan conjecture which was proved by Deligne. Finally, $\sum_{j=1}^H b_{ij}(m)$ equals the degree of Hecke operator $T_m$ on a space of cusp forms of weight 2 on $\Gamma_0(N)$. Hence it is $\sigma_1(m)$.

**Remark.** Assume that $G$ is a $k$ regular multi graph (i.e. the degree of each vertex is $k$) with $k = 2^r + 1$. Then the above $B_r$ relation becomes

$$B_r B_1 = B_{r+1} + 2B_{r-1} \text{ for } r \geq 0.$$ 

This is exactly the recursion relation satisfied by the Hecke operators $B_r = T^{2r}$ acting on the space of modular forms of weight 2 on $\Gamma_0(N)$ when $2 \nmid N$. Thus if we are able to associate a $2^r + 1$ regular graph $G$ to the Hecke operator $T^{2r}$, the action of the Hecke operators $T^{2r}$ will determine the $A_r$ and hence give us an information about $G$. For example, the trace formula for $T^{2r}$ immediately yields on the girth of $G$. By varying the spaces on which the Hecke operators act, we will obtain a large family of interesting graphs. If $m$ is not a prime, we will be able to associate a graph $T_m$. These graphs will in general be "almost Ramanujan".

Let the notation and assumptions be as in section 3. Assume that all diagonal entries of $B(m; N)$ are even and let $G(m; N)$ denote the multigraph whose adjacency matrix is $B(m; N')$. In other words, let $G'(m; N)$ denote the graph whose vertices are identified with $H$ left ideal classes of $O$ represented by $I_1, \cdots, I_H$. $G'(m; N)$ has an edge connecting $I_i$ and $J_j$ for each pair $\pm \alpha \in I_j^{-1}I_i$ with $N(\alpha) = mN(I_i)/N(I_j)$.

**Theorem 4.2.** Let $M$ be an order of level $N = (q; L(2), \nu(2))$ in $A$ with class number $H$ given in the previous section and let $l$ be a prime with $l \nmid N$. Then the associated multigraph $G(l; N)$ is defined and is a $l + 1$ regular connected Ramanujan graph.

**Proof.** The results follow from Theorem 3.5. \qed
Remark. The above theorem does not mention the property of girth of a graph. As we will see examples, graphs constructed by the orders of level $N$ are containing usually loops.

Theorem 4.3. Let $M$, $A$, and $H$ be as in Theorem 3.5. Let $m, m_1, \ldots, m_r$ be positive, non square integers relatively prime to $N = 2^{2\nu}q$.

1. Let $G(m; N)$ be the multi-graph whose adjacency matrix $A_1$ is $B(m; N)$. Then $G(m; N)$ is a $\sigma_1(m)$ regular connected multi graph of order $H$. All eigenvalues $\lambda$ of $A_1$ not equal to $\sigma_1(m)$ satisfy $|\lambda| \leq \sigma_0(m)\sqrt{m}$.

2. Let $G(m^2; N)$ be the multi-graph whose adjacency matrix $A_1$ is $B(m^2; N) - I$ where $I$ is the $H$ by $H$ identity matrix. Then $G(m^2; N)$ is a $(\sigma_1(m^2) - 1)$ regular connected graph of order $H$. All eigenvalues $\lambda$ of $A_1$ not equal to $\sigma_1(m^2) - 1$ satisfy $|\lambda| \leq \sigma_0(m^2)m + 1$.

3. Assume $m_1 < m_2 < \cdots < m_r$ are relatively prime in pairs. Let $A_1 = \sum_{i=1}^r B(m_i; N)$. Then $A_1$ is the adjacency matrix of a regular connected graph of order $H$ and degree $\sum_{i=1}^r \sigma_1(m_i)$. All eigenvalues $\lambda$ of $A_1$ not equal to $\pm \sum_{i=1}^r \sigma_1(m_i)$ satisfy $|\lambda| \leq \sum_{i=1}^r \sigma_0(m_i)\sqrt{m_i}$.

Proof. (1) and (2) are immediately from Theorem 2.6. By the properties of Brandt matrices (Remark 2.26 in [17]), (3) is clear.

We call the graphs $G(m; N)$ in Theorem 4.1 almost Ramanujan because the nontrivial eigenvalues of the adjacency matrices of these graphs satisfy the Ramanujan conjecture bound $\sigma_0(m)\sqrt{m}$. The diameter of the graphs in Theorem 4.1 is bounded by the result of Chung [4].

References


Sungtae Jun
Division of Mathematics and Computer science,
Konkuk University,
Choongju, Choongbuk, 380-151, Korea
E-mail: sjun@kkku.ac.kr

Insuk Kim
Department of Mathematics education
Wonkwang University,
Iksan, Jeonbuk, 540-749, Korea
E-mail: iki@wonkwang.ac.kr