PROPERTIES OF DUAL RIESZ-NÁGY-TAKÁCS DISTRIBUTIONS

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Abstract. We give the relation between the Riemann-Stieltjes integrals with respect to the Riesz-Nágy-Takács(RNT) distribution $H_{a,p}$ satisfying the equation $a = (1-a)^m$ and those with respect to the dual RNT distribution $H_{1-a,1-p}$, which leads to a generalization of recent results for $a = \frac{1}{2}$.

1. Introduction

Recently moments of the Cantor distribution([7, 8, 9]) were investigated. More recently we([3]) also studied the moments of the RNT distribution([10]) which is a strictly increasing singular function using the so-called $(\tau, \tau-1)$-expansion([10]) of the unit interval. We note that the unit interval $[0, 1]$ is the attractor of the iterated function system of some similarities with the open set condition([1, 4, 5, 6]). Further we([2]) also discussed the Riemann-Stieltjes integrals with respect to the RNT distribution $H_{a,p}$([3]) satisfying the equation $1-a = a^m$ where $m$ is a positive integer over different intervals $[0, 1-a], [a, 1]$ and $[0, 1]$ omitting $[1-a, 1]$. We note that the Riemann-Stieltjes integral with respect to the RNT distribution over $[0,a]$ was also shown in [3]. In this paper, we study the relation between the Riemann-Stieltjes integrals with respect to the RNT distribution $H_{a,p}$ satisfying the equation $a = (1-a)^m$ and the Riemann-Stieltjes integrals with respect to $H_{1-a,1-p}$ over $[0,a], [1-a,1], [0,1-a]$ and $[0,1]$. This comparison gives the properties of the Riemann-Stieltjes integrals with respect to the RNT distribution $H_{a,p}$ satisfying the equation $a = (1-a)^m$ over $[1-a,1], [0,a], [a,1]$ and $[0,1]$ omitting $[0,1-a]$. This gives a generalization of recent results([7]) for $a = \frac{1}{2}$.

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2. Preliminaries

Let \( \mathbb{N} \) be the set of the positive integers. Consider \( a \in (0, 1) \) and \( p \in (0, 1) \). We([2, 3]) recall the RNT distribution

\[
H_{a,p}(x) = \sum_{j=1}^{\infty} \frac{(1-p)^{j-1}}{p^{j-1}} x^j
\]

for

\[
x = \sum_{j=1}^{\infty} \frac{(1-a)^{j-1}}{a^{j-1}} a^j \in (0, 1]
\]

with integers \( 1 \leq a_1 < a_2 < \cdots < a_j < \cdots \) and \( H_{a,p}(0) = 0 \). We note that if \( a \neq p \) then it is a singular function([4, 10]) whereas it is the identity function if \( a = p \).

3. Main results

We define \( F(x) = H_{a,p}(x) \) and \( G(x) = H_{1-a,1-p}(x) \). From now on, we consider the Riemann-Stieltjes integral \( \int_{\alpha}^{\beta} \phi(t) dF(t) \) of a continuous function \( \phi \) with respect to \( F \) over \( [\alpha, \beta] \subset [0, 1] \) and \( \int_{\alpha}^{\beta} \phi(t) dG(t) \) similarly. We introduce the following propositions related to the RNT distribution \( H_{1-a,1-p} \).

**Proposition 1.** ([3]) \( G((1-a)x) = (1-p)G(x) \) for \( x \in [0, 1] \).

**Proposition 2.** ([3]) \( G((1-a) + ax) = (1-p) + p G(x) \) for \( x \in [0, 1] \).

From now on, we assume that \( a = (1-a)^m \) for some \( m \in \mathbb{N} \). In this case, we note that \( G(1-a) = 1-p \). We give some scaling and translation properties already obtained of the \((\tau, \tau-1)\)-expansion of the unit interval.

**Theorem 3.** ([2]) \( G(ax) = (1-p)^m G(x) \) for \( x \in [0, 1] \).

**Theorem 4.** ([2]) \( G((1-a) + x) = (1-p) + \frac{p}{1-p^m} G(x) \) for \( x \in [0, a] \).

Using the scaling and translation properties of the \((\tau, \tau-1)\)-expansion of the unit interval, we had the following Riemann-Stieltjes integrals over different intervals.

**Theorem 5.** ([2])

\[
\int_{0}^{a} \phi(t) dG(t) = (1-p)^m \int_{0}^{1} \phi(at) dG(t).
\]
Theorem 6. ([2])

\[ \int_{1-a}^{1} \phi(t) dG(t) = \frac{p}{(1-p)m} \int_{0}^{a} \phi((1-a) + t) dG(t). \]

We give some relation between two dual distributions.

Proposition 7.

\[ F(y) - F(x) = G(1 - x) - G(1 - y) \]

for \( 0 \leq x < y \leq 1. \)

Proof. It is clear to see that

\[ F(x) = 1 - G(1 - x) \]

from the dual graphs. \( \square \)

Using the above property, we get the following relation between the Riemann-Stieltjes integrals with respect to two dual distributions.

Theorem 8.

\[ \int_{0}^{1} \phi(t) dF(t) = \int_{0}^{1} \phi(1 - t) dG(t). \]

Proof. Consider a partition

\[ t_0 = 0 < t_1 = \frac{1}{n} < t_2 = \frac{2}{n} < \ldots < t_n = 1 \]

of \([0, 1].\) Putting \( s_i = 1 - t_i, \) we have a partition

\[ s_n = 0 < s_{n-1} = \frac{1}{n} < s_{n-2} = \frac{2}{n} < \ldots < s_1 = 1 - \frac{1}{n} < s_0 = 1 \]

of \([0, 1].\) From Proposition 3, we have

\[ \int_{0}^{1} \phi(t) dF(t) = \lim_{n \to \infty} \sum_{i=1}^{n} \phi(t_i) [F(t_i) - F(t_{i-1})] \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \phi(t_i) [G(1 - t_{i-1}) - G(1 - t_i)] \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \phi(1 - s_i) [G(s_{i-1}) - G(s_i)] \]

\[ = \int_{0}^{1} \phi(1 - s) dG(s). \]

\( \square \)
Similarly we obtain the following relations between the Riemann-Stieltjes integrals with respect to two dual distributions $F$ and $G$.

**Theorem 9.**

\[
\int_{1-\alpha}^{1} \phi(t) dF(t) = \int_{0}^{\alpha} \phi(1-t) dG(t) = (1-p)^m \int_{0}^{1} \phi(1-at) dG(t).
\]

**Proof.** Using similar arguments in the proof of Theorem 8, we get

\[
\int_{1-\alpha}^{1} \phi(t) dF(t) = \int_{0}^{\alpha} \phi(1-t) dG(t).
\]

It follows from Theorem 5. \hfill \Box

**Theorem 10.**

\[
\int_{0}^{\alpha} \phi(t) dF(t) = \int_{1-\alpha}^{1} \phi(1-t) dG(t) = \frac{p}{(1-p)^m} \int_{0}^{\alpha} \phi(a-t) dG(t).
\]

**Proof.** Using similar arguments in the proof of Theorem 8, we get

\[
\int_{0}^{\alpha} \phi(t) dF(t) = \int_{1-\alpha}^{1} \phi(1-t) dG(t).
\]

It follows from Theorem 6. Precisely it follows from

\[
\int_{1-\alpha}^{1} \phi(1-t) dG(t) = \frac{p}{(1-p)^m} \int_{0}^{\alpha} \phi(1-t-(1-\alpha)) dG(t).
\]

\hfill \Box

**Theorem 11.**

\[
\int_{0}^{1} \phi(t) dF(t) = \int_{0}^{1-\alpha} \phi(1-t) dG(t) = (1-p) \int_{0}^{1} \phi(1-(1-\alpha)t) dG(t).
\]

**Proof.** Using similar arguments in the proof of Theorem 8, we get

\[
\int_{0}^{1} \phi(t) dF(t) = \int_{0}^{1-\alpha} \phi(1-t) dG(t).
\]

It follows from Theorem 6. Precisely it follows from

\[
\int_{0}^{1-\alpha} \phi(t) dG(t) = (1-p) \int_{0}^{1} \phi((1-\alpha)t) dG(t).
\]

\hfill \Box

We obtain the following relations between the Riemann-Stieltjes integrals with respect to $F$ over $[0,1]$ and different intervals.
Corollary 12.
\[ \int_{1-a}^{1} \phi(t) dF(t) = (1 - p)^m \int_{0}^{1} \phi(at + (1 - a)) dF(t). \]

**Proof.** It follows from Theorem 9 and Theorem 8 with
\[ \phi(1 - a(1 - t)) = \phi(at + (1 - a)). \]

\[ \square \]

We give proofs of the following two Corollaries for their relations between our Theorems, even though they were shown already.

**Corollary 13.** ([3])
\[ \int_{0}^{a} \phi(t) dF(t) = p \int_{0}^{1} \phi(at) dF(t). \]

**Proof.** From Theorem 10,
\[ \int_{0}^{a} \phi(t) dF(t) = \frac{p}{(1 - p)^m} \int_{0}^{a} \phi(a - t) dG(t). \]

Further from Theorem 5
\[ \int_{0}^{a} \phi(a - t) dG(t) = (1 - p)^m \int_{0}^{1} \phi(a - at) dG(t). \]

From Theorem 8,
\[ \int_{0}^{1} \phi(a - at) dG(t) = \int_{0}^{1} \phi(a - a(1 - t)) dF(t). \]

\[ \square \]

**Corollary 14.** ([3])
\[ \int_{a}^{1} \phi(t) dF(t) = (1 - p) \int_{0}^{1} \phi((1 - a)t + a) dF(t). \]

**Proof.** It follows from Theorems 11 and 8, that is,
\[ (1 - p) \int_{0}^{1} \phi(1 - (1 - a)t) dG(t) = (1 - p) \int_{0}^{1} \phi((1 - a)t + a) dF(t). \]

\[ \square \]
References


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