SOME PROPERTIES OF PARALLEL SURFACES IN EUCLIDEAN 3-SPACES

DAE WON YOON

Abstract. In this paper, we study some properties about the parallel surfaces of ruled surfaces in a Euclidean 3-space. Furthermore, we classify the parallel surfaces of ruled surfaces in a Euclidean 3-space satisfying a linear type and a quadric type with respect to the Gaussian curvature and the mean curvature.

1. Introduction

Let $f$ and $g$ be smooth functions on a surface $M$ in a Euclidean 3-space. The Jacobi function $\Phi(f, g)$ formed with $f, g$ is defined by

$$\Phi(f, g) = \det \begin{pmatrix} f_s & f_t \\ g_s & g_t \end{pmatrix}$$

where $f_s = \frac{\partial f}{\partial s}$ and $f_t = \frac{\partial f}{\partial t}$. In particular, a surface satisfying the Jacobi condition $\Phi(K, H) = 0$ with respect to the Gaussian curvature $K$ and the mean curvature $H$ on a surface $M$ is called a Weingarten surface or a W-surface. The classification of the Weingarten surfaces in a Euclidean space is almost completely open today. These surfaces were introduced by the very Weingarten in the context of the problem of finding all surfaces isometric to a given surface of revolution ([cf. 8]). Also, if a surface satisfies a linear type with respect to $K$ and $H$, that is, $aK + bH = c$ $(a, b, c \in \mathbb{R})$, then it is said to be a linear Weingarten surface and we abbreviate it by LW-surface. The first examples of LW-surfaces are those with constant mean curvature($a = 0$) and those with constant Gaussian curvature($b = 0$).

Although these two kinds of surfaces have been extensively studied in the literature, the classification of LW-surfaces in the general case is almost completely open today. Several geometors ([3,4,7,8,12,13,14,16])

Received June 23, 2008. Accepted October 8, 2008.

2000 Mathematics Subject Classification: 53A05, 53B25.

Key words and phrases: Gaussian curvature, mean curvature, ruled surface, parallel surface, Weingarten surface, linear Weingarten surface.
have studied \( W \)-surfaces and \( LW \)-surfaces and obtained many interesting results. Recently, N.G. Kim and D.W. Yoon ([6]) studied the ruled surfaces in a Euclidean 3-space satisfying the quadric type with respect to the Gaussian curvature, the mean curvature and the second mean curvature. The second mean curvature is the mean curvature of non-degenerate second fundamental form of a surface. Also, Y.H. Kim and D.W. Yoon ([5]) investigated the ruled surfaces in a Minkowski 3-space satisfying the quadric type with respect to the Gaussian curvature, the mean curvature and the second Gaussian curvature. The second Gaussian curvature is that of the non-degenerate second fundamental form of a surface.

In this paper, we will study the parallel surface of a ruled surface in a Euclidean 3-space \( \mathbb{R}^3 \) satisfying the conditions

\[
(1.1) \quad a\overline{K} + b\overline{H} = c, \quad b \neq 0,
\]

\[
(1.2) \quad a\overline{K}^2 + b\overline{K}\overline{H} + c\overline{H}^2 = d, \quad b^2 - 4ac > 0,
\]

where \( a, b, c, d \) are constants, and \( \overline{K}, \overline{H} \) the Gaussian curvature and the mean curvature of the parallel surface of a ruled surface. If a surface satisfies the equation (1.2), then a surface is said to be \( \overline{K} \overline{H} \)-quadric.

On the other hand, the parallel surfaces of a cylindrical ruled surface are ruled surfaces, but the parallel surfaces of a non-cylindrical ruled surface cannot be ruled surfaces ([10]).

Throughout this paper, we assume that all objects are smooth and all surfaces are connected unless stated otherwise.

2. Preliminaries

A surface \( \overline{M} \) whose points are at a constant distance along the normal from another surface \( M \) is said to be parallel to \( M \). So, there are infinite numbers of parallel surfaces because we choose the constant distance along the normal arbitrarily. From the definition it follows that a parallel surface can be regarded as the locus of points which are on the normals to \( M \) at a non-zero constant distance \( \lambda \) from \( M \).

First, we obtain the representation of points on \( \overline{M} \) using the representations of points on \( M \).

Let \( x \) be the position vector of a point \( P \) on \( M \) and \( \mathbf{X} \) be the position vector of a point \( \overline{P} \) on the parallel surface \( \overline{M} \). Then \( \overline{P} \) is at a constant distance \( \lambda \) from \( P \) along the normal to the surface \( M \). Therefore the
parametrization for $\overline{M}$ is given by

\begin{equation}
\overline{x}(s, t) = x(s, t) + \lambda n(s, t),
\end{equation}

where $\lambda$ is a constant scalar and $n$ is the unit normal vector field on $M$.

Let $I, II, K, H$ be the first fundamental, the second fundamental form, the Gaussian curvature and the mean curvature of $M$, respectively, and let $\overline{I}, \overline{II}, \overline{K}, \overline{H}$ be the corresponding ones for $\overline{M}$. With the parametrization for a parallel surface, the following proposition holds.

**Proposition 2.1** ([cf. 11]). Let $\overline{M}$ be a parallel surface of a surface $M$ in a Euclidean 3-space. Then we have

1. $\overline{I} = (1 - \lambda^2 K) I - 2\lambda(1 - \lambda H) II,$
2. $\overline{II} = \lambda K I + (1 - 2\lambda H) II,$
3. $\overline{K} = \frac{K}{1 - 2\lambda H + \lambda^2 K},$
4. $\overline{H} = \frac{H - \lambda K}{1 - 2\lambda H + \lambda^2 K}.$

From Proposition 2.1, differentiating $\overline{K}$ and $\overline{H}$ with respect to $s$ and $t$ respectively, we get

\begin{equation}
\begin{aligned}
(\overline{K})_s &= \frac{1}{(1 - 2\lambda H + \lambda^2 K)^2} (K_s - 2\lambda K_s H + 2\lambda KH_s), \\
(\overline{K})_t &= \frac{1}{(1 - 2\lambda H + \lambda^2 K)^2} (K_t - 2\lambda K_t H + 2\lambda KH_t), \\
(\overline{H})_s &= \frac{1}{(1 - 2\lambda H + \lambda^2 K)^2} (H_s - \lambda^2 KH_s - \lambda K_s + \lambda^2 HK_s), \\
(\overline{H})_t &= \frac{1}{(1 - 2\lambda H + \lambda^2 K)^2} (H_t - \lambda^2 KH_t - \lambda K_t + \lambda^2 HK_t),
\end{aligned}
\end{equation}

which imply the Jacobian function of the Gaussian curvature $\overline{K}$ and the mean curvature $\overline{H}$ is given by

\[ \Phi(\overline{K}, \overline{H}) = \frac{1}{1 - 2\lambda H + \lambda^2 K} (K_s H_t - K_t H_s) \]

\[ = \frac{1}{1 - 2\lambda H + \lambda^2 K} \Phi(K, H). \]

From the relationship of the above Jacobian function, we have thus the following theorem.

**Theorem 2.2.** Let $\overline{M}$ be a parallel surface of a surface $M$ in a Euclidean 3-space. If $\overline{M}$ is a Weingarten surface if and only if $M$ is a Weingarten surface.
3. Main Results

In this section, we study the parallel surface $\bar{M}$ of a ruled surface in a Euclidean 3-space $\mathbb{R}^3$ which satisfies a linear Weingarten equation (1.1) and a quadric equation (1.2) with respect to the Gaussian curvature $\bar{K}$ and the mean curvature $\bar{H}$ of the parallel surface $\bar{M}$. It is well known that a cylindrical ruled surface is developable, i.e., the Gaussian curvature $\bar{K}$ is identically zero. Therefore, from (3) of Proposition 2.1 $\bar{K}$ is identically zero. Thus, non-cylindrical ruled surfaces are meaningful for our study.

Let $M$ be a non-cylindrical ruled surface in $\mathbb{R}^3$. Then the parametrization for $M$ is given by

$$x = x(s, t) = \alpha(s) + t\beta(s),$$

where $(\beta, \beta) = 1$, $(\beta', \beta') = 1$ and $(\alpha', \beta') = 0$. In this case $\alpha$ is the striction curve of $x$, and the parameter $s$ is the arc-length on the spherical curve $\beta$. And we have the natural frame $\{x_s, x_t\}$ given by $x_s = \alpha' + t\beta'$ and $x_t = \beta$. Then, the components of the first fundamental form of $M$ are given by $E = \langle \alpha', \alpha' \rangle + t^2$, $F = \langle \alpha', \beta \rangle$, $G = 1$. We put $D = \sqrt{EG - F^2}$. In terms of the orthonormal frame $\{\beta, \beta', \beta \times \beta'\}$ we obtain

$$\alpha' = F \beta + Q \beta \times \beta', \quad \beta'' = -\beta - J \beta \times \beta', \quad \alpha' \times \beta = Q \beta',$$

where $Q = \langle \alpha', \beta \times \beta' \rangle$, $J = \langle \beta'', \beta \times \beta \rangle$. Thus, we get $D = \sqrt{Q^2 + t^2}$, from which the unit normal vector $n$ of $M$ is written as

$$n = \frac{1}{D} (\alpha' \times \beta + t\beta' \times \beta) = \frac{1}{D} (Q\beta' - t\beta \times \beta').$$

This leads to the components $e$, $f$ and $g$ of the second fundamental form of $M$

$$e = \frac{1}{D} (Q(F + QJ) - Q't + Jt^2), \quad f = \frac{Q}{D}, \quad g = 0.$$

Therefore, using the data described above, the Gaussian curvature $K$ and the mean curvature $H$ of $M$ are given respectively by

$$K = -\frac{Q^2}{D^4}, \quad H = \frac{1}{2D^3} A,$$

where we put $A = Jt^2 - Q't + Q(QJ - F)$. Differentiating $K$ and $H$ with respect to $t$ respectively, we get

$$K_t = \frac{4Q^2}{D^5} t, \quad H_t = \frac{1}{2D^3} B,$$

where we put $B = -Jt^3 + 2Q't^2 + Q(-QJ + 3F)t - Q^2$. 
Let $\overline{M}$ be a parallel surface of a non-cylindrical ruled surface $M$ in \(\mathbb{R}^3\). Then, by the definition of a parallel surface, the parametrization for $\overline{M}$ is given by

$$\overline{x}(s, t) = x(s, t) + \lambda n(s, t).$$

Suppose that a parallel surface $\overline{M}$ in $\mathbb{R}^3$ is a linear Weingarten surface. Then by (1.1), (2.2), (3.2) and (3.3) we have

\[(3.4)\]

$$(-8a\lambda Q^2 At - 2a\lambda Q^2 B + bB D^4 + b\lambda^2 Q^2 B + 4b\lambda^2 Q^2 At)^2 = 64(a - b\lambda)^2 A^4 t^2 D^6.$$

From the functions $D$, $A$ and $B$ the equation (3.4) becomes the polynomial in $t$ whose coefficients are functions of variable $s$. Then, by the coefficient of the highest order $t^{14}$, we have $b^2 J^2 = 0$, from which $J = 0$ because of $b \neq 0$. Therefore, the functions $A$ and $B$ can be rewritten in the form

\[(3.5)\]

$$A = -Q' t - QF, \quad B = 2Q' t^2 + 3QF t - Q^2 Q'.$$

By (3.5) and the coefficient of $t^{12}$ of (3.4), we have $4b^2 Q'^2 = 0$, from which $Q' = 0$. In this case the coefficient of $t^{10}$ of (3.4) is given by $9b^2 Q^2 F = 0$, which implies $F = 0$ because the ruled surface $M$ is non-developable, that is, $Q \neq 0$. Thus, from (3.2) $M$ is minimal, that is, it is a helicoid. On the other hand, the coefficients of $t^8, t^6, t^4$ and $t^2$ are given as follows:

$$t^8 : -64Q^{10}(a - b\lambda)^2 = 0, \quad t^6 : -192Q^8(a - b\lambda)^2 = 0,$$

$$t^4 : -192Q^6(a - b\lambda)^2 = 0, \quad t^2 : -64Q^4(a - b\lambda)^2 = 0.$$

From which we have $\lambda = \frac{a}{b}$.

Thus, we have

**Theorem 3.1.** Let $\overline{M}$ be a parallel surface of non-cylindrical ruled surface $M$ in a Euclidean 3-space. If $\overline{M}$ is a linear Weingarten surface satisfying $a\overline{K} + b\overline{H} = c$ ($a, b \neq 0, c \in \mathbb{R}$). Then $\overline{M}$ is parametrized by

$$\overline{x}(s, t) = (s \cos t, s \sin t, ht) + \frac{a}{b\sqrt{t^2 + h^2}}(h \sin t, -h \cos t, s), \quad h \neq 0.$$

Next, we consider parallel surfaces of non-cylindrical ruled surfaces satisfying the condition (1.2).

**Theorem 3.2.** Let $\overline{M}$ be a parallel surface of non-cylindrical ruled surface $M$ in a Euclidean 3-space and let $a, b, c, d$ be constants such that $b^2 - 4ac > 0$ and $c \neq 0$. If $\overline{M}$ is a $\overline{K} \overline{H}$-quadric surface satisfying
\[ aK^2 + bKK + cH^2 = d. \] Then \( M \) is parametrized by
\[
\mathbf{r}(s, t) = (s \cos t, s \sin t, h t) + \frac{b \pm \sqrt{b^2 - 4ac}}{2c\sqrt{t^2 + h^2}}(h \sin t, -h \cos t, s), \quad h \neq 0.
\]

**Proof.** Let \( M \) be a parallel surface of a non-cylindrical ruled surface \( \mathbf{x}(s, t) = \alpha(s) + t\beta(s) \) in \( \mathbb{R}^3 \). Then the parametrization for \( M \) is given by
\[
\mathbf{r}(s, t) = \mathbf{r}(s) + t\mathbf{b}(s) + \lambda \mathbf{n}(s, t),
\]
where \( \langle \mathbf{b}, \mathbf{b} \rangle = 1, \langle \mathbf{b}', \mathbf{b}' \rangle = 1 \) and \( \langle \alpha', \beta' \rangle = 0 \).

Suppose that a parallel surface \( \overline{M} \) is \( \overline{K} \overline{H} \)-quadric. Then, by using (2.2) the equation (1.2) implies
\[
(-16aQ^4tD^2 - 4b\lambda Q^2tA^2D^4 - b\lambda Q^2ABD^4 + 16b\lambda Q^4tD^2
\]
\[+ cAB + 4c\lambda^2Q^2tA^2D^4 - 16c\lambda^2Q^4tD^2)^2
\]
\[(3.6) \quad = (16a\lambda Q^4tAD^4 + 4a\lambda Q^4BD^4 + 4bQ^2tA - 8b\lambda^2Q^4tAD^4
\]
\[- 3b\lambda^2Q^2BD^4 - bQ^2B - 4b\lambda^2Q^4tAD^4 + c\lambda^2Q^2ABD^4
\]
\[ - 8c\lambda Q^2tA + 2c\lambda^2Q^3B + 2c\lambda^3Q^4BD^4 + 8c\lambda^3Q^4tAD^4)^2 D^2.
\]

From the functions \( D, A \) and \( B \) the equation (3.6) becomes the polynomial in \( t \) whose coefficients are functions of variable \( s \). Then, by the coefficient of the highest order \( t^{20} \), we have \( c^2\lambda^2Q^4t^{14} = 0 \), from which \( J = 0 \) because \( c \neq 0 \). In this case, by the coefficient of \( t^{16} \) of the equation (3.6), we have \( 4\lambda^2Q^4t^8 = 0 \), which implies \( Q = 0 \). From \( J = Q' = 0 \), the coefficient of \( t^{12} \) of the equation (3.6) is given by \( 12c^2\lambda^2Q^6F^3 = 0 \), so we have \( F = 0 \). Thus, the surface \( M \) is minimal by (3.2). In this case, we can show that the other coefficients of the equation (3.6) are given as follows:
\[
256Q^8(a - b\lambda + c\lambda^2)^2 = 0, \quad 512Q^{10}(a - b\lambda + c\lambda^2)^2 = 0,
\]
\[
256Q^{12}(a - b\lambda + c\lambda^2)^2 = 0,
\]
which imply \( a - b\lambda + c\lambda^2 = 0 \), that is, \( \lambda = \frac{b \pm \sqrt{b^2 - 4ac}}{2c} \). This completes the proof. \( \square \)

**References**

Figure 1. Surfaces parallel to a helicoid with $\lambda = 0, 0.5, 1, 2, 4$.

Dae Won Yoon
Department of Mathematics Education and RINS,
Gyeongsang National University,
Jinju 660-701, Korea
E-mail: dwyon@gnu.ac.kr