ON WEAKLY $\gamma$-CONTINUOUS FUNCTIONS

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Abstract. We introduce the concepts of weakly $\gamma$-continuity, strongly $\gamma$-closed graph and $\gamma$-$T_2$ spaces. And we study some characterizations and properties of such concepts.

1. Introduction

Let $X, Y$ and $Z$ be topological spaces on which no separation axioms are assumed unless explicit stated. Let $S$ be a subset of $X$. The closure (resp. interior) of $S$ will be denoted by $cl(S)$ (resp. $int(S)$). A subset $S$ of $X$ is called semi-open set [2] (resp. $\alpha$-set [4]) if $S \subseteq cl(int(S))$ (resp. $S \subseteq int(cl(int(S)))$). The complement of a semi-open set (resp. $\alpha$-set) is called semi-closed set (resp. $\alpha$-closed set).

A subset $M(x)$ of a space $X$ is called a semi-neighborhood of a point $x \in X$ if there exists a semi-open set $S$ such that $x \in S \subseteq M(x)$. In [1], Latif introduced the notion of semi-convergence of filters. And he investigated some characterizations related to semi-open continuous functions. Now we recall the concept of semi-convergence of filters. Let $S(x) = \{A \in SO(X) : x \in A\}$ and let $S_x = \{A \subseteq X : \text{there exists } \mu \subseteq S(x) \text{ such that } \mu \text{ is finite and } \cap \mu \subseteq A\}$. Then $S_x$ is called the semi-neighborhood filter at $x$. For any filter $F$ on $X$, we say that $F$ semi-converges to $x$ if and only if $F$ is finer than the semi-neighborhood filter at $x$. A subset $U$ of $X$ is called a $\gamma$-open set [4] in $X$ if whenever a filter $F$ semi-converges to $x$ and $x \in U$, then $U \in F$. The class of all $\gamma$-open sets in $X$ will be denoted by $\gamma(X)$.

The $\gamma$-interior [4] of a set $A$ in $X$, denoted by $I_\gamma(A)$, is the union of all $\gamma$-open sets contained in $A$.

The $\gamma$-closure [4] of a set $A$ in $X$, denoted by $cl_\gamma(A)$, $cl_\gamma(A) = \{x \in X : A \cap U \neq \emptyset \text{ for all } U \in S_x\}$.

Received May 6, 2008. Accepted August 29, 2008.

2000 Mathematics Subject Classification: 54A05, 54B10, 54C10, 54D30.

Key words and phrases: weakly $\gamma$-continuous, $\gamma$-compact, $\gamma$-$T_2$-space, strongly $\gamma$-closed graph.
Theorem 1.1 ([4]). Let \((X, \tau)\) be a topological space and \(A \subseteq X\).
(a) \(I_\gamma(A) = \{x \in A : A \in S_\tau\}\).
(b) \(A\) is \(\gamma\)-open set if and only if \(A = I_\gamma(A)\).
(c) A set \(G\) is \(\gamma\)-closed if and only if whenever \(F\) semi-converges to \(x\) and \(A \in F\), then \(x \in A\).

Theorem 1.2 ([4]). Let \((X, \tau)\) be a topological space and \(A\) be a subset of \(X\).
(1) \(A \subseteq Cl_\gamma(A)\).
(2) \(A\) is \(\gamma\)-closed if and only if \(A = Cl_\gamma A\).
(3) \(I_\gamma(A) = X - Cl_\gamma(X - A)\).
(4) \(Cl_\gamma(A) = X - I_\gamma(X - A)\).

2. Weakly \(\gamma\)-continuous functions

Definition 2.1. Let \((X, \tau)\) and \((Y, \mu)\) be two topological spaces. Then \(f : X \rightarrow Y\) is said to be weakly \(\gamma\)-continuous if for \(x \in X\) and each open subset \(V\) containing \(f(x)\), there is a \(\gamma\)-open subset \(U\) containing \(x\) such that \(f(U) \subseteq cl(V)\).

We get the following implications but the converses are not true:
continuous \(\Rightarrow\) semi-continuous \(\Rightarrow\) \(\gamma\)-continuous \(\Rightarrow\) weakly \(\gamma\)-continuous

Example 2.2. Let \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, \{a, b\}, X\}\) be a topology on \(X\). Then \(\gamma(X) = \{\emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}\). Consider a function \(f : (X, \tau) \rightarrow (Y, \mu)\) defined as follows: \(f(a) = c, f(b) = d, f(c) = a\) and \(f(d) = b\). Then \(f\) is weakly \(\gamma\)-continuous. But \(f\) is not \(\gamma\)-continuous because for a \(\gamma\)-open set \(\{a, b\}\), \(f^{-1}(\{a, b\}) = \{c, d\}\) is not \(\gamma\)-open.

Theorem 2.3. Let \(f : (X, \tau) \rightarrow (Y, \mu)\) be a function on topological spaces \((X, \tau)\) and \((Y, \mu)\). Then the following statements are equivalent:
(1) \(f\) is weakly \(\gamma\)-continuous.
(2) \(f^{-1}(V) \subseteq I_\gamma(f^{-1}(cl(V)))\) for every open subset \(V\) of \(Y\).
(3) \(Cl_\gamma(f^{-1}(int(A))) \subseteq f^{-1}(A)\) for every closed set \(A\) of \(Y\).
(4) \(Cl_\gamma(f^{-1}(int(cl(B)))) \subseteq f^{-1}(cl(B))\) for every closed set \(B\) of \(Y\).
(5) \(f^{-1}(int(B)) \subseteq I_\gamma(f^{-1}(cl(int(B))))\) for every closed set \(B\) of \(Y\).
(6) \(Cl_\gamma(f^{-1}(V)) \subseteq f^{-1}(cl(V))\) for every open subset \(V\) of \(Y\).

Proof. (1) \(\Rightarrow\) (2) Let \(V\) be an open subset in \(Y\) and \(x \in f^{-1}(V)\). There exists a \(\gamma\)-open set \(U\) of \(X\) containing \(x\) such that \(f(U) \subseteq cl(V)\).
Since \( x \in U \subseteq f^{-1}(\text{cl}(V)) \), by definition of \( \gamma \)-interior, \( x \in I_\gamma(f^{-1}(\text{cl}(V))) \).
Hence \( f^{-1}(V) \subseteq I_\gamma(f^{-1}(\text{cl}(V))) \).

(2) \implies (3) Let \( A \) be a closed subset in \( Y \). Then \( Y - A \) in open in \( Y \) and, by (2)
\[
f^{-1}(Y - A) \subseteq I_\gamma(f^{-1}(\text{cl}(Y - A)))
= I_\gamma(f^{-1}(Y - \text{int}(A)))
\subseteq X - \text{Cl}_\gamma(f^{-1}(\text{int}(A))).
\]
Thus \( \text{Cl}_\gamma(f^{-1}(\text{int}(A))) \subseteq f^{-1}(A) \).

(3) \implies (4) Let \( B \) be a subset of \( Y \). Since \( \text{cl}(B) \) is closed in \( Y \), from (3), it follows \( \text{Cl}_\gamma(f^{-1}(\text{int}(\text{cl}(B)))) \subseteq f^{-1}(\text{cl}(B)) \).

(4) \implies (5) Let \( B \) be a subset of \( Y \). Then
\[
f^{-1}(\text{int}(B)) = X - f^{-1}(\text{cl}(Y - B))
\subseteq X - \text{Cl}_\gamma(f^{-1}\text{int}(\text{cl}(Y - B)))
= I_\gamma(f^{-1}\text{cl}(\text{int}(B))).
\]
Thus we get the result.

(5) \implies (6) Let \( V \) be an open subset of \( Y \). Suppose \( x \notin f^{-1}(\text{cl}(V)) \).
Then \( f(x) \notin \text{cl}(V) \) and so there exists an open set \( U \) containing \( f(x) \) such that \( U \cap V = \emptyset \) and so \( \text{cl}(U) \cap V = \emptyset \). By (5), \( x \in f^{-1}(U) \subseteq I_\gamma(f^{-1}(\text{cl}(U))) \). Then by definition of \( \gamma \)-interior, there exists an open set \( G \) containing \( x \) such that \( x \in G \subseteq f^{-1}(\text{cl}(U)) \). Since \( \text{cl}(U) \cap V = \emptyset \) and \( f(G) \subseteq \text{cl}(U) \), we have \( G \cap f^{-1}(V) = \emptyset \) and so \( x \notin \text{Cl}_\gamma(f^{-1}(V)) \).
Hence \( \text{Cl}_\gamma(f^{-1}(V)) \subseteq f^{-1}(\text{cl}(V)) \).

(6) \implies (1) Let \( x \in X \) and \( V \) an open set in \( Y \) containing \( f(x) \). Since \( V = \text{int}(V) \subseteq \text{int}(\text{cl}(V)) \), by (6),
\[
x \in f^{-1}(V) \subseteq f^{-1}(\text{int}(\text{cl}(V)))
= X - f^{-1}(\text{cl}(Y - \text{cl}(V)))
\subseteq X - \text{Cl}_\gamma(f^{-1}(\text{cl}(Y - \text{cl}(V))))
= I_\gamma(f^{-1}(\text{cl}(V))).
\]
So there exists a \( \gamma \)-open subset \( U \) in \( X \) such that \( U \subseteq f^{-1}(\text{cl}(V)) \).
Hence \( f \) is weakly \( \gamma \)-continuous.
\[\square\]

We recall that a point \( x \) of a topological space \( X \) is said to be \( \theta \)-adherent of \( A \) if \( A \cap \text{cl}(V) \neq \emptyset \) for every open set \( V \) containing \( x \). The set of all \( \theta \)-adherent points of \( A \) is called \( \theta \)-closure of \( A \) [6] and is denoted by
\( \text{cl}_\theta(A) \). If \( A = \text{cl}_\theta(A) \), then \( A \) is called \( \theta \)-closed. The complement of a \( \theta \)-closed set is said to be \( \theta \)-open. It is shown in [6] that \( \text{cl}(A) = \text{cl}_\theta(A) \) for every open set \( A \) and \( \text{cl}_\theta(B) \) is closed for every subset \( B \) of \( X \).

**Theorem 2.4.** Let \( f : (X, \tau) \to (Y, \mu) \) be a function on topological spaces \((X, \tau)\) and \((Y, \mu)\). Then the following statements are equivalent:

1. \( f \) is weakly \( \gamma \)-continuous.
2. \( \text{Cl}_\gamma(f^{-1}(\text{int}(\text{cl}_\theta(B)))) \subset f^{-1}(\text{cl}_\theta(B)) \) for every set \( B \) of \( Y \).
3. \( \text{Cl}_\gamma(f^{-1}(\text{int}(\text{cl}(B)))) \subset f^{-1}(\text{cl}(B)) \) for every set \( B \) of \( Y \).
4. \( \text{Cl}_\gamma(f^{-1}(\text{int}(\text{cl}(G)))) \subset f^{-1}(\text{cl}(G)) \) for every open subset \( G \) of \( Y \).
5. \( f(\text{Cl}_\gamma(A)) \subset \text{cl}_\theta(f(A)) \) for every set \( A \) of \( X \).
6. \( \text{Cl}_\gamma(f^{-1}(B)) \subset f^{-1}(\text{cl}_\theta(B)) \) for every set \( B \) of \( Y \).

**Proof.** (1) \( \implies \) (2) Let \( B \) be any subset in \( Y \); then \( \text{cl}_\theta(B) \) is closed, by Theorem 2.3 (3), we get the result.

(2) \( \implies \) (3) It is obvious since \( \text{cl}(B) \subset \text{cl}_\theta(B) \) for every subset \( B \) of \( Y \).

(3) \( \implies \) (4) It is obvious since \( \text{cl}(G) = \text{cl}_\theta(G) \) for every open subset \( G \) of \( Y \).

(4) \( \implies \) (1) Since \( G \subset \text{int}(\text{cl}(G)) \) for every open set \( G \) of \( Y \), from Theorem 2.3 (6), it follows \( f \) is weakly \( \gamma \)-continuous.

(1) \( \implies \) (5) Let \( A \) be a subset of \( X \). Let \( x \in \text{Cl}_\gamma(A) \) and \( G \) be an open subset of \( Y \) containing \( f(x) \). Since \( f \) is weakly \( \gamma \)-continuous, there exists a \( \gamma \)-open set \( U \) containing \( x \) in \( X \) such that \( f(U) \subset \text{cl}(G) \). Since \( x \in \text{Cl}_\gamma(A) \), we have \( U \cap A \neq \emptyset \) and so \( 0 \neq f(U) \cap f(A) \subset \text{cl}(G) \cap f(A) \). Hence \( f(x) \in \text{cl}_\theta(f(A)) \).

(5) \( \implies \) (6) Let \( B \) be a subset of \( Y \); then by (5), we have \( f(\text{Cl}_\gamma(f^{-1}(B))) \subset \text{cl}_\theta(f(f^{-1}(B))) \subset \text{cl}_\theta(B) \) and so we get the result.

(6) \( \implies \) (1) Let \( B \) be a subset of \( Y \); then by (6),

\[
\text{Cl}_\gamma(f^{-1}(\text{int}(\text{cl}(B)))) \subset f^{-1}(\text{cl}_\theta(\text{int}(\text{cl}(B))))
= f^{-1}(\text{cl}(\text{int}(\text{cl}(B))))
\subset f^{-1}(\text{cl}(B)).
\]

Hence \( f \) is weakly \( \gamma \)-continuous by Theorem 2.3 (4). \( \square \)

**Theorem 2.5.** Let \( f : (X, \tau) \to (Y, \mu) \) be a function on topological spaces \((X, \tau)\) and \((Y, \mu)\). Then the following statements are equivalent:

1. \( f \) is weakly \( \gamma \)-continuous.
2. \( \text{Cl}_\gamma(f^{-1}(\text{int}(K))) \subset f^{-1}(\text{cl}(K)) \) for every regular closed set \( K \) of \( Y \).
3. \( \text{Cl}_\gamma(f^{-1}(\text{int}(\text{cl}(G)))) \subset f^{-1}(\text{cl}(G)) \) for every \( \beta \)-open set \( G \) of \( Y \).
(4) \( Cl_\gamma(f^{-1}(\text{int}(cl(G)))) \subseteq f^{-1}(cl(G)) \) for every semiopen set \( G \) of \( Y \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( K \) be any regular closed set of \( Y \). Then by Theorem 2.3(6), we have \( Cl_\gamma(f^{-1}(\text{int}(K))) \subseteq f^{-1}(\text{cl}(\text{int}(K))) \). Since \( K \) is regular closed, we have \( Cl_\gamma(f^{-1}(\text{int}(K))) \subseteq f^{-1}(K) \).

(2) \( \Rightarrow \) (3) Let \( G \) be any \( \beta \)-open set. From \( cl(G) \subseteq cl(\text{int}(cl(G))) \subseteq cl(G) \), it follows \( cl(G) \) is regular closed. By (2), we have \( Cl_\gamma(f^{-1}(\text{int}(cl(G)))) \subseteq f^{-1}(cl(G)) \).

(3) \( \Rightarrow \) (4) It is obvious since every semiopen set is \( \beta \)-open.

(4) \( \Rightarrow \) (1) Let \( V \) be any open set of \( Y \). Then by (4),
\[
Cl_\gamma(f^{-1}(V)) \subseteq Cl_\gamma(f^{-1}(\text{int}(cl(V)))) \subseteq f^{-1}(cl(V)).
\]
Hence from Theorem 2.3(6), \( f \) is weakly \( \gamma \)-continuous. \( \square \)

**Theorem 2.6.** Let \( f : (X, \tau) \rightarrow (Y, \mu) \) be a function on topological spaces \( (X, \tau) \) and \( (Y, \mu) \). Then the following statements are equivalent:

1. \( f \) is weakly \( \gamma \)-continuous.
2. \( Cl_\gamma(f^{-1}(\text{int}(cl(G)))) \subseteq f^{-1}(cl(G)) \) for every preopen set \( G \) of \( Y \).
3. \( Cl_\gamma(f^{-1}(G)) \subseteq f^{-1}(cl(G)) \) for every preopen set \( G \) of \( Y \).
4. \( f^{-1}(G) \subseteq \text{Int}_\gamma(f^{-1}(cl(G))) \) for every preopen set \( G \) of \( Y \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( G \) be any preopen set in \( Y \). Then \( cl(G) = cl(\text{int}(cl(G))) \). Let \( A = \text{int}(cl(G)) \). Then from Theorem 2.3 (6), it follows that \( Cl_\gamma(f^{-1}(A)) \subseteq f^{-1}(cl(A)) \). Since \( cl(A) = cl(G) \), we have \( Cl_\gamma(f^{-1}(\text{int}(cl(G)))) \subseteq f^{-1}(cl(G)) \).

(2) \( \Rightarrow \) (3) Obvious.

(3) \( \Rightarrow \) (4) Let \( G \) be any preopen set in \( Y \). Then from definition of preopen sets and (3), it follows that
\[
f^{-1}(G) \subseteq f^{-1}(\text{int}(cl(G))) = X - f^{-1}(cl(Y - cl(G))) \subseteq X - (Cl_\gamma(f^{-1}(Y - cl(G)))) = \text{Int}_\gamma(f^{-1}(cl(G))).
\]
Hence we have (4).

(4) \( \Rightarrow \) (1) Since every open set is preopen, from (4) and Theorem 2.3(6), \( f \) is weakly \( \gamma \)-continuous. \( \square \)
Definition 2.7. Let $X$ be a topological space. Then $X$ is said to be $\gamma$-$T_2$ if for every two distinct points $x$ and $y$ in $X$, there exist two disjoint $\gamma$-open sets $U$ and $V$ such that $x \in U$ and $y \in V$.

Let $X$ be a topological space. Then $X$ is said to be Urysohn if for every two distinct points $x$ and $y$ in $X$, there exist two open sets $U$ and $V$ such that $\text{cl}(U) \cap \text{cl}(V) = \emptyset$.

Theorem 2.8. Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces $(X, \tau)$ and $(Y, \mu)$. If $f$ is a weakly $\gamma$-continuous injection and $Y$ is Urysohn, then $X$ is $\gamma$-$T_2$.

Proof. Let $x_1$ and $x_2$ be two distinct elements in $X$, then $f(x_1) \neq f(x_2)$. There exist two open sets $U$ and $V$ in $Y$ containing $f(x_1)$, $f(x_2)$, respectively, such that $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. Since $f$ is weakly $\gamma$-continuous, there exist $\gamma$-open sets $U_1$, $V_2$ containing $x_1$, $x_2$, respectively, such that $f(U_1) \subseteq \text{cl}(U)$, $f(V_2) \subseteq \text{cl}(V)$. It follows $U_1 \cap V_2 = \emptyset$. Hence $X$ is $\gamma$-$T_2$. □

Definition 2.9. Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces $(X, \tau)$ and $(Y, \mu)$. We call $f$ has a strongly $\gamma$-closed graph if for each $(x, y) \notin G(f)$, there exist a $\gamma$-open set $U$ and an open set $V$ containing $x$ and $y$, respectively, such that $(U \times \text{cl}(V)) \cap G(f) = \emptyset$.

Lemma 2.10. Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces $(X, \tau)$ and $(Y, \mu)$. Then $f$ has a strongly $\gamma$-closed graph if for each $(x, y) \notin G(f)$, there exist a $\gamma$-open set $U$ containing and an open set $V$ containing $x$ and $y$, respectively, such that $f(U) \cap \text{cl}(V) = \emptyset$.

Proof. Obvious. □

Theorem 2.11. Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces $(X, \tau)$ and $(Y, \mu)$. If $f$ is weakly $\gamma$-continuous and $Y$ is Urysohn, then $f$ has a strongly $\gamma$-closed graph.

Proof. Let $(x, z) \notin G(f)$. Then $z \neq f(x)$ and since $Y$ is Urysohn, there exist two open sets $U$ and $V$ containing $z$ and $f(x)$, respectively, such that $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. Since $f$ is weakly $\gamma$-continuous, there exists a $\gamma$-open set $H$ containing $x$ such that $f(H) \subseteq \text{cl}(V)$. It implies $f(H) \cap \text{cl}(U) = \emptyset$. Hence $f$ has a strongly $\gamma$-closed graph. □

Theorem 2.12. Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces $(X, \tau)$ and $(Y, \mu)$. If $f$ is a weakly $\gamma$-continuous injection with a strongly $\gamma$-closed graph, then $X$ is $\gamma$-$T_2$. 
Proof. Let \( x_1 \) and \( x_2 \) be two distinct elements in \( X \), then \( f(x_1) \neq f(x_2) \). This implies that \((x_1, f(x_2)) \in (X \times Y) - G(f)\). Since \( f \) has a strongly \( \gamma \)-closed graph, there exist a \( \gamma \)-open set \( U \) and an open set \( V \) containing \( x_1 \) and \( f(x_2) \), respectively, such that \( f(U) \cap \text{cl}(V) = \emptyset \). Since \( f \) is weakly \( \gamma \)-continuous, there exists a \( \gamma \)-open set \( W \) containing \( x_2 \) such that \( f(W) \subseteq \text{cl}(V) \). It implies \( f(W) \cap f(U) = \emptyset \). Therefore \( W \cap U = \emptyset \) and so \( X \) is a \( \gamma \)-\( T_2 \) space.

**Definition 2.13.** A subset \( A \) of a topological space \((X, \tau)\) is called a \( \gamma \)-compact relative to \( A \) if every collection \( \{U_i : i \in J\} \) of \( \gamma \)-open subsets of \( X \) such that \( A \subseteq \bigcup \{U_i : i \in J\} \), there exists a finite subset \( J_0 \) of \( J \) such that \( A \subseteq \bigcup \{U_i : i \in J_0\} \).

A subset \( A \) of a topological space \( X \) is said to be quasi \( H \)-closed relative to \( A \) [6] if every collection \( \{U_i : i \in J\} \) of open subsets of \( X \) such that \( A \subseteq \bigcup \{U_i : i \in J\} \), there exists a finite subset \( J_0 \) of \( J \) such that \( A \subseteq \bigcup \{\text{cl}(U_i) : i \in J_0\} \).

**Theorem 2.14.** Let \( f : (X, \tau) \to (Y, \mu) \) be a function on topological spaces \((X, \tau)\) and \((Y, \mu)\). If \( f \) is weakly \( \gamma \)-continuous and \( A \) is a \( \gamma \)-compact subset of \( X \), then \( f(A) \) is quasi \( H \)-closed relative to \((Y, \mu)\).

Proof. Let \( \{V_i : i \in J\} \) be a cover of \( f(A) \) by open subsets of \( Y \). For each \( x \in A \), there exists \( i(x) \in J \) such that \( f(x) \in V_{i(x)} \). Since \( f \) is weakly \( \gamma \)-continuous, there exists a \( \gamma \)-open set \( U(x) \) containing \( x \) such that \( f(U(x)) \subseteq \text{cl}(V_{i(x)}) \). The family \( \{U(x) : x \in A\} \) is a cover of \( A \) by \( \gamma \)-open sets in \( X \). Since \( A \) is \( \gamma \)-compact, there is a finite subcover \( \{U(x_1), U(x_2), \ldots, U(x_n) : x_j \in A, j = 1, 2, \ldots, n\} \) such that \( A \subseteq \bigcup U(x_j) \). Then

\[
1 \leq j \leq n.
\]

Thus \( f(A) \) is quasi \( H \)-closed relative to \((Y, \mu)\).

References


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