EXTREMAL CASES OF SN-MATRICES

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Abstract. We denote by $\mathcal{Q}(A)$ the set of all real matrices with the same sign pattern as a real matrix $A$. A matrix $A$ is an SN-matrix provided there exists a set $\mathcal{S}$ of sign patterns such that the set of sign patterns of vectors in the null-space of $A$ is $\mathcal{S}$, for each $A \in \mathcal{Q}(A)$. Some properties of SN-matrices are investigated.

1. Introduction

The sign of a real number $a$ is defined by

$$\text{sign}(a) = \begin{cases} 
-1 & \text{if } a < 0, \\
0 & \text{if } a = 0, \\
1 & \text{if } a > 0.
\end{cases}$$

The sign pattern of a real matrix $A$ is the $(0,1,-1)$-matrix obtained from $A$ by replacing each entry by its sign. We denote by $\mathcal{Q}(A)$ the set of all real matrices with the same sign pattern as $A$. The zero pattern of a matrix $A$ is the $(0,1)$ matrix obtained from $A$ by replacing each nonzero entry by 1.

A vector is mixed if it has a positive entry and a negative entry. A matrix is row-mixed if each of its rows is mixed. A vector is balanced if it is the zero vector or is mixed. The notion of a row-balanced matrix is defined analogously. A signing is a nonzero, diagonal $(0,1,-1)$-matrix. A signing is strict if each of its diagonal entries is nonzero. A matrix $B$ is strictly row-mixable provided there exists a strict signing $D$ such that $BD$ is row-mixed.

Let $A$ be an $m$ by $n$ matrix and $b$ an $m$ by 1 vector. The linear system $Ax = b$ has signed solutions provided there exists a collection $\mathcal{S}$ of $n$ by 1 sign patterns such that the set of sign patterns of the solutions to $Ax = b$...
is $S$, for each $\tilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$. This notion generalizes that of a sign-solvable linear system (see [1] and references therein). The linear system, $Ax = b$, is sign-solvable provided each linear system $\tilde{A}x = \tilde{b}$ ($\tilde{A} \in \mathcal{Q}(A)$, $\tilde{b} \in \mathcal{Q}(b)$) has a solution and all solutions have the same sign pattern. Thus, $Ax = b$ is sign-solvable if and only if $Ax = b$ has signed solutions and the set $S$ has cardinality 1.

A matrix $A$ is an SN-matrix provided $Ax = 0$ has signed solutions. Thus, $A$ is an SN-matrix if and only if there exists a set $S$ of sign patterns such that the set of sign patterns of vectors in the null-space of $A$ is $S$, for each $\tilde{A} \in \mathcal{Q}(A)$. An L-matrix is a matrix, $A$, with the property that each matrix in $\mathcal{Q}(A)$ has linearly independent rows. A square L-matrix is a sign-nonsingular, or SNS-matrix for short. A totally L-matrix is an $m \times n$ matrix such that each $m$ by $m$ submatrix is an SNS-matrix. It is known that totally L-matrices are SN-matrices[2].

Some properties of SN-matrices have been studied in [2, 3, 4, 5]. In [6] we proved that if a strictly row-mixable $m$ by $n$ SN-matrix is not conformally contractible, then it is permutation equivalent to

$$\begin{bmatrix}
I_k & B \\
O & C
\end{bmatrix}$$

where $2 \leq k \leq m$.

In this paper, considering a strictly row-mixable $m$ by $n$ SN-matrix of the form in (1) we find the range of $n$ and characterize the matrices satisfying the extremal cases of $n$ for $k = m$.

We use the following standard notations throughout the paper. If $k$ is a positive integer, then $\langle k \rangle$ denotes the set $\{1, 2, \ldots, k\}$. Let $A$ be an $m \times n$ matrix. If $\alpha$ is a subset of $\{1, 2, \ldots, m\}$ and $\beta$ is a subset of $\{1, 2, \ldots, n\}$, then $A[\alpha|\beta]$ denotes the submatrix of $A$ determined by the rows whose indices are in $\alpha$ and the columns whose indices are in $\beta$. The submatrix complementary to $A[\alpha|\beta]$ is denoted by $A(\alpha|\beta)$. In particular, $A(\alpha|-)$ denotes the submatrix obtained from $A$ by deleting the rows whose indices are in $\alpha$. Let $J_{m,n}$ denote the $m$ by $n$ matrix all of whose entries are 1 and let $e_i$ denote the column vector all of whose entries are 0 except for the $i$th entry which is 1. $O$ denotes a zero matrix.

2. Main Results

Let $A$ be an $m$ by $n$ $(0, 1, -1)$-matrix. The matrix $B$ is conformally contractible to $A$ provided there exists an index $k$ such that the rows
and columns of $B$ can be permuted so that $B$ is $m+1$ by $n+1$ matrix of the form

$$\begin{bmatrix}
A[(m)][(n) \setminus \{k\}] & x & y \\
0 & \cdots & 0 & 1 & -1
\end{bmatrix},$$

where $x = [x_1, \ldots, x_m]^T$ and $y = [y_1, \ldots, y_m]^T$ are $(0, 1, -1)$ vectors such that $x_i y_i \geq 0$ for $i = 1, 2, \ldots, m$, and the sign pattern of $x+y$ is the $k$th column of $A$. In this case we say that the zero pattern of $A$ is obtained from the zero pattern of $B$ by a contraction. More precisely, let $A = [a_{ij}]$ be an $m$ by $n$ $(0, 1)$-matrix such that the row $p$ of $A$ contains exactly two 1’s, say $a_{pr} = a_{ps} = 1$ whenever $r \neq s$. Let $u$ be the $m$ by 1 $(0, 1)$ column vector whose $i$th entry is 1 if and only if $a_{ir} = 1$ or $a_{is} = 1$. Let $B$ be the $m-1$ by $n-1$ matrix obtained from $A$ by replacing column $s$ by $u$ and then deleting row $p$ and column $r$. We say that $B$ is the matrix obtained from $A$ by the contraction of columns $r$ and $s$ on row $p$. It is known that if $B$ is conformally contractible to $A$, then $A$ is an $SN$-matrix if and only if $B$ is an $SN$-matrix[2].

It is easy to show that if an $SN$-matrix has two nonzero columns which are identical up to multiplication by $-1$, then the columns have exactly one nonzero entry. Whenever we consider a matrix $A$ of the form in (1), we may assume that $A$ is a strictly row-mixable $SN$-matrix which is not conformally contractible to a matrix, and each column of $A$ is distinct. That is, each column of $\begin{bmatrix} B \\ C \end{bmatrix}$ has at least two nonzero entries.

Let $A = (a_1, \ldots, a_s) = (c_1, \ldots, c_m)^T$ be an $m$ by $s$ matrix and $B = (b_1, \ldots, b_t) = (d_1, \ldots, d_n)^T$ an $n$ by $t$ matrix. Write $A \square B$ as

$$\begin{bmatrix}
a_1 & \cdots & a_{s-1} & a_s & 0 & \cdots & 0 \\
0 & \cdots & 0 & b_1 & b_2 & \cdots & b_t
\end{bmatrix}$$

and $A \diamond B$ as

$$\begin{bmatrix}
c_1 \\
\vdots & O \\
c_m & d_1 \\
O & \vdots \\
d_n
\end{bmatrix}.$$

Then $A \square B$ is an $m+n$ by $s+t-1$ matrix and $A \diamond B$ is an $m+n-1$ by $s+t$ matrix.
Now we want to investigate an $m$ by $n \ (m < n)$ matrix $A$ of the form in (1) with $k = 2$ or $k = m$. Let $\sigma(A)$ be the number of nonzero entries of $A$.

Let $k = 2$. Since $A$ is an SN-matrix and every row of $A$ has at least three nonzero entries, $n \geq m + 2$. The equality holds if and only if $A$ is a totally $L$-matrix (see proposition 2 in [6]). In this case $A$ can be obtained from
\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & -1 & 0 & 1
\end{bmatrix}
\]
by a sequence of single extensions up to row and column permutations and multiplication of rows and columns by $-1$ (for definition see p.88 in [1]). What are the maximum value $\xi$ of $n$ and the matrices corresponding to $\xi$? Let $J_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Let $H = J_2 \cdot \cdots \cdot J_2$ and $A = [e_1 \ H \ e_m]$. It is easy to show that $A$ is a matrix of the form in (1) with $k = 2$. Hence $\xi \geq 2m$.

At present we cannot find the value $\xi$ but we conjecture that $\xi = 2m$ for $m \geq 2$ and a matrix corresponding to $\xi = 2m$ is permutation equivalent to the matrix $A = [e_1 \ H \ e_m]$ up to multiplication of rows and columns by $-1$.

Let $\lfloor a \rfloor$ denote the smallest integer no less than $a$. Let $\mathcal{M}_m$ be the set of all $m$ by $\lfloor \frac{m}{2} \rfloor + 1$ (0, 1)-matrices defined inductively as follows:

For $m = 2$, let $\mathcal{M}_2 = \{J_{2,2}\}$. For any even number $m( \geq 4)$, $M_m \in \mathcal{M}_m$ if and only if $M_m$ is permutation equivalent to
\[
\begin{bmatrix}
M_{m-2} & O \\
C & 1
\end{bmatrix},
\]
where $M_{m-2} \in \mathcal{M}_{m-2}$, and $C$ has a column $(1,1)^T$ and other columns are all zero.

For odd number $m$, $M_m \in \mathcal{M}_m$ if and only if every row of $M_m$ has exactly two nonzero entries and every column of $M_m$ has at least two nonzero entries, and there exists a row $i$ of $M_m$ such that the contraction of $M_m$ on the row $i$ is contained in $\mathcal{M}_{m-1}$. Thus $\mathcal{M}_3$ is the set of all matrices which are permutation equivalent to
\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]

and \(M_m \in \mathcal{M}_m(m \geq 5)\) if and only if \(M_m\) is permutation equivalent to one of the following matrices

\[
\begin{bmatrix}
M' & O \\
a & b \\
O & M''
\end{bmatrix}
\]

(2)

where \(M' \in \mathcal{M}_k\), \(M'' \in \mathcal{M}_{m-k-1}\) for some even number \(k\) and the vectors \(a\) and \(b\) have exactly one nonzero entry respectively, or

\[
\begin{bmatrix}
M' & O & O \\
S & 1 & T \\
O & O & M''
\end{bmatrix}
\]

(3)

where \(M' \in \mathcal{M}_k\), \(M'' \in \mathcal{M}_{m-k-3}\) for some even number \(k\), \(S\) and \(T\) have columns \((1,0,1)^T\) and \((0,1,1)^T\) respectively and other columns are zero.

Proposition 2.4 in [6] states that if \(A\) is strictly row-mixable \(m\) by \(n\) \(SN\)-matrix with no duplicate columns up to multiplication by \(-1\) and every row has at least three non-zero elements, then \(A\) has at least two rows with exactly three nonzero entries. Using this property we can obtain the range of \(\sigma(A)\) for a matrix \(A\) of the form in (1) with \(k = m\) and we can characterize the matrices in the extremal cases of \(\sigma(A)\). Let \(\mathcal{N}_m\) be the set of all \(m\) by \(2m - 2\) matrices \(B\) with \(B^T \in \mathcal{M}_{2m-2}\).

**Proposition 1.** Let \(A\) be a matrix of the form in (1) with \(k = m(m \geq 2)\). Then \(3m \leq \sigma(A) \leq 5m - 4\). Moreover, \(\sigma(A) = 5m - 4\) if and only if the zero pattern of the matrix obtained from \(A\) by deleting the identity submatrix \(I_m\) is contained in \(\mathcal{N}_m\).

**Proof.** Since every row of \(A\) has at least three nonzero entries, \(3m \leq \sigma(A)\). By the remark mentioned above, we may assume that \(A\) is of the
form

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & \cdots & 0 \\
0 \\
\vdots \\
0 & B
\end{bmatrix}
\]  

(4)

Notice that all columns of $B(-1,2)$ are distinct. If all columns of $B$ are distinct, then $\sigma(B) \leq 5(m-1) - 4$ by induction hypothesis. Hence $\sigma(A) = \sigma(B) + 3 \leq 5(m-1) - 4 + 3 < 5m - 4$. Let $B$ have duplicate columns up to multiplication by $-1$. Since such columns have only one nonzero entry, the zero pattern of $A$ is permutation equivalent to one of the following matrices

\[
\begin{bmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 \\
0 & 0 \\
\vdots \\
0 & 0 \\
& B'
\end{bmatrix}
\]  

(5)

\[
\begin{bmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 \\
0 & 0 \\
\vdots \\
0 & 0 \\
& B'
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 \\
0 & 0 \\
\vdots \\
0 & 0 \\
& B'
\end{bmatrix}
\]  

(6)

where the submatrix $B'$ has all distinct columns. It is easy to show that $\sigma(B') \leq 5(m-1) - 4$. Since $\sigma(B) \leq \sigma(B') + 2$, $\sigma(A) = \sigma(B) + 3 \leq \sigma(B') + 2 + 3 \leq 5(m-1) - 4 + 5 = 5m - 4$.

Let $A$ be a matrix of the form in (1) with $\sigma(A) = 5m - 4$. If the zero pattern of $A$ is of the form in (5), then $\sigma(B') = 5m - 8 > 5(m-1) - 4$, which is impossible. Hence the zero pattern of $A$ should be one of the matrices in (6). Consider a matrix $A$ of the second form in (6). Suppose that a matrix of the second form in (6) is the zero pattern of $A$. If the number of nonzero entries in the first row or the second row of $B'$ is 2, it easy to show that $\sigma(B') < 5(m-1) - 4$ and hence $\sigma(A) < 5m - 4$. Thus each row of $B'$ has at least three nonzero entries. Therefore the matrix obtained from $B'$ by deleting the columns corresponding to the identity submatrix $I_{m-1}$ is contained in $N_{m-1}$ by induction hypothesis.
Then $J_{2,2}$ is the zero pattern of a submatrix of the conformal contraction on the first row of $A(-|1)$. By Theorem B in [6] $A$ is not $SN$-matrix, which is a contradiction. Next consider a matrix of the first form in (6).

If the number of nonzero entries in the first row of $B'$ is 1 or 2, it is also easy to show that $\sigma(B') < 5(m-1)-4$. Thus each row of $B'$ has at least three nonzero entries. Since $\sigma(B') = 5(m-1)-4$, the matrix obtained from $B'$ by deleting the columns corresponding to the identity submatrix $I_{m-1}$ is contained in $N_{m-1}$ by induction hypothesis. Clearly the matrix obtained from the zero pattern of $A$ by deleting the columns corresponding to $I_m$ is contained in $N_m$.  

Lemma 2. Let $A$ be an $m$ by $n$ $(0,1)$-matrix such that each row of $A$ has exactly two nonzero entries ($m \geq 3$). If $n \leq \left\lfloor \frac{m}{2} \right\rfloor$, then there exists a matrix $B$ obtained from $A$ by a finite sequence of contractions on rows such that $J_{3,2}$ is a submatrix of $B$.

Proof. We will prove it by induction on $m$. For $m = 3, 4$, there is nothing to prove. Let $m \geq 5$. If $A$ has $J_{2,2}$ as a submatrix, we may assume that $A$ is of the form

$$A = [a_{ij}] = \begin{bmatrix} A_1 & A_2 \\ O & J_{2,2} \end{bmatrix}.$$ 

Suppose that $J_{3,2}$ is not a submatrix of $A$. Then $a_{i,n-1} \cdot a_{in} = 0$ for all $i \in \{1, 2, \cdots, m-2\}$. Let $B$ be the contraction of $A$ on the row $m$ and let $B'$ be the matrix obtained from $B$ by deleting the last row. Then $B'$ is an $m-2$ by $n-1$ matrix such that each row of $B'$ has exactly two nonzero entries. Since $n \leq \left\lfloor \frac{m}{2} \right\rfloor$, $n-1 \leq \left\lfloor \frac{m-1}{2} \right\rfloor = \left\lfloor \frac{m-2}{2} \right\rfloor$. By induction there exists a matrix $C$ obtained from $B'$ by a finite sequence of contractions such that $J_{3,2}$ is a submatrix of $C$. Hence $J_{3,2}$ is a submatrix of a matrix obtained from $A$ by a finite sequence of contractions. This is a contradiction.

If $A$ does not have $J_{2,2}$ as a submatrix, we may assume that $A = [a_{ij}]$ is one of the forms

$$\begin{bmatrix} A_1 & O \\ O & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} A_1 & A_2 \\ E & 0 \\ 1 & 1 \end{bmatrix}$$

where $E$ has only one nonzero entry in the first row and $a_{i,n-1} \cdot a_{in} = 0$ for all $i \in \{1, 2, \cdots, m-2\}$. In the former case, clearly $A_1$ satisfies the hypothesis. By induction $A_1$ and hence $A$ satisfies the result. In the latter case, let $B$ be the contraction of $A$ on row $m$. Then $B$ is an $m-1$
by \( n - 1 \) matrix such that each row of \( B \) has exactly two nonzero entries and \( n - 1 = \left\lfloor \frac{m-2}{2} \right\rfloor \leq \left\lfloor \frac{m-1}{2} \right\rfloor \). By induction, \( B \) and hence \( A \) satisfies the result. \( \square \)

**Proposition 3.** Let \( A \) be an \( m \) by \( n \) matrix containing \( I_m \) as a submatrix such that each row of \( A \) has exactly three nonzero entries and \( m \geq 2 \). If \( A \) is an \( SN \)-matrix, then \( n \geq \left\lceil \frac{m}{2} \right\rceil + m + 1 \).

**Proof.** The result is clear for \( m = 2 \). Let \( m \geq 3 \). Suppose that \( n < \left\lceil \frac{m}{2} \right\rceil + m + 1 \). Without loss of generality we may assume that \( A = [I_m \ B] \). Then \( B \) is \( m \) by \( n - m \) matrix such that each row of \( B \) has exactly two nonzero entries and \( n - m \leq \left\lceil \frac{m}{2} \right\rceil \). By Lemma 2, there exists a matrix \( B \) obtained from \( A \) by a finite sequence of contractions on rows such that \( J_{3,2} \) is a submatrix of \( B \). Since \( A \) is an \( SN \)-matrix if and only if a conformal contraction of the matrix \( A \) is an \( SN \)-matrix, \( A \) is not an \( SN \)-matrix by Theorem B in [6]. Hence we have the result. \( \square \)

In the following we can get matrices on which equality in Proposition 3 holds.

**Proposition 4.** Let \( A = [I_m \ B] \) be an \( m \) by \( n \) \( SN \)-matrix with no duplicate columns up to multiplication by \(-1\) such that each row of \( A \) has exactly three nonzero entries, then \( n = \left\lceil \frac{m}{2} \right\rceil + m + 1 \) if and only if the zero pattern of \( B \) is in \( \mathcal{M}_m \).

**Proof.** (*Sufficiency.*) It is clear.

(*Necessity.*) We will prove it by induction on \( m \). Since \( A \) is an \( SN \)-matrix with no duplicate columns up to multiplication by \(-1\), each column of the matrix \( B \) has at least two nonzero entries. Let \( Z(X) \) denote the zero pattern of a matrix \( X \). It is easy to show that \( Z(B) \) is \( J_{2,2} \) if \( m = 2 \), and \( Z(B) \) is permutation equivalent to

\[
\begin{bmatrix}
    1 & 1 & 0 \\
    0 & 1 & 1 \\
    1 & 0 & 1
\end{bmatrix}
\]

if \( m = 3 \). Thus we are done for \( m = 2, 3 \). Let \( m \geq 4 \). First, let \( Z(B) \) have no submatrix of the form \( J_{2,2} \). Without loss of generality, we may assume that \( Z(B) \) is of the form

\[
\begin{bmatrix}
    1 & 1 & 0 & \cdots & 0 \\
    0 & 1 & 1 & \cdots & 0 \\
    1 & 0 & & & \\
    * & * & & &
\end{bmatrix}
\]
Let $C$ be the $m - 1$ by $\left\lfloor \frac{m}{2} \right\rfloor$ matrix obtained from $Z(B)$ by contraction on the first row. Then each row of $C$ has exactly two nonzero entries. If $m$ is even, the matrix $A$ is not an $SN$-matrix by Lemma 2. Hence $m$ is odd and by induction hypothesis $C \in \mathcal{M}_{m-1}$. Thus there exist permutation matrices $P$, $Q$ such that $PZ(B)Q$ has a submatrix of the form

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1
\end{bmatrix}
\]

since $m \geq 5$. This is impossible since $A$ is an $SN$-matrix.

Next, let $Z(B)$ have $J_{2,2}$ as a submatrix. Without loss of generality, we may assume that $Z(B)[1,2][1,2] = J_{2,2}$. Suppose that $B[3, \ldots, m][1,2] = O$. Then $[I_{m-2} B(1,2)[1,2]]$ is an $m - 2$ by $\left\lfloor \frac{m}{2} \right\rfloor + m - 3$ matrix each row of which has exactly three nonzero entries. Clearly $[I_{m-2} B(1,2)[1,2]]$ is an $SN$-matrix. Hence the total number of columns of $[I_{m-2} B(1,2)[1,2]]$ is no less than $\left\lfloor \frac{m}{2} \right\rfloor + m - 2$ by Proposition 3. This is a contradiction. Thus $B[3, \ldots, m][1,2] \neq O$.

Let $\sigma(B[3, \ldots, m][1,2]) = 1$. Then we may assume that $Z(B)$ is of the form

\[
\begin{bmatrix}
1 & 1 & O \\
1 & 1 \\
0 & 1 & 10 \cdots 0 \\
O & M
\end{bmatrix}
\]

Let $D$ be the matrix obtained from $Z(B)$ by the contraction on the third row. Then each column of the matrix $D$ has at least two nonzero entries and each row of $D$ has exactly two nonzero entries. Clearly $[I_{m-2} D]$ is the zero pattern of an $m - 1$ by $\left\lfloor \frac{m}{2} \right\rfloor + m - 1$ $SN$-matrix. By Proposition 3, we have $\left\lfloor \frac{m}{2} \right\rfloor \geq \left\lfloor \frac{m-1}{2} \right\rfloor + 1$. This implies that $m$ is odd and $\left\lfloor \frac{m}{2} \right\rfloor = \left\lfloor \frac{m-1}{2} \right\rfloor + 1$. Hence $D \in \mathcal{M}_{m-1}$ by induction hypothesis and $M \in \mathcal{M}_{m-3}$. Thus $Z(B) \in \mathcal{M}_m$.

Let $\sigma(B[3, \ldots, m][1,2]) \geq 2$. Let $B'$ be the matrix obtained from $B$ by the conformal contraction on the first row and then by deleting the first row. Then $[I_{m-2} B']$ is satisfies the hypothesis. If $m$ is even, then we may assume that $Z(B)$ is either
By contracting on the third row in the first matrix, we have $J_{3,2}$ as its submatrix. This implies that $A$ is not an $SN$-matrix by Theorem B in [6]. It is impossible. Hence $Z(B)$ is of the form in the second matrix. It is contained in $M_m$.

Let $m$ be odd. Then $Z(B')$ is one of matrices in (2) or (3) where $M' \in M_p$ and $M'' \in M_q$ for some even numbers $p$, $q$ and one of $M'$, $M''$ in (3) may be vacuous.

If $Z(B')$ is a matrix of the form in (2), then $Z(B)$ has a submatrix which is permutation equivalent to one of the matrices in (7). The first matrix in (7) also does not occur and hence $Z(B) \in M_m$. Let $Z(B')$ be a matrix of the form in (3). If $M'$ is not vacuous, it is easy to show that $Z(B) \in M_m$ by the similar method above. If $M'$ is vacuous, then $Z(B)$ has a submatrix which is permutation equivalent to

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1
\end{bmatrix}
\] or
\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1
\end{bmatrix}.
\]

The first matrix in (8) does not occur and hence $Z(B) \in M_m$. Thus we have the result. \hfill \Box

Let

\[
B = \begin{bmatrix}
B_1 & B_2 \\
B_3 & 1
\end{bmatrix}
\] and

\[
C = \begin{bmatrix}
1 & C_1 \\
C_2 & C_3
\end{bmatrix}.
\]

Let $B \ast C$ denote

\[
\begin{bmatrix}
B_1 & B_2 & O \\
B_3 & 1 & C_1 \\
O & C_2 & C_3
\end{bmatrix}.
\]

**Proposition 5.** Let $m > 1$ be an integer. For any $s$ with $3m \leq s \leq 3m - 4$, there exists an $m$ by $n$ matrix $A$ in (1) with $\sigma(A) = s$ and $k = m$. 
Proof. First of all, we show that there exists such a matrix $A$ with
\[
\sigma(A) = s \text{ for } 4m - 2 \leq s \leq 5m - 4. \text{ Let } A_t = (J_2 \cdots J_2) \otimes (J_2 \cdots J_2)
\]
where $1 \leq t \leq m - 1$. Then $[I_m \ A_t]$ satisfies the conditions in (1) and
$\sigma([I_m \ A_t]) = 5m - t - 3$.

Let $m$ be any even integer. If $m = 2$, then $A = [I_2 \ J_2]$ is an $SN$-matrix
with $\sigma(A) = s(3m \leq s \leq 4m - 2)$. Let $m \geq 4$. Let $B = J_2 \otimes \cdots \otimes J_2$.
Then $A = [I_m \ B]$ is an $SN$-matrix with $\sigma(A) = 3m$. Consider a column $c$ of $B$ with at least three nonzero entries. We choose a nonzero entry $a$ of $c$. Then the matrix $B$ has the unique submatrix $J_2$ which contains $a$. Let $C$ be the unique submatrix $J_2$. Let $d$ be the column vector obtained from $c$ by replacing every nonzero entry which is not an entry of $C$ with 0. Let $B_1$ be the matrix obtained from $B$ by replacing $a$ with 0 and by adding the column $d$ as the last column. Then the matrix $A_1 = [I_m \ B_1]$ is an $SN$-matrix with $\sigma(A_1) = 3m + 1$. Applying the process above to the matrix $B_1$, we choose a matrix $B_2$ which is an $SN$-matrix with $\sigma(A_2) = 3m + 2$. We can continue the process until every column has exactly two nonzero entries. That is, we can find matrices $B_1, B_2, \ldots, B_{m-2}$. Let $A_i = [I_m \ B_i]$ for $i = 1, 2, \ldots, m - 2$. Then $A_i$ is
an $SN$-matrix with $\sigma(A_i) = 3m + i$ for $i = 1, 2, \ldots, m - 2$. Thus we have the result.

Let $m$ be any odd integer. First, if $m = 3$, then $A = [I_3 \ B_3]$ is an
$SN$-matrix with $\sigma(A) = s(3m \leq s < 4m - 2)$ where
\[
B_3 = \begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
1 & 0 & 1
\end{bmatrix}.
\]

For $m \geq 5$, let $B = J_2 \otimes J_1 \otimes J_3 \otimes \cdots \otimes J_2$. Then the matrix $A = \[I_m \ B]$ is an $SN$-matrix with $\sigma(A) = 3m$. By the similar method shown in the case of even $m$, we can find matrices $B_1, B_2, \ldots, B_{m-3}$ such that
$A_i = [I_m \ B_i]$ is an $SN$-matrix with $\sigma(A_i) = 3m + i$ for $i = 1, 2, \ldots, m - 3$. Thus we have the result.

\[
\square
\]

References


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