p-PRECONVEX SETS ON PRECONVEXITY SPACES

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Abstract. In this paper, we introduce the concept of p-preconvex sets on preconvexity spaces. We study some properties for p-preconvex sets by using the co-convexity hull and the convexity hull. Also we introduce and study the concepts of pc-convex function, p* c-convex function, pf-convex function and p* f -convex function.

1. Introduction

In [1], Guay introduced the concept of preconvexity spaces defined by a binary relation on the power set $P(X)$ of a nonempty set $X$ and investigated some properties. He showed that a preconvexity on a nonempty set yields a convexity space in the same manner as a proximity [6] yields a topological space. In [3], we introduced the concepts of co-convexity hull and co-convex sets on preconvexity spaces. And we characterized c-convex functions and c-concave functions by using the co-convexity hull and the convexity hull.

Semi-preconvex sets, sc-convex functions and s* c-convex functions are introduced in [4]. In [5], we introduced the $\beta$-preconvex set on a preconvexity space and studied some properties. And we introduced the concepts of $\beta$ c-convex functions and $\beta$* c-convex functions which are defined by the $\beta$-preconvex sets.

In this paper, we introduce the concept of p-preconvex set on a preconvexity space and study some basic properties. And we introduce and study the concepts of pc-convex functions, p* c-convex functions, pc-convex functions and p* c-convex functions which are defined by the p-preconvex sets. In particular, for two preconvexity spaces $(X, \sigma), (Y, \mu)$, (a) if a function $f : (X, \sigma) \to (Y, \mu)$ is c-concave and pc-convex, then $f$ is p* c-convex;

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(b) if a function \( f : (X, \sigma) \to (Y, \mu) \) is \( c \)-convext and \( pI \)-convex, then \( f \) is \( p^*I \)-convex.

2. Preliminaries

**Definition 2.1** ([1]). Let \( X \) be a nonempty set. A binary relation \( \sigma \) on \( P(X) \) is called a preconvexity on \( X \) if the relation satisfies the following properties; we write \( x \sigma A \) for \( \{x\} \sigma A \):

1. If \( A \subset B \), then \( A \sigma B \).
2. If \( A \sigma B \) and \( B = \emptyset \), then \( A = \emptyset \).
3. If \( A \sigma B \) and \( b \sigma C \) for all \( b \in B \), then \( A \sigma C \).
4. If \( A \sigma B \) and \( x \in A \), then \( x \sigma B \).

The pair \((X, \sigma)\) is called a preconvexity space. Let \((X, \sigma)\) be a preconvexity space and \( A \subset X \). \( G(A) = \{x \in X : x \sigma A \} \) is called the convexity hull of a subset \( A \). \( A \) is called convex [1] if \( G_{\sigma}(A) = A \) (simply, \( G(A) \)).

\( I_\sigma(A) = \{x \in A : x \notin (X - A)\} \) (simply, \( I(A) \)) is called the co-convexity hull [3] of a subset \( A \). And \( A \) is called a co-convex set if \( I(A) = A \) [3].

Let \( T(X) = \{A \subset X : I(A) = A\} \) and \( G(X) = \{A \subset X : G(A) = A\} \).

**Theorem 2.2** ([1], [3]). For a preconvexity space \((X, \sigma)\),

1. \( G(\emptyset) = \emptyset, \ I(X) = X \).
2. \( A \subset G(A), \ I(A) \subset A \) for all \( A \subset X \).
3. If \( A \subset B \), then \( G(A) \subset G(B) \), \( I(A) \subset I(B) \).
4. \( G(G(A)) = G(A), \ I(I(A)) = I(A) \) for \( A \subset X \).
5. \( I(A) = X - G(X - A) \) and \( G(A) = X - I(X - A) \).

**Theorem 2.3** ([1], [3]). Let \( \sigma \) be a preconvexity on \( X \) and \( A, B \subset X \). Then

1. \( A \sigma B \iff A \subset G(B) \iff I(X - B) \subset X - A \).
2. \( A \sigma B \iff G(A) \sigma G(B) \iff I(X - B) \sigma I(X - A) \).

**Definition 2.4** ([4]). Let \((X, \sigma)\) be a preconvexity space and \( A \subset X \). \( A \) is called a semi-preconvex set if \( A \sigma I(A) \). And \( A \) is called a cosemi-preconvex set if the complement of \( A \) is a semi-preconvex set.

Let \( S_\sigma(X) \) (resp., \( SC_\sigma(X) \)) denote the set of all semi-preconvex sets (resp., cosemi-preconvex sets) in a preconvexity space \((X, \sigma)\).
Definition 2.5 ([5]). Let \((X, \sigma)\) be a preconvexity space and \(A \subset X\). \(A\) is called a \(\beta\)-preconvex set if \(A \sigma I(G(A))\). And \(A\) is called a \(\text{cop}\)-preconvex set if the complement of \(A\) is a \(\beta\)-preconvex set.

We recall that the notions of \(c\)-convex function and \(c\)-concave function: Let \((X, \sigma)\) and \((Y, \mu)\) be two preconvexity spaces. A function \(f : X \to Y\) is said to be \(c\)-concave [2] if for \(C, D \subset Y\) whenever \(C \mu D\), \(f^{-1}(C) \sigma f^{-1}(D)\). A function \(f : X \to Y\) is said to be \(c\)-convex [1] if \(A \sigma B\) implies \(f(A) \mu f(B)\). And \(f\) is \(c\)-convex iff for each \(U \in I(Y)\), \(f^{-1}(U) \in I(X)\) [3].

3. \(p\)-preconvex sets

Definition 3.1. Let \((X, \sigma)\) be a preconvexity space and \(A \subset X\). \(A\) is called a \(p\)-preconvex set if \(A \subset I(G(A))\). And \(A\) is called a \(\text{cop}\)-preconvex set if the complement of \(A\) is a \(p\)-preconvex set.

Let \(P_\sigma(X)\) (resp., \(PC_\sigma(X)\)) denote the set of all \(p\)-preconvex sets (resp., \(\text{cop}\)-preconvex sets) in a preconvexity space \((X, \sigma)\).

Now we get the following implications but the converses are not true in general as shown in the next example:

\[
\begin{array}{c|c}
\text{co-convex} & \beta\text{-preconvex} \\
\downarrow & \downarrow \\
\text{semi-preconvex} & \text{p-preconvex}
\end{array}
\]

Example 3.2. (1) Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, X, \{b, c\}\}\). Define \(A \sigma B\) to mean \(A \subset cl(B)\), the closure of \(B\) in \(X\). Then \(\sigma\) is a preconvexity on \(X\). In the preconvexity space \((X, \sigma)\), \(G(X) = \{\emptyset, X, \{a\}\}\), \(\mathcal{I}(X) = \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}\) and 
\(S_{\sigma}(X) = \{\emptyset, X, \{b, c\}\}\). Hence we know that a \(p\)-convex set \(\{a, b\}\) is neither co-convex nor semi-convex.

(2) Let \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}\}\). Define \(A \sigma B\) to mean \(A \subset cl(B)\), the closure of \(B\) in \(X\). Then \(\sigma\) is a preconvexity on \(X\). In the preconvexity space \((X, \sigma)\), \(G(X) = \{\emptyset, X, \{b, c, d\}, \{a, b, c\}, \{b, c\}\}\), \(\mathcal{I}(X) = \tau\). Then \(\{a, b\}\) is semi-convex and \(\beta\)-preconvex but not preconvex.

From Theorem 2.2, we get the following:

Theorem 3.3. Let \((X, \sigma)\) be a preconvexity space and \(A \subset X\). Then \(A\) is a \(\text{cop}\)-preconvex set if and only if \(G(I(A)) \subset A\).
Theorem 3.4. Every $p$-preconvex set is a $\beta$-preconvex set in a preconvexity space $(X, \sigma)$.

Proof. Let $A$ be a $p$-preconvex set; then by definition of $p$-preconvex sets, $A \subseteq I(G(A))$. By Definition 3.1 (1), $A \sigma I(G(A))$. □

Corollary 3.5. Every cop-preconvex set is co$\beta$-preconvex in a preconvexity space $(X, \sigma)$.

Proof. Obvious. □

Theorem 3.6. In a preconvexity space $(X, \sigma)$, $X$ and $\emptyset$ are both $p$-preconvex and cop-preconvex.

Proof. By Theorem 2.2, it is obvious. □

Theorem 3.7. In a preconvexity space $(X, \sigma)$, the arbitrary union of $p$-preconvex sets is a $p$-preconvex set.

Proof. Let $\mathcal{F} = \{A_\alpha : A_\alpha \in \mathcal{P}_\sigma(X)\}$ be any subfamily of $\mathcal{P}_\sigma(X)$ and $x \in \bigcup \mathcal{F}$. Then there exists a $p$-preconvex set $A_\alpha$ containing $x$ such that $x \in A_\alpha \subseteq I(G(A_\alpha))$. And from Theorem 2.2 and $A_\alpha \subseteq \bigcup \mathcal{F}$, it follows $I(G(A_\alpha)) \subseteq I(G(\bigcup \mathcal{F}))$ and so $x \in I(G(\bigcup \mathcal{F}))$. Hence, $\bigcup \mathcal{F} \subseteq I(G(\bigcup \mathcal{F}))$. □

Theorem 3.8. In a preconvexity space $(X, \sigma)$, the arbitrary intersection of cop-preconvex sets is a cop-preconvex set.

Proof. From Theorem 2.2 and Theorem 3.7, it is obvious. □

Definition 3.9. Let $(X, \sigma)$ be a preconvexity space and $A \subseteq X$.
1. $pG(A) = \cap \{F : A \subseteq F, F \in \mathcal{P}_\sigma(X)\}$.
2. $pI(A) = \cup \{U : U \subseteq A, U \in \mathcal{P}_\sigma(X)\}$.

Theorem 3.10. Let $(X, \sigma)$ be a preconvexity space and $A, B \subseteq X$.
1. $I(A) \subseteq pI(A) \subseteq A$.
2. $A \subseteq pG(A) \subseteq G(A)$.
3. $A$ is $p$-preconvex iff $A = pI(X)$.
4. $A$ is cop-$p$-preconvex iff $A = pG(X)$.

Proof. (1) and (2) are obvious from Theorem 3.4 and Corollary 3.5.
(3) It is obtained from Theorem 3.7.
(4) It is obtained from Theorem 3.8. □

Theorem 3.11. Let $(X, \sigma)$ be a preconvexity space and $A, B \subseteq X$.
1. $pI(X) = X$.
2. $pI(A) \subseteq A$. 
3. If $A \subseteq B$, then $pI(A) \subseteq pI(B)$.
4. $pI(pI(A)) = pI(A)$.

Proof. (1), (2) and (3) are obvious.
(4) Since $pI(A) \subseteq A$, by (3), $pI(pI(A)) \subseteq pI(A)$.
For the converse, let $x \in pI(A)$; then since $x \in pI(A) \subseteq pI(A)$ and
$pI(A)$ is a $p$-preconvex set, we get $x \in pI(pI(A))$ by Definition 3.9. \qed

**Theorem 3.12.** Let $(X, \sigma)$ be a preconvexity space and $A, B \subseteq X$.

1. $pG(\emptyset) = \emptyset$.
2. $A \subseteq pG(A)$.
3. If $A \subseteq B$, then $pG(A) \subseteq pG(B)$.
4. $pG(pG(A)) = pG(A)$.

Proof. It is similar to the proof of Theorem 3.11. \qed

4. **pc-convex functions and $pI$-convex functions**

**Definition 4.1.** Let $(X, \sigma)$ and $(Y, \mu)$ be two preconvexity spaces. A function $f : X \to Y$ is said to be pc-convex if for each $A \in \mathcal{I}(Y)$, $f^{-1}(A) \in \mathcal{P}_\sigma(X)$.

Every pc-convex function is $\beta c$-convex but the converse is not always true as follows:

**Example 4.2.** In Example 3.2 (2), consider a function $f : (X, \sigma) \to (X, \sigma)$ defined as follows: $f(a) = f(b) = a$, $f(d) = b$ and $f(c) = c$. Then $f$ is $\beta c$-convex but not pc-convex because for co-convex set $\{a\}$, $f^{-1}(\{a\}) = \{a, b\}$ is not $p$-preconvex.

**Theorem 4.3.** Let $(X, \sigma)$ and $(Y, \mu)$ be two preconvexity spaces and $f : X \to Y$ a function. Then the following things are equivalent:

1. $f$ is pc-convex.
2. $f^{-1}(I(B)) \subseteq I(G(f^{-1}(B)))$ for all $B \subseteq Y$.
3. $G(I(f^{-1}(B))) \subseteq f^{-1}(G(B))$ for all $B \subseteq Y$.
4. $f(G(I(A))) \subseteq G(f(A))$ for all $A \subseteq X$.
5. For each $U \in \mathcal{G}(Y)$, $f^{-1}(U) \in \mathcal{P}_\sigma(X)$.

Proof. (1) $\Rightarrow$ (2) Suppose that $f$ is pc-convex and let $A \subseteq Y$; then since $I(A) \subseteq A$, by Theorem 2.2, we get $I(G(f^{-1}(I(A)))) \subseteq I(G(f^{-1}(A)))$. Since $I(A) \in \mathcal{I}(Y)$ and $f$ is pc-convex, $f^{-1}(I(A)) \subseteq I(G(f^{-1}(I(A))))$. Hence, we have $f^{-1}(I(A)) \subseteq I(G(f^{-1}(A)))$. 

(2) $\Rightarrow$ (1) Suppose that $f$ is not pc-convex. Then there exists a set $B \subseteq Y$ such that $f^{-1}(I(B)) \not\subseteq I(G(f^{-1}(B)))$.

(2) $\Rightarrow$ (3) Since $I(B) \subseteq B$, we have $I(G(f^{-1}(B))) \subseteq G(f^{-1}(B))$. Therefore, $G(I(f^{-1}(B))) \subseteq f^{-1}(G(B))$.

(3) $\Rightarrow$ (1) Suppose that $f$ is not pc-convex. Then there exists a set $B \subseteq Y$ such that $G(I(f^{-1}(B))) \not\subseteq f^{-1}(G(B))$.

(3) $\Rightarrow$ (4) Since $I(A) \subseteq A$, we have $G(I(A)) \subseteq G(A)$. Therefore, $f(G(I(A))) \subseteq f(G(A))$.

(4) $\Rightarrow$ (1) Suppose that $f$ is not pc-convex. Then there exists a set $A \subseteq X$ such that $f(G(A)) \not\subseteq G(f(A))$.

(4) $\Rightarrow$ (2) Since $G(A) \subseteq A$, we have $f(G(A)) \subseteq f(A)$. Therefore, $f^{-1}(I(B)) \subseteq f^{-1}(G(f^{-1}(B)))$.

(2) $\Rightarrow$ (5) Suppose that $f$ is not pc-convex. Then there exists a set $B \subseteq Y$ such that $f^{-1}(U) \not\subseteq f^{-1}(U)$ for some $U \in \mathcal{G}(Y)$.

(5) $\Rightarrow$ (1) Suppose that $f$ is not pc-convex. Then there exists a set $A \subseteq X$ such that $f^{-1}(U) \not\subseteq f^{-1}(U)$ for some $U \in \mathcal{G}(Y)$.

(5) $\Rightarrow$ (2) Since $f^{-1}(U) \subseteq f^{-1}(U)$, we have $f^{-1}(I(B)) \subseteq f^{-1}(G(f^{-1}(B)))$. Therefore, $G(I(f^{-1}(B))) \subseteq f^{-1}(G(B))$.

(2) $\Rightarrow$ (3) Since $I(B) \subseteq B$, we have $I(G(f^{-1}(B))) \subseteq G(f^{-1}(B))$. Therefore, $G(I(B)) \subseteq f^{-1}(G(B))$.
(2) $\Rightarrow$ (3) Let $B \subset Y$; then by (2), we have
\[ X - f^{-1}(G(B)) = f^{-1}(I(Y - B)) \subset I(G(f^{-1}(Y - B))) = X - G(I(f^{-1}(B))). \]
Hence (3) is obtained.
(3) $\iff$ (4) It is obvious.
(3) $\Rightarrow$ (5) It is obvious.
(5) $\Rightarrow$ (1) It is obvious by Theorem 2.2.

From Theorem 3.10 and Theorem 4.3, we get the following:

**Theorem 4.4.** Let $(X, \sigma)$ and $(Y, \mu)$ be two preconvexity spaces and $f : X \to Y$ a function. Then the following things are equivalent:

1. $f$ is pc-convex.
2. $f^{-1}(I(B)) \subset pI(f^{-1}(B))$ for all $B \subset Y$.
3. $pG(f^{-1}(B)) \subset f^{-1}(G(B))$ for all $B \subset Y$.
4. $f(pG(A)) \subset G(f(A))$ for all $A \subset X$.

**Definition 4.5.** Let $(X, \sigma)$ and $(Y, \mu)$ be two preconvexity spaces. A function $f : X \to Y$ is said to be $p^*c$-convex if for each $A \in \mathcal{P}_\mu(Y)$, $f^{-1}(A) \in \mathcal{P}_\sigma(X)$.

Every $p^*c$-convex function is pc-convex but the converse is not always true as shown in the next example:

**Example 4.6.** In Example 3.2 (1), consider a function $f : (X, \sigma) \to (X, \sigma)$ defined as follows: $f(a) = b, f(b) = c$ and $f(c) = a$. Then $f$ is pc-convex but not $p^*c$-convex because $f^{-1} \{\{b\}\} = \{a\}$ is not p-preconvex for a p-preconvex set $\{b\}$.

We have the following:
\[
\begin{array}{c}
\text{pc-convex} \iff \beta c-convex \\
\uparrow & \uparrow \\
c-convex \Rightarrow \text{pc-convex} \iff p^*c-convex
\end{array}
\]

**Theorem 4.7.** Let $(X, \sigma)$ and $(Y, \mu)$ be two preconvexity spaces. A function $f : X \to Y$ is $p^*c$-convex iff for $A \subset Y$ whenever $A \subset I(G(\mathcal{A}))$, $f^{-1}(A) \subset I(G(f^{-1}(\mathcal{A})))$.

**Proof.** From Definition 4.5, it is obvious.

**Theorem 4.8 ([3]).** Let $(X, \sigma)$ and $(Y, \mu)$ be two preconvexity spaces and $f : X \to Y$ a function. Then the following things are equivalent:
1. $f$ is $c$-concave.
2. $f^{-1}(G(A)) \subseteq G(f^{-1}(A))$ for all $A \subseteq Y$.
3. $I(f^{-1}(A)) \subseteq f^{-1}(I(A))$ for all $A \subseteq Y$.

Theorem 4.9 ([3]). Let $(X, \sigma)$ and $(Y, \mu)$ be two preconvexity spaces and $f : X \to Y$ a function. Then the following things are equivalent:

1. $f$ is $c$-convex.
2. $f(G(A)) \subseteq G(f(A))$ for all $A \subseteq X$.
3. $G(f^{-1}(B)) \subseteq f^{-1}(G(B))$ for all $B \subseteq Y$.
4. $f^{-1}(I(B)) \subseteq I(f^{-1}(B))$ for all $B \subseteq Y$.
5. For each $U \in \mathcal{I}(Y)$, $f^{-1}(U) \in \mathcal{I}(X)$.
6. For each $C \in \mathcal{G}(Y)$, $f^{-1}(U) \in \mathcal{G}(X)$.

Lemma 4.10. Let $f : (X, \sigma) \to (Y, \mu)$ be a function on two preconvexity spaces. If $f$ is $c$-convex and $c$-concave, then we have the following:

1. $f^{-1}(I(B)) = I(f^{-1}(B))$ for all $B \subseteq Y$.
2. $G(f^{-1}(B)) = f^{-1}(G(B))$ for all $B \subseteq Y$.

Proof. From the above Theorem 4.8 and Theorem 4.9, the results are obtained.

Theorem 4.11. Let $f : (X, \sigma) \to (Y, \mu)$ be a function on two preconvexity spaces. If $f$ is $c$-concave and $c$-convex, then $f$ is $p^*c$-convex.

Proof. Let $A \in \mathcal{P}_\mu(Y)$; then $A \subseteq I(G(A))$. Since $f$ is $c$-concave and $c$-convex, from Lemma 4.10, it follows

$$f^{-1}(A) \subseteq f^{-1}(I(G(A))) = I(G(f^{-1}(A))).$$

Hence by Theorem 4.7, $f$ is $p^*c$-convex.

Corollary 4.12. Let $f : (X, \sigma) \to (Y, \mu)$ be a function on two preconvexity spaces. If $f$ is $c$-concave and $c$-convex, then $f$ is $pc$-convex.

Proof. Since every $p^*c$-convex function is $p$-convex, $f$ is $pl$-convex.

Theorem 4.13. Let $f : (X, \sigma) \to (Y, \mu)$ be a function on two preconvexity spaces. If $f$ is $c$-concave and $pc$-convex, then $f$ is $p^*c$-convex.

Proof. Suppose $f$ is $c$-concave and $pc$-convex and let $A \in \mathcal{P}_\mu(Y)$; then $A \subseteq I(G(A))$. From Theorem 4.3 (2) and Theorem 4.8 (2), it follows

$$f^{-1}(A) \subseteq f^{-1}(I(G(A))) \subseteq I(G(f^{-1}(G(A)))) \subseteq I(G(G(f^{-1}(A)))) \subseteq I(G(f^{-1}(A))).$$

Hence by Theorem 4.7, $f$ is $p^*c$-convex.

Theorem 4.14. Let $(X, \sigma)$ and $(Y, \mu)$ be two preconvexity spaces and $f : X \to Y$ a function. Then the following things are equivalent:
1. $f$ is $p^*c$-convex.
2. For each $U \in \mathcal{PC}(Y)$, $f^{-1}(U) \in \mathcal{PC}(X)$.
3. $f(pG(A)) \subset pG(f(A))$ for all $A \subset X$.
4. $pG(f^{-1}(B)) \subset f^{-1}(pG(B))$ for all $B \subset Y$.
5. $f^{-1}(pI(B)) \subset pI(f^{-1}(B))$ for all $B \subset Y$.

Proof. Obvious. \qed

**Definition 4.15.** Let $(X, \sigma)$ and $(Y, \mu)$ be two preconvexity spaces and $f : X \to Y$ a function. Then

1. $f$ is said to be $pI$-convex if for each $U \in \mathcal{I}(X)$, $f(U) \in \mathcal{P}(Y)$.
2. $f$ is is said to be $p^*I$-convex if for each $U \in \mathcal{P}(X)$, $f(U) \in \mathcal{P}(Y)$.

**Example 4.16.** In Example 4.6, the function $f$ is $pI$-convex but not $I$-convex. And $f$ is not $p^*I$-convex because $f(\{c\}) = \{a\}$ is not $p$-convex for a $p$-convex set $\{c\}$ in $X$.

$I$-convex $\Rightarrow pI$-convex $\iff p^*I$-convex

**Theorem 4.17.** Let $f : (X, \sigma) \to (Y, \mu)$ be a function on two preconvexity spaces. Then $f$ is $pI$-convex if $f(I(A)) \subset I(G(f(A)))$ for all $A \subset X$.

Proof. Let $f$ be $pI$-convex; then since $I(A) \in \mathcal{I}(X)$, $f(I(A)) \subset I(G(f(I(A)))) \subset I(G(f(A)))$.

Suppose that $f(I(A)) \subset I(G(f(A)))$ for all $A \subset X$. Since $U \in \mathcal{I}(X)$ iff $I(U) = U$, we have $f$ is $pI$-convex. \qed

**Theorem 4.18.** Let $(X, \sigma)$ and $(Y, \mu)$ be two preconvexity spaces and $f : X \to Y$ a function. Then $f$ is $c$-concave iff $f(I(U)) \subset I(f(U))$ for all $U \subset X$.

Proof. Let $U$ be a subset of $X$; then from Theorem 4.8, it follows

$I(U) \subset I(f^{-1}(f(U)) \subset f^{-1}(I(f(U)))$.

Hence $f(I(U)) \subset I(f(U))$.

Similarly, we have the converse. \qed

**Theorem 4.19.** Let $f : (X, \sigma) \to (Y, \mu)$ be a function on two preconvexity spaces. Then if $f$ is $c$-convex and $c$-concave, then $f$ is $p^*I$-convex.

Proof. Let $U \in \mathcal{P}(X)$; then $U \subset I(G(U))$. From Theorem 4.9 (2) and Theorem 4.18, we have
\[ f(U) \subset f(I(G(U))) \subset I(f(G(U))) \subset I(G(f(U))). \]
Hence \( f(U) \in \mathcal{P}(Y). \)

\[ \square \]

**Corollary 4.20.** Let \( f : (X, \sigma) \to (Y, \mu) \) be a function on two preconvexity spaces. Then if \( f \) is c-convex and c-concave, then \( f \) is \( \text{p}_1 \)-convex.

**Proof.** Since every \( \text{p}_1 \)-convex function is \( \text{p}_1 \)-convex, by Theorem 4.19, \( f \) is \( \text{p}_1 \)-convex. \[ \square \]

**Theorem 4.21.** Let \( f : (X, \sigma) \to (Y, \mu) \) be a function on two preconvexity spaces. Then \( f \) is \( \text{p}_1 \)-convex and c-convex, then \( f \) is \( \text{p}_1 \)-convex.

**Proof.** Let \( U \in \mathcal{P}(X) \); then since \( U \subset I(G(U)) \), from Theorem 4.17 and c-convexity, we have
\[ f(U) \subset f(I(G(U))) \subset I(G(f(G(U)))) \subset I(G(G(f(U)))) \subset I(G(f(U))). \]
Hence \( f(U) \in \mathcal{P}(Y). \) \[ \square \]

**References**


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