CHARACTERIZATIONS OF REAL HYPERSURFACES
OF TYPE A IN A COMPLEX SPACE FORM USED BY
THE ξ-PARALLEL STRUCTURE JACOBI OPERATOR

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Abstract. Let $M$ be a real hypersurface of a complex space form with almost contact metric structure $(\phi, \xi, \eta, g)$. In this paper, we study real hypersurfaces in a complex space form whose structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ is $\xi$-parallel. In particular, we prove that the condition $\nabla_\xi R_\xi = 0$ characterize the homogeneous real hypersurfaces of type A in a complex projective space $P_n\mathbb{C}$ or a complex hyperbolic space $H_n\mathbb{C}$ when $g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant and not equal to $-c/24$ on $M$, where $c$ is a constant holomorphic sectional curvature of a complex space form.

1. Introduction

Let $(M_n(c), J, \bar{g})$ be a complex $n$-dimensional complex space form with Kähler structure $(J, \bar{g})$ of constant holomorphic sectional curvature $c$ and let $M$ be an orientable real hypersurface in $M_n(c)$. Then $M$ has an almost contact metric structure $(\phi, \xi, \eta, g)$ induced from $(J, \bar{g})$.

It is known that there are no real hypersurfaces with parallel Ricci tensors in a nonflat complex space form (see [6], [9]). This result says that there does not exist locally symmetric real hypersurfaces in a nonflat complex space form. The structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ has a fundamental role in contact geometry. Cho and the second author started the study on real hypersurfaces in a complex space form by using the operator $R_\xi$ in [3], [4] and [5]. Recently Ortega, Pérez and Santos [13] have proved that there are no real hypersurfaces in a complex projective space $P_n\mathbb{C}, n \geq 3$ with parallel structure Jacobi operator $\nabla R_\xi = 0$. More generally, such a result has been extended by [14].

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Now in this paper, motivated by results mentioned above we consider the parallelism of the structure Jacobi operator $R_\xi$ in the direction of the structure vector field, that is $\nabla_\xi R_\xi = 0$.

In 1970’s, Takagi ([15], [16]) classified the homogeneous real hypersurfaces of $P_n \mathbb{C}$ into six types. On the other hand, Cecil and Ryan [2] extensively studied a Hopf hypersurface (whose structure vector field $\xi$ is principal), which is realized as tubes over certain submanifolds in $P_n \mathbb{C}$, by using its focal map. By making use of those results and the mentioned work of Takagi, Kimura [10] proved the local classification theorem for Hopf hypersurfaces of $P_n \mathbb{C}$ whose all principal curvatures are constant. For the case of a complex hyperbolic space $H_n \mathbb{C}$, Berndt [1] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant. Among the several types of real hypersurfaces appeared in Takagi’s list or Berndt’s list, a particular type of tubes over totally geodesic $P_k \mathbb{C}$ or $H_k \mathbb{C}$ ($0 < k < n - 1$) adding a horosphere in $H_n \mathbb{C}$, which is called type A, has a lot of nice geometric properties. For example, Okumura [12] (resp. Montiel and Romero [11]) shows that a real hypersurface of type A if and only if the structure operator $\phi$ commutes with the shape operator ($A\phi = \phi A$).

In this paper we study a real hypersurface in a nonflat complex space form $M_n(c)$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant and not equal to $-c/24$ on $M$. We give another characterization of real hypersurfaces of type A in $M_n(c)$ by above two conditions. In particular, in the case of $P_n \mathbb{C}$, it is not necessary to the condition $g(\nabla_\xi \xi, \nabla_\xi \xi)$ is not equal to $-c/24$. The main purpose of the present paper is to establish Theorem 2 stated in section 5. We note that the condition $g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant on $M$ is a much weaker condition. Indeed, every Hopf hypersurface always satisfies this condition.

All manifolds in this paper are assumed to be connected and of class $C^\infty$ and the real hypersurfaces are supposed to be oriented.

2. Preliminaries

Let $M$ be a real hypersurface of a nonflat complex space form $M_n(c)$, $c \neq 0$ and $C$ be a unit normal vector on $M$. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Kähler metric $\tilde{g}$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)C, \quad \tilde{\nabla}_X C = -AX$$
for any vector fields $X$ and $Y$ on $M$, where $g$ denotes the Riemannian metric of $M$ induced from $\tilde{g}$ and $A$ is the shape operator of $M$ in $M_n(c)$. For any vector field $X$ tangent to $M$, we put

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where $J$ is the almost complex structure of $M_n(c)$. Then we may see that $M$ induces an almost contact metric structure $(\phi, \xi, \eta, g)$, namely

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

for any vector fields $X$ and $Y$ on $M$.

Since $J$ is parallel, we verify, using the Gauss and Weingarten formulas, that

$$(2.1) \quad \nabla_X \xi = \phi AX,$$
$$(2.2) \quad (\nabla_X \phi) Y = \eta(Y)AX - g(AX, Y)\xi.$$

Since the ambient space is of constant holomorphic sectional curvature $c$, we have the following Gauss and Codazzi equations respectively:

$$(2.3) \quad R(X, Y)Z = \frac{c}{4} \{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY,$$
$$(2.4) \quad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4} \{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}$$

for any vector fields $X$, $Y$ and $Z$ on $M$, where $R$ denotes the Riemannian curvature tensor of $M$.

In the sequel, to write our formulas in convention forms, we denote by $\alpha = \eta(AX)$, $\beta = \eta(A^2)\xi$, and for a function $f$ we denote by $\nabla f$ the gradient vector field of $f$.

If we put $U = \nabla \xi$, then $U$ is orthogonal to the structure vector $\xi$. From (2.1), we get

$$\phi U = -A\xi + \alpha \xi,$$

which enables us to obtain $g(U, U) = \beta - \alpha^2$. If we put

$$\phi U = -A\xi + \alpha \xi,$$

we have

$$(2.5) \quad A\xi = \alpha \xi + \mu W,$$

where $W$ is a unit vector field orthogonal to $\xi$. Then we get $U = \mu \phi W$, which shows that $W$ is also orthogonal to $U$. Further we have

$$\mu^2 = \beta - \alpha^2.$$
Thus we see that $\xi$ is principal curvature vector, that is $A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$.

In this paper, we basically use the technical computations with the orthogonal triplet $\{\xi, U, W\}$ and their associated scalars $\alpha, \beta$ and $\mu$.

Because of (2.1), (2.5) and (2.6), it is seen that
\begin{equation}
(2.8)
 g(\nabla_X\xi, U) = \mu g(AW, X)
\end{equation}
and
\begin{equation}
(2.9)
 \mu g(\nabla_X W, \xi) = g(AU, X)
\end{equation}
for any vector field $X$ on $M$.

Differentiating (2.5) covariantly along $M$ and making use of (2.1) and (2.2), we find
\begin{equation}
(2.10)
 (\nabla_X A)\xi = -\phi\nabla_X U + g(AU + \nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX,
\end{equation}
which enables us to obtain
\begin{equation}
(2.11)
 (\nabla_{\xi} A)\xi = 2AU + \nabla\alpha,
\end{equation}
where we have used (2.4). From (2.1) and (2.10), it is verified that
\begin{equation}
(2.12)
 \nabla_{\xi} U = 3\phi AU + \alpha AX - \beta\xi + \phi\nabla\alpha.
\end{equation}

The curvature equation (2.3) gives the structure Jacobi operator $R_{\xi}$:
\begin{equation}
(2.13)
 R_{\xi}(X) = R(X, \xi)\xi = \frac{c}{4} \{X - \eta(X)\xi\} + \alpha AX - \eta(AX)AX
\end{equation}
for any vector field $X$ on $M$.

3. Real hypersurfaces satisfying $\nabla_{\xi} R_{\xi} = 0$

We set $\Omega = \{p \in M; \mu(p) \neq 0\}$ and suppose that $\Omega$ is non-empty, that is, $\xi$ is not a principal curvature vector on $M$. Hereafter, unless otherwise stated, we discuss our arguments on the open subset $\Omega$ of $M$.

Differentiating (2.13) covariantly, and using (2.11), we find
\begin{equation}
(3.1)
 g((\nabla_{\xi} R_{\xi}) Y, Z) = -\frac{c}{4} \{u(Y)\eta(Z) + u(Z)\eta(Y)\} + (\xi\alpha) g(AY, Z)
 + \alpha g((\nabla_{\xi} A) Y, Z) - \eta(AX) \{3g(AX, Y) + A\alpha\}
 - \eta(AY) \{3g(AU, Z) + Z\alpha\},
\end{equation}
where $u$ is a 1-form dual to $U$ with respect to $g$, that is $u(X) = g(U, X)$. 
We assume that $\nabla_\xi R_\xi = 0$. Then from (3.1) we have
\begin{equation}
\alpha(\nabla_\xi A)X + (\xi \alpha)AX = \frac{c}{4}(u(X)\xi + \eta(X)U) \\
+ \eta(AX)(3A\nu + \nabla \alpha) + \{3g(AX, X) + X\alpha\}A\xi.
\end{equation}
Putting $X = \xi$ in this and using (2.11), we find
\begin{equation}
\alpha AU + \frac{c}{4}U = 0,
\end{equation}
which shows that $\alpha \neq 0$ on $\Omega$.
Differentiating (3.3) covariantly and using itself, we obtain
\begin{equation}
-\frac{c}{4}(X\alpha)U + \alpha^2(\nabla X A)U + \alpha^2 A\nabla_X U + \frac{c}{4}\alpha \nabla_X U = 0,
\end{equation}
or, using (2.4) and (2.5)
\begin{equation}
\frac{c}{4}\{(Y\alpha)u(X) - (X\alpha)u(Y)\} + \frac{c}{4}\alpha^2\mu\{\eta(X)w(Y) - \eta(Y)w(X)\} \\
+ \alpha^2\{g(A\nabla_X U, Y) - g(A\nabla_Y U, X)\} + \frac{c}{4}\alpha du(X, Y) = 0,
\end{equation}
where $w$ is a dual 1-form of $W$ with respect to $g$, that is $w(X) = g(W, X)$. Here, $du$ is the exterior derivative of a 1-form $u$ given by
$$du(X, Y) = - Yu(X) + Xu(Y) - u([X, Y]).$$
If we replace $X$ by $U$ in (3.5), then it follows that
\begin{equation}
\frac{c}{4}\mu^2\nabla\alpha - (U\alpha)U\} + \alpha^2 A\nabla_U U + \frac{c}{4}a \nabla_U U = 0,
\end{equation}
because $U$ and $W$ are mutually orthogonal.
Combining (2.10) to (3.2) and using (2.4), we obtain
\begin{equation}
\alpha^2\phi\nabla_X U = \alpha^2(X\alpha)\xi - \frac{c}{4}\alpha u(X)\xi + (\xi \alpha)AX + \frac{c}{4}\alpha^2\phi X \\
- \eta(AX)\left(\alpha \nabla \alpha - \frac{3}{4}cU\right) - \left(\alpha(X\alpha) - \frac{3}{4}cu(X)\right)A\xi \\
- \frac{c}{4}\alpha\{u(X)\xi + \eta(X)U\} - \alpha^2 A\phi AX + \alpha^3\phi AX.
\end{equation}
Applying $\phi$ to this and using (2.8), we have
\begin{equation}
\alpha^2\nabla_X U + \alpha^2\mu g(W, X)\xi - \alpha\eta(AX)\phi \nabla \alpha \\
= -\alpha(\xi \alpha)\phi AX + \frac{c}{4}\alpha^2(X - \eta(X)\xi) + \frac{3}{4}c\mu\eta(AX)W + \alpha(X\alpha)U \\
- \frac{3}{4}cu(X)U + \alpha^3 AX - \frac{c}{4}\alpha \mu \eta(X)W - \alpha^3 \eta(AX)\xi + \alpha^2 \phi A\phi AX.
\end{equation}
On the other hand, differentiating (2.6) covariantly, and using (2.1), we find
\[(\nabla_X A)\xi - \frac{c}{4}\phi X + A\phi AX = (X\alpha)\xi + \alpha\phi AX + (X\mu)W + \mu\nabla_X W,\]
or using (2.4) and (3.2),
\[(3.9)\]
\[\alpha\mu\nabla_X W = \frac{c}{4}\{u(X)\xi + \eta(X)U\} + \eta(A)A(3AU + \nabla\alpha) + 3g(AU, X)A\xi + \mu(X\alpha)W - \frac{c}{4}\alpha\phi X + \alpha A\phi AX - \alpha^2\phi\xi - (\xi\alpha)AX - \alpha(X\mu)W.\]
From (2.12) and (3.3) we have
\[(3.10)\]
\[\alpha\nabla_\xi U = \frac{3}{4}c\mu W + \alpha^2 A\xi - \alpha\beta\xi + \alpha\phi\nabla\alpha.\]
Putting \(X = \alpha U\) in (3.2) and using (2.4) and (3.3), we get
\[(3.11)\]
\[\alpha^2(\nabla_\xi A)U - \frac{c}{4}(\xi\alpha)U = \frac{c}{4}\alpha\mu^2\xi + \left\{\alpha(U\alpha) - \frac{3}{4}c\mu^2\right\}A\xi.\]
If we put \(X = \alpha\xi\) in (3.5) and make use of (3.10) and (3.11), then we have
\[(3.12)\]
\[\alpha A\phi\nabla\alpha + \frac{c}{4}\phi\nabla\alpha + (U\alpha)A\xi + \mu\left\{\alpha^2 + \frac{3}{4}c\right\}AW - \mu\xi - \frac{1}{\alpha}\left(\mu^2 - \frac{c}{4}\right)W = 0.\]

4. Real hypersurfaces satisfying \(\nabla_\xi R_\xi = 0\) and \(g(U, U)\) is constant

In this section we assume that \(\nabla_\xi R_\xi = 0\) and at the same time \(g(\nabla_\xi \xi, \nabla_\xi \xi)\) is constant, i.e. \(g(U, U) = \mu^2\) is constant. Then we get
\[(4.1)\]
\[\nabla \mu = 0,\]
on \(\Omega\). Note that above equation implies that
\[g(\nabla_X U, U) = 0.\]
If we take a inner product \(U\) to (3.8), then we have
\[(4.2)\]
\[(W\alpha)A\xi = (\xi\alpha)AW + \frac{3}{4\alpha}c\mu U - \mu\nabla\alpha + \alpha A\phi AW,\]
which shows that \(W\alpha = 0\) on \(\Omega\) because of (3.3). Moreover we take a inner product \(W\) to (4.2), then we get
\[(4.3)\]
\[(\xi\alpha)g(AW, W) = 0.\]
Thus, (4.2) turns out to be

\[ \mu \left( \nabla \alpha - \frac{3}{4 \alpha} cU \right) = (\xi \alpha) AW + \alpha A \phi AW, \]

which implies that

\[ \alpha(U \alpha) = \frac{3}{4} c \mu^2 - \frac{c}{4} \alpha \lambda, \]

where we have put \( \lambda = g(AW, W) \).

Putting \( X = U \) in (3.8) and using (3.3), we also find

\[ \alpha^2 \nabla_U U = -\frac{c}{4} (\xi \alpha) \mu W + \left\{ \alpha(U \alpha) - \frac{3}{4} c \mu^2 \right\} U + \frac{c}{4} \alpha \mu \phi AW, \]

or using (4.5)

\[ \alpha \nabla_U U = -\frac{c}{4} \mu (\xi \alpha) W - \frac{c}{4} \mu \lambda U + \frac{c}{4} \mu \phi AW. \]

Hence we have

\[ \alpha^2 A \nabla U U = -\frac{c}{4} \mu (\xi \alpha) AW + \left( \frac{c}{4} \right)^2 \mu \lambda U + \frac{c}{4} \mu A \phi AW. \]

Combining last two equations, it follows that

\[ \alpha^2 A \nabla_U U + \frac{c}{4} \alpha \nabla_U U = -\frac{c}{4} \mu (\xi \alpha) \left( \alpha AW + \frac{c}{4} W \right) + \frac{c}{4} \mu \left( \alpha A \phi AW + \frac{c}{4} \phi AW \right), \]

which together with (3.6) yields

\[ \mu^2 \nabla \alpha - (U \alpha) U = \frac{\mu}{\alpha} (\xi \alpha) \left( \alpha AW + \frac{c}{4} W \right) - \mu \left( \alpha A \phi AW + \frac{c}{4} \phi AW \right). \]

If we take a inner product \( W \) to this and take account of (4.3), then we obtain \( \xi \alpha = 0 \). So we see, using (4.5), that

\[ \mu \nabla \alpha = \left( \frac{3}{4 \alpha} c \mu - \frac{c \lambda}{4 \mu} \right) U - \alpha A \phi AW - \frac{c}{4} \phi AW. \]

Combining this to (4.4), we have

\[ \alpha A \phi AW = -\frac{c}{8} \left( \frac{\lambda}{\mu} U + \phi AW \right) \]

by virtue of \( \xi \alpha = 0 \). Thus, (4.7) turns out to be

\[ \mu \left( \alpha \nabla \alpha - \frac{3}{4} cU \right) = -\frac{c}{8} \left( \alpha \phi AW + \frac{\alpha \lambda}{\mu} U \right). \]

Putting \( X = \xi \) in (3.9) and using (3.3) and (4.9), we obtain

\[ \mu^2 \nabla_{\xi} W = -\frac{\mu \alpha + \frac{c \lambda}{8 \mu}}{\alpha} U - \frac{c}{8} \phi AW, \]
which implies
\[ g(AW, \nabla_\xi W) = 0. \]

We verify, using (3.2) and \( \omega = 0 \), that \( g((\nabla_\xi A)W, W) = 0 \). By the definition of \( \lambda \), we see that
\[ \xi \lambda = g((\nabla_\xi A)W, W) + 2g(AW, \nabla_\xi W). \]
Therefore we obtain \( \xi \lambda = 0 \). Summing up we have

**Lemma 1.** \( \xi \alpha = \xi \lambda = \omega = 0 \) on \( \Omega \).

Applying (4.8) by \( \phi \), we get
\[ \alpha \phi \nabla \phi = \frac{c}{8} (\lambda W + AW - \mu \xi). \]
If we apply (4.7) by \( \phi \) and take account of the last equation, then we obtain
\[ \mu \phi \nabla \alpha = \frac{c}{8} (AW - \mu \xi) + \frac{c}{8} \left( \lambda - \frac{6}{\alpha} \mu^2 \right) W. \]
Accordingly we have
\[ \mu \alpha A \phi \nabla \alpha = \frac{c}{8} (\alpha (A^2 W - \mu A \xi) + \frac{c}{8} (\alpha \lambda - 6 \mu^2) AW. \]
Substituting (4.5), (4.11) and this into (3.12), we obtain
\[ -\frac{c}{8} \alpha A^2 W = \left\{ \mu^2 \left[ \frac{c^2}{2} + \frac{c}{8} \alpha \xi \right] \right\} AW + \mu \left( \frac{3}{4 \alpha} \mu^2 - \frac{c}{4} \lambda - \frac{c}{8} \alpha \right) A \xi \]
\[ -\mu \left\{ \frac{c^2}{32} + \mu^2 \left( \alpha^2 + \frac{3}{4} \right) \right\} \xi \]
\[ -\frac{1}{\alpha} \left\{ \mu^2 \left( \alpha^2 + \frac{3}{4} c \right) \left( \mu^2 - \frac{c}{4} \right) + \frac{3}{16} \mu^2 \right\} W, \]
which implies
\[ \frac{c}{8} \alpha g(A^2 W, W) = \frac{c}{8} \mu^2 \left( \alpha + \frac{3}{2 \alpha} c \right) - \frac{3 c}{4 \alpha} A \xi \]
\[ + \mu^2 \left[ \alpha + \frac{3}{4 \alpha} c \right] \left( \mu^2 - \frac{c}{4} \right) + \lambda \left( \alpha^2 \mu^2 - \frac{1}{4} c \mu^2 + \frac{c^2}{16} \right) - \frac{c}{8} \alpha^2 \lambda^2. \]

If we take account of Lemma 1, we can write the equation (3.8) as
\[ \alpha(\nabla_\xi u)(Y) + (4 \alpha \mu g(AX, W) \eta(Y) - \eta(AX)g(\phi \nabla \alpha, Y) \]
\[ = \frac{c}{4} \{ g(X, Y) - \eta(X) \eta(Y) \} + \frac{3}{4 \alpha} c \mu \eta(AX) w(Y) + (X\alpha) u(Y) - \frac{3}{4 \alpha} cu(X) u(Y) \]
\[ + \alpha^2 g(AX, Y) - \frac{c}{4} \mu \eta(X) w(Y) - \alpha^2 \eta(AX) \eta(Y) + \alpha g(\phi \nabla \alpha AX, Y). \]
From this, we have a Codazzi-type for $u$:

$$
\alpha\{\nabla_X u(Y) - \nabla_Y u(X)\} + \alpha \mu \{ \eta(Y)w(AX) - \eta(X)w(AY) \}
- \eta(AX)g(\phi \nabla \alpha, Y) + \eta(AY)g(\phi \nabla \alpha, X)
= \mu \left( \alpha^2 + \frac{c}{2} \right) (\eta(X)w(Y) - \eta(Y)w(X))
+ (X \alpha)u(Y) - (Y \alpha)u(X)
+ \alpha g((\phi A \phi A - A \phi A \phi)X, Y),
$$

where we have used (2.6). Putting $X = \xi$ in this and using (3.3), (4.10) and Lemma 1, we get

$$
(\nabla_\xi u)(Y) - (\nabla_Y u)(\xi)
= \left( \mu + \frac{c}{8\mu} \right) w(AY) - \left( \mu^2 + \frac{c}{8} \right) \eta(Y) + \left( \mu \alpha + \frac{c\lambda}{8\mu} \right) w(Y).
$$

(4.14)

Applying (4.10) by $\alpha A$ and using (3.3) and (4.8), we have

$$
\alpha \mu^2 A \nabla_\xi W = \frac{c}{4} \left( \mu \alpha + \frac{3\lambda}{16\mu} \right) U + \left( \frac{c}{8} \right)^2 \phi AW,
$$

which shows that

$$
\alpha \mu^2 A \nabla_\xi W = -\frac{c}{4} \left( \alpha \mu^2 + \frac{3}{16} c\lambda \right) W - \left( \frac{c}{8} \right)^2 (AW - \mu \xi).
$$

(4.15)

If we replace $X$ by $W$ in (3.2) and make use of (3.3) and Lemma 1, then we obtain

$$
\alpha (\nabla_\xi A) W = \mu \left( \nabla \alpha - \frac{3}{4\alpha} cU \right),
$$

which together with (4.9) yields

$$
\alpha \phi (\nabla_\xi A) W = \frac{c}{8} (AW - \mu \xi + \lambda W).
$$

(4.16)

5. Lemmas and theorems

We will continue our arguments under the hypotheses $\nabla_\xi R_\xi = 0$ and at the same time $g(U, U)$ is constant. Then (4.9) is rewritted as

$$
\frac{4}{c} \mu^2 (Y \alpha^2) = (6\mu^2 - \alpha \lambda)u(Y) - \mu \alpha g(\phi AW, Y)
$$

(5.1)

for any vector field $Y$. Since $\mu$ is constant, if we differentiate this with respect to a vector field $X$ again, and take the skew-symmetric part for
$X$ and $Y$, then we eventually have
\begin{equation}
0 = Y(\alpha \lambda)u(X) - X(\alpha \lambda)u(Y) + (6\mu^2 - \alpha \lambda)((\nabla_X u)(Y) - (\nabla_Y u)(X)) \\
+ \frac{c}{8} \left( \frac{\lambda}{\mu} - \frac{6\mu}{\alpha} \right) \{u(X)g(\phi AW, Y) - u(Y)g(\phi AW, X)\} \\
+ \mu (g(A^2 W, X)\eta(Y) - g(A^2 W, Y)\eta(X) + g(\phi(\nabla X A)W, X) \\
- g(\phi(\nabla Y A)W, Y) + g(\phi A \nabla Y W, X) - g(\phi A \nabla X W, Y). \}
\end{equation}
Putting $X = \xi$ in this, and using (4.14), (4.15), (4.16) and Lemma 1, we have
\begin{equation}
\mu \alpha A^2 W = \left\{ (6\mu^2 - \alpha \lambda) \left( \mu + \frac{c}{8\mu} \right) - \frac{c}{8} \mu + \frac{1}{\mu} \left( \frac{c}{8} \right)^2 \right\} AW \\
+ \left\{ \alpha \mu^2 (\alpha + \lambda) - (6\mu^2 - \alpha \lambda) \left( \mu^2 + \frac{c}{8} \right) + \frac{c}{8} \mu^2 - \left( \frac{c}{8} \right)^2 \right\} \xi \\
+ \left\{ (6\mu^2 - \alpha \lambda) (\mu + \frac{c}{8} \mu) - \frac{c}{8} \mu \lambda + \frac{c}{4} \alpha \mu + 3 \left( \frac{c}{8} \right)^2 \frac{\lambda}{\mu} \right\} W,
\end{equation}
which implies that
\begin{equation}
\mu \alpha g(A^2 W, W) = (6\mu^2 - \alpha \lambda) \left( \mu \lambda + \mu \alpha + \frac{c\lambda}{4\mu} \right) + \frac{c}{4} \alpha \mu - \frac{c}{4} \mu \lambda + \left( \frac{c}{4} \right)^2 \frac{\lambda}{\mu},
\end{equation}
Combine (4.12) and (5.3), we find
\begin{equation}
\left\{ \alpha^2 \mu^2 - \frac{3}{4} \alpha \mu^2 + \frac{5}{64} \frac{\alpha \lambda}{\mu^2} - \left( \frac{c}{8} \right)^2 \frac{\alpha}{\mu^2} - \left( \frac{c}{8} \right)^3 \frac{1}{\mu^2} \right\} AW = f_1 \xi + f_2 W
\end{equation}
for some smooth functions $f_1$ and $f_2$ on $\Omega$.
Now, we are going to prove the following:

**Lemma 2.** $AW = \mu \xi + \lambda W$ on $\Omega$.

**Proof.** If not, then we have from (5.5)
\begin{equation}
\left( \frac{c}{8} \right)^2 \alpha \lambda = \mu^4 \alpha^2 - \frac{3}{4} \alpha \mu^4 + \frac{5}{64} \alpha^2 \mu^2 - \left( \frac{c}{8} \right)^3.
\end{equation}
If we combine (4.13) to (5.4), then we get
\begin{align*}
\frac{c}{8} \left\{ (6\mu^2 - \alpha \lambda) \left( \mu \lambda + \mu \alpha + \frac{c\lambda}{4\mu} \right) + \frac{c}{4} \alpha \mu - \frac{c}{4} \mu \lambda + \left( \frac{c}{4} \right)^2 \frac{\lambda}{\mu} \right\} \\
= \mu^3 \left( -\alpha \mu^2 + \frac{c}{8} \alpha - \frac{3}{4\alpha} \alpha \mu^2 + \frac{3}{16\alpha} \lambda \mu + \frac{c}{4} \lambda \right) + \frac{c}{8} \mu \left( \alpha \lambda + \frac{c}{2} \right).
\end{align*}
Comparing this with (5.6), we obtain
\[
\left\{ \mu^{10} + \frac{3c}{8} \mu^8 + \left( \frac{c}{8} \right)^2 \mu^6 \right\} \alpha^4 \\
+ \left\{ \frac{12c}{8} \mu^{10} + 32 \left( \frac{c}{8} \right)^2 \mu^8 - 9 \left( \frac{c}{8} \right)^4 \mu^4 - 3 \left( \frac{c}{8} \right)^5 \mu^2 \right\} \alpha^2 \\
+ 42 \left( \frac{c}{8} \right)^3 \mu^8 + 27 \left( \frac{c}{8} \right)^4 \mu^6 - 38 \left( \frac{c}{8} \right)^5 \mu^4 - 29 \left( \frac{c}{8} \right)^6 \mu^2 + 6 \left( \frac{c}{8} \right)^7 = 0,
\]
which implies that \( \alpha \) is a root of the algebraic equation with constant coefficient, because \( \mu \) is constant. Consequently \( \alpha \) is constant and hence \( 3\mu^2 = \alpha \lambda \) by virtue of (4.5). Thus, (4.9) is reduced to
\[\mu \phi A W = \lambda U.\]
So we have \( AW = \mu \xi + \lambda W \), a contradiction. Thus, Lemma 2 is proved. \( \square \)

Using Lemma 2, we have
\[(\nabla_X A) W + A \nabla_X W = \mu \phi A X + (X \lambda) W + \lambda \nabla_X W,\]
which yields
\[g((\nabla_X A) W, Y) + g(A \nabla_X W, Y) = \mu g(\phi A X, Y) + (X \lambda) w(Y) + \lambda g(\nabla_X W, Y).\]
Putting \( Y = W \) in this and using (2.9) and (3.3), we find
\[g((\nabla_X A) W, W) = \frac{c}{2\alpha} \nu(X) + X \lambda,\]
which together with (2.4) implies that
\[(\nabla_W A) W = \frac{c}{2\alpha} U + \nabla \lambda.\]
If we put \( X = W \) in (5.7) and make use of Lemma 2 and the last equation, then we obtain
\[(5.8) \quad A \nabla_W W - \lambda \nabla_W W = \left( \lambda - \frac{c}{2\alpha} \right) U - \nabla \lambda.\]
Indeed, it is, using Lemma 2, seen that \( g(A^2 W, W) = \lambda^2 + \mu^2 \). Hence (4.13) becomes
\[\frac{c}{4} \alpha \lambda^2 - \left( \frac{c}{2} \alpha \lambda - \alpha^2 \mu^2 + \frac{c}{4} \mu^2 - \frac{c^2}{16} \right) \lambda - \alpha \mu^2 \left( \mu^2 - \frac{c}{8} \right) = 0.
\]
From this and Lemma 1 we verify that \( W \lambda = 0 \). Thus, (5.8) is accomplished on \( \Omega \).
Because of Lemma 2, (4.9) turns out to be

$$\mu^2 \alpha \nabla \alpha = \frac{c}{4} (3\mu^2 - \alpha \lambda) U. \quad (5.9)$$

If we take account of (2.6), (5.9), Lemma 1 and Lemma 2, then (3.9) is reduced to

$$\mu^2 \alpha \nabla_X W = -\frac{c}{4} \alpha \phi X + \alpha \phi AX - \alpha^2 \phi AX - \frac{c}{2} u(X) \xi$$

$$\quad + \frac{c}{4\mu} (\mu^2 - \alpha \lambda) \eta(X) U - \frac{c\lambda}{4} (u(X) W + w(X) U). \quad (5.10)$$

Putting $X = W$ in this and using (3.3) and Lemma 2, we have

$$\mu^2 \alpha \nabla_W W = -\left(\frac{c}{2} + \frac{c}{4\alpha} + \alpha^2 \lambda\right) U.$$

Combining this to (5.8), we obtain

$$\alpha \mu^2 \nabla \lambda = \left\{ \alpha \mu^2 \left(\lambda - \frac{c}{2\alpha}\right) - (\lambda + \frac{c}{4\alpha}) \left(\alpha^2 \lambda + \frac{c}{2} \lambda + \frac{c}{4\alpha}\right) \right\} U. \quad (5.11)$$

In the next place, we prove

**Lemma 3.** $\alpha$ and $\lambda$ are constant on $\Omega$.

**Proof.** (5.9) is rewritten as

$$\mu^2 \alpha (Y \alpha) = \frac{c}{4} (3\mu^2 - \alpha \lambda) u(Y) \quad (5.12)$$

for any vector field $Y$. Using the same method as that used to derive (5.2) from (5.1), we can deduce from the last equation the following:

$$Y(\alpha \lambda) u(X) - X(\alpha \lambda) u(Y) + (3\mu^2 - \alpha \lambda)((\nabla_X u)(Y) - (\nabla_Y u)(X)) = 0,$$

which together with Lemma 1 gives

$$(3\mu^2 - \alpha \lambda)((\nabla \xi u)(Y) - (\nabla_Y u)(\xi)) = 0.$$

Now, we suppose that $3\mu^2 - \alpha \lambda \neq 0$ on $\Omega$, and that we restrict the arguments on such a place. Then we have from the last equation

$$g(\nabla \xi U, Y) + g(\nabla_Y \xi, U) = 0.$$

Further, we get from (4.14)

$$\left(\mu + \frac{c}{8\mu}\right) AW - \left(\mu^2 + \frac{c}{8}\right) \xi + \left(\mu \alpha + \frac{c}{8\mu} \lambda\right) W = 0$$

and hence

$$\mu^2 (\lambda + \alpha) + \frac{c}{4} \lambda = 0 \quad (5.13)$$
with the aid of Lemma 2, which implies
\[ (\mu^2 + \frac{c}{4}) \nabla \lambda = -\mu^2 \nabla \alpha. \]

From this and (5.12) we see that
\[ \alpha(\mu^2 + \frac{c}{4}) \nabla \lambda = \frac{c}{4} (\alpha \lambda - 3\mu^2) U, \]

which together with (5.11) yields
\[ \frac{c}{4} \mu^2 (\alpha \lambda - 3\mu^2) \]
\[ = \alpha \mu^2 \left( \mu^2 + \frac{c}{4} \right) \left( \lambda - \frac{c}{2\alpha} \right) - \left( \lambda + \frac{c}{4\alpha} \right) \left( \mu^2 + \frac{c}{4} \right) \left( \alpha^2 \lambda + \frac{c}{2} \lambda + \frac{c}{4} \alpha \right). \]

If we make use of (5.13), then we get
\[ \mu^4 \alpha^4 + \left( \mu^6 - \frac{c^2}{8} \mu^2 \right) \alpha^2 + f(\mu) = 0, \]

where \( f(\mu) \) is certain polynomial with respect to \( \mu \). Since \( \mu \) is constant, the last equation tells us that \( \alpha \) is constant and hence \( 3\mu^2 = \alpha \lambda \) because of (5.12), a contradiction. Therefore, we arrive at the conclusion. \( \square \)

**Lemma 4.** \( \alpha^2 + (3/4)c = 0 \) on \( \Omega \).

**Proof.** Replacing \( X \) by \( U \) in (5.10) and making use of (3.3) and Lemma 2, we find
\[ \alpha \nabla_U W = -\frac{c}{4} \mu \xi. \]

If we take a inner product (5.10) with \( U \) and use (3.3), Lemma 2 and Lemma 3, then we also obtain
\[ \alpha g(\nabla_X W, U) = -\mu \left( \alpha^2 + \frac{3}{4} \right) \eta(X) = \left( \frac{c}{4\alpha} + \frac{c}{2} \alpha + \alpha^2 \lambda \right) w(X). \]

From (5.7) we have a Codazzi-type formula for \( w \):
\[ \lambda((\nabla_X w)(Y) - (\nabla_Y w)(X)) = \frac{c}{4\mu} (u(X)\eta(Y) - u(Y)\eta(X)) \]
\[ + g(A\nabla_X W, Y) - g(A\nabla_Y W, X) - \mu g((\phi A + A\phi)X, Y), \]

where we have used (2.4) and Lemma 3. If we replace \( X \) by \( U \) and take account of (3.3) and (5.15), then we obtain
\[ \left( \lambda + \frac{c}{4\alpha} \right) g(\nabla_X W, U) = -\mu^2 g(AW, X) + \frac{c}{2\alpha} \mu^2 w(X) - \frac{c}{4\alpha} \lambda \mu \eta(X), \]
or make use of (5.16),
\[
\left( \lambda + \frac{c}{4\alpha} \right) \left( \mu \left( \alpha^2 + \frac{3}{4}c \right) \xi + \left( \frac{c}{4} \alpha + \frac{c}{2} \lambda + \alpha^2 \lambda \right) W \right) = \mu^2 \alpha W - \frac{c}{2} \mu^2 W + \frac{c}{4} \lambda \mu \xi.
\]

From this we have
\[
(5.17) \quad \left( \lambda + \frac{c}{4\alpha} \right) \left( \alpha^2 + \frac{3}{4}c \right) - \alpha \mu^2 - \frac{c}{4} \lambda = 0
\]
and
\[
(5.18) \quad \left( \lambda + \frac{c}{4\alpha} \right) \left( \frac{c}{4} \alpha + \frac{c}{2} \lambda + \alpha^2 \lambda \right) + \mu^2 \left( \frac{c}{2} - \alpha \lambda \right) = 0
\]
because of Lemma 2. Since we have
\[
(5.19) \quad 3\mu^2 = \alpha \lambda
\]
by virtue of (5.9) and Lemma 3, we can deduce (5.17) as
\[
\left( \alpha^2 + \frac{3}{4}c \right) \left( 2\mu^2 + \frac{c}{4} \right) = 0.
\]

Now, we suppose that \(2\mu^2 + c/4 = 0\) on \(\Omega\), then we have \(c < 0\).
However combining this and (5.19) to (5.18) we get \(\alpha^2 = c/4\), which
implies that \(c > 0\), a contradiction. Therefore, Lemma 4 is proved. \(\square\)

From (5.19) and Lemma 4, we have
\[
(5.20) \quad \mu^2 \alpha = -\frac{c}{4} \lambda,
\]
which together with (5.18) implies that \(6\lambda = \alpha\). So we see, using (5.20),
that \(6\mu^2 + c/4 = 0\). Developed as above we conclude that \(\mu = 0\) or
\(6\mu^2 + c/4 = 0\) on \(M\) because \(\mu\) is constant. Thus we have

**Proposition 1.** Let \(M\) be a real hypersurface of a complex space
form \(M_{n}(c)\), \(c \neq 0\) which satisfies \(\nabla_{\xi} R_{\xi} = 0\). If \(g(\nabla_{\xi} \xi, \nabla_{\xi} \xi) = \mu^2\) is constant on \(M\), then \(\mu = 0\), that is, \(A\xi = \alpha \xi\) or \(6\mu^2 + c/4 = 0\).

If \(6\mu^2 + c/4 \neq 0\) holds on Proposition 1, then we have \(A\xi = \alpha \xi\)
on whole space \(M\). So we verify that \(\alpha\) is constant on \(M\) (see [8]).
Thus it follows from (3.2) that \(\alpha \nabla_{\xi} A = 0\). Consequently, we see that
\(\alpha(A\phi - \phi A) = 0\) by virtue of (2.1) and (2.4).

Here, we note the case \(\alpha = 0\) corresponds to the case of tube of radius \(\pi/4\)
in \(P_{n}\mathbb{C}\) (see [2]). But, in the case of \(H_{n}\mathbb{C}\) it is known that \(\alpha\) never
vanishes for Hopf hypersurfaces (cf. [1]). Thus, owing to Okumura's
work for \(P_{n}\mathbb{C}\) or Montiel and Romero's work for \(H_{n}\mathbb{C}\) we have
Theorem 2. Let $M$ be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$ which satisfies $\mu^2 = g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant and $6\mu^2 + c/4 \neq 0$. Then $M$ holds $\nabla_\xi R_\xi = 0$ if and only if $M$ is locally congruent to one of the following:

(I) in case that $M_n(c) = P_n \mathbb{C}$ with $\eta(A\xi) \neq 0$,

$(A_1)$ a geodesic hypersphere of radius $r$, where $0 < r < \pi/2$ and $r \neq \pi/4$,

$(A_2)$ a tube of radius $r$ over a totally geodesic $P_k \mathbb{C}(1 \leq k \leq n-2)$, where $0 < r < \pi/2$ and $r \neq \pi/4$;

(II) in case that $M_n(c) = H_n \mathbb{C}$,

$(A_0)$ a horosphere,

$(A_1)$ a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1} \mathbb{C}$,

$(A_2)$ a tube over a totally geodesic $H_k \mathbb{C}(1 \leq k \leq n-2)$.

Corollary 3. Let $M$ be a real hypersurface of a complex projective space $P_n \mathbb{C}$ which satisfies $\mu^2 = g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant. Then $M$ holds $\nabla_\xi R_\xi = 0$ if and only if $M$ is locally congruent to one of the following:

$(A_1)$ a geodesic hypersphere of radius $r$, where $0 < r < \pi/2$ and $r \neq \pi/4$,

$(A_2)$ a tube of radius $r$ over a totally geodesic $P_k \mathbb{C}(1 \leq k \leq n-2)$, where $0 < r < \pi/2$ and $r \neq \pi/4$;

provided that $\eta(A\xi) \neq 0$.

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