GROUND STATE MASS CONCENTRATION IN THE $L^2$-CRITICAL NONLINEAR HARTREE EQUATION BELOW $H^1$

MYEONGJU CHAE

Abstract. We consider finite time blowup solutions of the $L^2$-critical focusing Hartree equation on $\mathbb{R}^n$, $n \geq 3$ below $H^1$.

1. Introduction

In this paper we study the initial value problem of the $L^2$-critical focusing Hartree equation,

$$
\begin{cases}
    i\partial_t \phi + \frac{1}{2} \Delta \phi = -(|x|^{-2} * |\phi|^2)\phi, & x \in \mathbb{R}^n, \ t > 0, \\
    \phi(x, 0) = \phi_0(x) \in H^s(\mathbb{R}^n).
\end{cases}
$$

Here $H^s(\mathbb{R}^n)$ denotes the usual inhomogeneous Sobolev space. (1.1) is meaningful in dimension $n \geq 3$, where the Hartree potential is locally integrable. Hartree type equation arises in atomic and nuclear physics and is related to the mean-field theory with respect to wave functions describing boson systems. ([15], [26])

The local well-posedness results for $s \geq 0$ can be shown by the Strichartz estimates similarly as in the mass critical NLS with the polynomial nonlinearity $|\phi|^{\frac{4}{n}} \phi$. For $s > 0$ (1.1), which is the case we consider, is locally well-posed in the subcritical sense. More precisely, for any $\phi_0 \in H^s(\mathbb{R}^n)$, the lifetime span of the solution depends on the norm of the initial data, $||\phi_0||_{H^s}$. Whereas, for $s = 0$ the lifetime depends on the profile of the initial data as well.

The classical solutions to (1.1) enjoy the mass conservation law,

$$
||\phi(\cdot, t)||_{L^2(\mathbb{R}^n)} = ||\phi_0(\cdot)||_{L^2(\mathbb{R}^n)},
$$

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and the energy conservation law,

\begin{equation}
E[t] := \int_{\mathbb{R}^n} |\nabla \phi|^2 + (|x|^{-2} + |\phi|^2)|\phi|^2 dx.
\end{equation}

When $s \geq 1$, the energy conservation law (1.2) together with the subcritical local theory immediately yields the global well-posedness for defocusing case. When $0 \leq s < 1$, the energy could be infinite and the mass conservation law cannot imply the global well-posedness. In this case the gwp results is extended to $\frac{2(n-2)}{3n-4} < s < 1$ in [4] employing the $I$-method and the interaction Morawetz inequality developed in [10], and later in [8, 13] modified for mass critical situation. Also the small data theory asserts that sufficiently small initial $L^2$ data leads to a unique global $L^2$ solution regardless with the focusing/defocusing signs. However, blowup may occur in focusing case for large data.

The equation (1.1) is known to have a ground state solution $Q$, which solves

\[ \Delta Q - Q = -(|x|^{-2} * |Q|^2)Q. \]

The existence of $Q$ is proven in [23] with the decisive property of being the sharp constant of the Gagliardo-Nirenberg inequality such as

\[ \int_{\mathbb{R}^n} (|x|^{-2} + |u|^2)|u|^2(x)dx \leq \frac{2}{\|Q\|_{L^2}^2} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2. \]

The uniqueness is open except $n=4$ [29]. The mass concentration phenomenon of $H^s$- blow up solution is considered for the $L^2$-critical cubic NLS on $\mathbb{R}^2$ in [12] and [20]. Combining the argument in [12, 20] and the almost conservation law established in [4], we can reach the similar conclusion for Hartree type (1.1). We state the main theorem as follows.

**Theorem 1.1.** There exist $0 < s_Q < 1$ such that the following is true for any $s_Q < s < 1$. Let $\phi_0 \in H^s(\mathbb{R}^n)$ be such that the corresponding solution $\phi$ of (1.1) blows up in finite time $T^* > 0$ and let $\lambda(t) > 0$ such that $\frac{(T^*-t)^2}{\lambda(t)} \to 0$ as $t \to T^*$. Then there exists $x(t) \in \mathbb{R}^n$ such that

\[ \limsup_{t \to T^*} \int_{|x-x(t)| \leq \lambda(t)} |\phi(t,x)|^2 dx \geq \int Q^2. \]

The number $s_Q = \frac{1}{4} + \frac{\sqrt{185}}{20} = 0.9\ldots$ arises from the almost conservation law for the modified energy. For $L^2$ critical cubic case, the almost conservation law ((3.25) in [12]) reads

\[ \sup_{t \in [0,T_{\text{lwp}}]} |E[I_N \phi(t)]| \leq |E[I_N \phi_0]| + CN^{-\alpha_4} \|I_N(D)\phi_0\|_{L^2}^4 + CN^{-\alpha_6} \|I_N(D)\phi_0\|_{L^2}^6 \]
with \( \alpha_4 = \frac{3}{2} - \) and \( \alpha_6 = 2 - \). In Hartree case, we have the similar estimate with \( \alpha_4 = \alpha_6 = 1 - \) as seen in Proposition 3.1 in Section 3. The slow decay rate in Hartree case is partly due that the bilinear estimate used in [12] seems not to make a meaningful gain in higher dimension greater than 2, while \( L^2 \)-critical focusing Hartee equation makes sense in \( n \geq 3 \). The paper is organized as follows.

**Notations**

Given \( A, B \), we write \( A \lesssim B \) to mean that for some universal constant \( K > 2 \), \( A \leq K \cdot B \). We write \( A \sim B \) when both \( A \lesssim B \) and \( B \lesssim A \).

The notation \( A \ll B \) denotes \( B > 3 \cdot A \). We write \( \langle A \rangle \equiv (1 + A^2)^{1/2} \), and \( \langle \nabla \rangle \) for the operator with Fourier multiplier \( (1 + |\xi|^2)^{1/2} \). The symbol \( \nabla \) denote the spatial gradient. We will often use the notation \( \frac{1}{2} + \equiv \frac{1}{2} + \epsilon \) for some universal \( 0 < \epsilon \ll 1 \). Similarly, we write \( \frac{1}{2} - \equiv \frac{1}{2} - \epsilon \).

We use the function space \( L^q_t L^r_x \) and \( H^{s,p} \) given norms by

\[
\|F\|_{L^q_t L^r_x(\mathbb{R}^{n+1})} \equiv \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} |F(x,t)|^r \, dx \right)^{\frac{q}{r}} \, dt \right)^{\frac{1}{q}},
\]

\[
\|u\|_{H^{s,p}(\mathbb{R}^n)} \equiv \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{\frac{n}{2}} \mathcal{F} u]\|_{L^p(\mathbb{R}^n)},
\]

where \( \mathcal{F} \) is a fourier transform, \( 1 \leq p, q, r \leq \infty \).

**2. The local well-posedness**

We refer \( (q, r) \) the admissible pair when \( 2 \leq q \leq \infty, 2 \leq r \leq \frac{2n}{n-2} \) and

\[
\frac{2}{q} + \frac{n}{r} = \frac{n}{2},
\]

and state the Strichartz inequality in dimension \( n \).

**Proposition 2.1.** Suppose that \( (q, r) \), \( (\lambda, \eta) \) are any two admissible pairs. Suppose that \( u(x, t) \) is a solution of the problem

\[
(2.3) \quad i\partial_t u(x, t) + \Delta u(x, t) = F(x, t), \quad (x, t) \in \mathbb{R}^n \times [0, T],
\]

for a data \( u(0) \in H^s \), \( F \in L^\lambda_t H^\eta_x([0, T] \times \mathbb{R}^n)) \) where \( \lambda \) and \( \eta \) are the Hölder conjugates of \( \lambda \) and \( \eta \), respectively. Then \( u \) belongs to \( L^\lambda_t H^{s, r}_x([0, T] \times \mathbb{R}^n) \cap C_t H^{s, r}_x([0, T] \times \mathbb{R}^3) \) and we have the estimate

\[
\|u\|_{L^\lambda_t H^{s, r}_x([0, T] \times \mathbb{R}^n)} \lesssim \|u(0)\|_{H^s(\mathbb{R}^n)} + \|F\|_{L^\lambda_t H^{\eta, r}_x([0, T] \times \mathbb{R}^n)}.
\]
For the pure power nonlinearity $\lambda |u|^\alpha u$, the local well-posedness of
$i\partial_t u + \frac{1}{2} \Delta u = \lambda |u|^\alpha u$ with the rough data $u(0) \in H^s$, $0 < s < 1$ was
proven in [2] See also [3, 27].

We define the Strichartz norm of functions $\phi : [0, T] \times \mathbb{R}^n \to \mathbb{C}$ by
$$
\| \phi \|_{S_T^q} = \sup_{(q,r) \text{admissible}} \| \phi \|_{L^q_t L^r_x([0,T] \times \mathbb{R}^n)}.
$$

In particular $S_T^q \subset C_t L^2_x([0,T] \times \mathbb{R}^n)$. Then the Strihartz estimates may
be written as
$$
\| \phi \|_{S_T^q} \leq \| \phi \|_{L^2} + \| (i\partial_t + \Delta) \phi \|_{L^q_t L^r_x([0,T] \times \mathbb{R}^n)},
$$
where $(q, r)$ is any admissible pair. We define
$$
\| \phi \|_{S_T^s} = \sup_{(q,r) \text{admissible}} \| \langle \nabla \rangle^s \phi \|_{L^q_t L^r_x([0,T] \times \mathbb{R}^n)}.
$$

The local existence theorem of (1.1) is as follows.

**Theorem 2.1.** For a given $\phi_0 \in H^s(\mathbb{R}^n)$, $0 < s$, there exists a
positive time $T_{lwp} = T(\| \phi_0 \|_{H^s})$ and the unique solution $\phi$ of (1.1), in
$\phi \in C_t H^s([0, T_{lwp}] \times \mathbb{R}^n) \cap S_T^s$ for every admissible pair $(q, r)$ with

$$
T_{lwp} = c_0 \| \phi_0 \|_{H^s}^2
$$

for a constant $c_0$ and

$$
\| \phi \|_{S_{T_{lwp}}^s} \leq 2 \| \phi_0 \|_{H^s}.
$$

**Proof.** Let $S^L(t)$ be the flow map $e^{it\Delta}$ corresponding to the the linear
Schrödinger equation. Then the integral formulation of (1.1) is
$$
\phi(t) = S^L(t)\phi_0 - i \int_0^t S^L(t - \tau)V \ast |\phi|^2 \phi(\tau) d\tau.
$$

We will show that the map $A : \phi \to S^L(t)\phi_0 + \int_0^t S^L(t - \tau)(|x|^{-2} * |\phi|^2)\phi(\tau) d\tau$ is a contraction mapping on the ball $\| \phi \|_{S_T^s} \leq 2M$ when $T$
is chosen later and $\| \phi_0 \|_{H^s} < M$.

Let us show $A$ is well defined on $X$. Applying the linear and the dual
Strichartz estimates, we have

$$
\| A \phi \|_{S_T^s} \lesssim \| \phi_0 \|_{H^s} + \| |x|^{-2} * |\phi|^2 \phi \|_{L^2_t H^{2,\eta'}}([0,T] \times \mathbb{R}^n)}
$$

for any admissible $(\lambda, \eta)$. We recall the Leibnitz rule for fractional
Sobolev spaces [7]: For $s > 0$, $1 < p < \infty$,
$$
\| fg \|_{H^{s,p}} \lesssim \| f \|_{L^{r_1}} \| g \|_{H^{s_1, q_2}} + \| f \|_{L^{r_1}} \| g \|_{H^{s_2, r_2}}
$$
provided $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2}$, with $q_2, r_2 \in (1, \infty)$ and $q_1, r_1 \in (1, \infty]$.

Let us choose $(\lambda', \eta') = (\frac{4}{3+\xi}, \frac{2n}{n-\alpha+1})$. The fractional Leibnitz rule, Hardy-Sobolev and Hölder’s inequalities lead to

\[
\langle |(|x|^{-2} * |\phi|^2)\phi \rangle_{H^{s,\frac{2n}{n-\alpha+1}}} \\
\leq \|(|x|^{-2} * |\phi|^2)\|_{H^{s,\frac{2n}{n-\alpha+1}}} + \|(|x|^{-2} * |\phi|^2)\|_{L^{\frac{2n}{n-\alpha+1}}} \|\phi\|_{H^{s,\frac{2n}{n-\alpha+1}}} \\
\lesssim \|\phi\|^2_{H^{s,\frac{2n}{n-\alpha+1}}} \|\phi\|_{L^{\frac{2n}{n-\alpha+1}}} + \|\phi\|^2_{L^{\frac{2n}{n-\alpha+1}}} \|\phi\|_{H^{s,\frac{2n}{n-\alpha+1}}} \\
\lesssim 2\|\phi\|^2_{L^{\frac{2n}{n-\alpha+1}}} \|\phi\|_{H^{s,\frac{2n}{n-\alpha+1}}}.
\]

By use of the Sobolev embedding $\|\phi\|_{L^{\frac{2n}{n-\alpha+1}}} \lesssim \|\phi\|_{H^{s,\frac{2n}{n-\alpha+1}}}$, we have

\[
\|(|x|^{-2} * |\phi|^2)\phi \|_{H^{s,\frac{2n}{n-\alpha+1}}} \lesssim \|\phi\|^3_{H^{s,\frac{2n}{n-\alpha+1}}}.
\]

Combining this with (2.6) we find

\[
\|A\phi\|_{S_T^s} \lesssim \|\phi_0\|_{H^s} + \left(\int_0^T \|\phi\|_{H^{s,\frac{2n}{n-\alpha+1}}} \frac{dt}{T^{\frac{3+s}{4}}} \right)^{\frac{3+s}{4}} \\
\lesssim \|\phi_0\|_{H^s} + T^s \|\phi\|^3_{L^{\frac{4}{n-\alpha+1}}H^{\frac{2n}{n-\alpha+1}}(\times\mathbb{R}^n)} \\
\lesssim \|\phi_0\|_{H^s} + T^s \|\phi\|^3_{S_T^s}.
\]

We choose $T$ as $T \lesssim \|\phi_0\|_{H^s}^{\frac{2}{s}}$ so as to $\|A\phi\|_{S_T^s} \lesssim \|\phi_0\|_{H^s}$. It can be similarly argued that $A$ is a contraction. In particular the local solution $\phi$ satisfies as long as $T \leq T_{\text{lap}} = c_0 \|\phi_0\|_{H^s}^{\frac{2}{s}}$.

\[
\|\phi\|_{S_T^s} \leq 2\|\phi_0\|_{H^s}
\]

The uniqueness assertion follows in the similar way as Proposition 4.2 of [2]. \(\square\)

Let us define the smoothing operator $I_N$, which sends an $H^s$ function to an $H^1$ function.

\[
(2.8) \quad \hat{I_N}f(\xi) = m(\xi)\hat{f}(\xi),
\]
where the multiplier $m(\xi)$ is smooth, radially symmetric, nonincreasing in $|\xi|$ and

\[
m_N(\xi) = \begin{cases} 
1 & |\xi| \leq N \\
\left(\frac{N}{|\xi|}\right)^{1-s} & |\xi| \geq 2N.
\end{cases}
\]

A parameter $N \gg 1$ is sometimes dropped when there will not be a confusion. The operator $I_N$ is the same smoothing operator as in [10] introduced to replace the energy conservation law below $H^1$ space. As intended, the definition of $m(\xi)$ gives the following relations between $\|I\phi\|_{H^1}$ and $\|\phi\|_{H^s}$ for $0 < s < 1$;

\[
\begin{align*}
\|I_N\phi\|_{L^2} &\leq \|\phi\|_{L^2} \\
\|\phi\|_{H^s} &\leq \|I_N(\nabla)\phi\|_{L^2} \leq N^{1-s}\|\phi\|_{H^s}.
\end{align*}
\]

Now we state the modified local well posedness as follows.

**Proposition 2.2.** For $\phi_0 \in H^s$, $s > 0$ the (1.1) evolution $\phi_0 \mapsto \phi(t)$ is well-posed on the time interval $[0, \tilde{T}_{lw}]$ with

\[
\begin{align*}
\tilde{T}_{lw} &= c_0 \|I_N\nabla\phi_0\|_{L^2}^{-2}, \\
\|I_N(\nabla)\phi\|_{S^0_{\tilde{T}_{lw}}} &\leq 2\|I_N(\nabla)\phi_0\|_{L^2}.
\end{align*}
\]

**Proof.** The proof is a direct modification of the arguments used to prove Theorem 3. We use that the Leibnitz rule holds for the operator $I_N(\nabla)$ for $s > 0$.

\[
\square
\]

3. Almost conservation law of the modified energy

Let us define the iteration space $Z_I(t)$ as

\[
Z_I(t) := \|\langle \nabla \rangle I\phi\|_{S^0_{\tilde{T}_{lw}}} = \sup_{(q,r) \text{ admissible}} \|\langle \nabla \rangle I\phi\|_{L^q_t L^r_x([0,t] \times \mathbb{R}^n)}
\]

We show the *almost conservation law* of the modified energy. The usual energy (1.2) is shown to be conserved by differentiating in
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time

$$\frac{d}{dt} E(\phi(t))$$

$$= \int_{\mathbb{R}^n} 2 \text{Re} \partial_t \phi (2(|x|^{-2} * |\phi|^2) \phi - \Delta \phi - 2 \partial_t \phi) + (|x|^{-2} * \partial_t |\phi|^2) |\phi|^2$$

$$- (|x|^{-2} * |\phi|^2) \partial_t |\phi|^2 dx$$

$$= \int_{\mathbb{R}^n} (|x|^{-2} * \partial_t |\phi|^2) |\phi|^2 - (|x|^{-2} * |\phi|^2) \partial_t |\phi|^2 dx$$

$$= 0,$$

using the equation (1.1). Since $I \phi$ is not a solution to the equation (1.1), $E(I \phi)(t)$ is not conserved. But still we have a control of the time increment of the modified energy $E(I \phi)(t)$. Differentiating $E(I \phi)(t)$ in time, we obtain

$$\frac{d}{dt} E(I \phi)(t) = \int_{\mathbb{R}^n} 2 \text{Re} \partial_t I \bar{\phi} [2 (I(|x|^{-2} * |\phi|^2) \phi) - \Delta I \phi - 2 i \partial_t I \phi] dx.$$

Then we have

$$E(I \phi(t)) - E(I \phi(0)) = 4 \text{Re} \int_0^T \int_{\mathbb{R}^n} \partial_t I \bar{\phi} [(|x|^{-2} * |I \phi|^2) I \phi - I((|x|^{-2} * |\phi|^2) \phi)] dx dt$$

(3.13)

$$:= E_1(t)$$

The following proposition shows that $E(I \phi)$ is an "almost" conserved quantity.

**Proposition 3.1.** Assume we have $s > 0$, $N \gg 1$, $\phi_0 \in C_0^\infty(\mathbb{R}^n)$, and a solution of (1.1) on a time interval $[0, T]$, $T \leq T_{\text{w}}$. Then we have

(3.14)

$$\sup_{t \in [0, T_{\text{w}}]} |E(I N \phi)(t)| \leq E(I N \phi_0) + C N^{-1+} \|I N(\nabla)\phi_0\|_{L^2}^4 + C N^{-1+} \|I N(\nabla)\phi_0\|_{L^2}^6.$$

**Proof of Proposition 3.1.**

The proof is same as in the defocusing case ([4]) apart that here we can bound $Z_I(t) \leq 2 \|\nabla I \phi_0\|_{L^2}$ by Proposition 2.2. We compute in the frequency space. Applying the Parseval formula to $E_1$ in (3.13), we
obtain
\[ E_1 = \text{Re} \int_0^T \int_{\sum_{j=1}^4 \xi_j = 0} \left( 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) |\xi_2 + \xi_3|^{-(n-2)} \longdiv{\partial_t I\phi(\xi_1) \hat{I}\phi(\xi_2) \hat{I}\phi(\xi_3) \hat{I}\phi(\xi_4)} d\xi_1 d\xi_2 d\xi_3 d\xi_4 dt. \] (3.15)

Now if we use equation (1.1) to substitute for \( \partial_t I\phi \) in (3.15), then it is split into two terms as follows:

\[ E_{1a} \equiv \left| \int_0^T \int_{\sum_{j=1}^4 \xi_j = 0} \left( 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) |\xi_2 + \xi_3|^{-(n-2)} \right| \]

\[ \times \left| \Delta \hat{T}\phi(\xi_1) \hat{T}\phi(\xi_2) \hat{T}\phi(\xi_3) \hat{T}\phi(\xi_4) d\xi_1 d\xi_2 d\xi_3 d\xi_4 dt \right|, \]

\[ E_{1b} \equiv \left| \int_0^T \int_{\sum_{j=1}^4 \xi_j = 0} \left( 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) |\xi_2 + \xi_3|^{-(n-2)} \right| \]

\[ \times \left| (I(|x|^{\frac{n}{2}} + |\phi|^2 \phi))(\xi_1) \hat{I}\phi(\xi_2) \hat{I}\phi(\xi_3) \hat{I}\phi(\xi_4) d\xi_1 d\xi_2 d\xi_3 d\xi_4 dt \right| \]

In both cases, we break down \( \phi \) into Littlewood-Paley pieces \( \phi_j \), each localized in \( 2^k \), in frequency, \( \{\xi_j\} \sim 2^k \) = \( N_j \), \( k_j = 0, 1, 2, \cdots \), and then use a version of Coifman-Meyer estimate for a class of multiplier operators.

**Proposition 3.2** (Proposition 6.1 in [6]). Let \( \sigma(\xi) \) be infinitely differentiable so that for all \( \alpha \in \mathbb{N}^{nk} \) and all \( \xi = (\xi_1, \ldots, \xi_k) \in \mathbb{R}^{
 k} \). Then there is a constant \( c(\alpha) \) with

\[ |\partial_\xi^\alpha \sigma(\xi)| \leq c(\alpha) (1 + |\xi|)^{-|\alpha|}. \]

Let the multi-linear operator \( \Lambda \) be given

\[ \Lambda(f_1, \ldots, f_k)(x) = \int_{\mathbb{R}^nk} e^{ix(\xi_1 + \ldots + \xi_k)} \sigma(\xi_1, \ldots, \xi_k) |\xi_2 + \xi_3|^{-(n-2)} f_1(\xi_1) \hat{f}_2(\xi_2) \cdots \hat{f}_k(\xi_k) d\xi_1 \cdots d\xi_k \]

for \( k \geq 2 \). Then we have

\[ \|\Lambda(f_1, \cdots, f_k)\|_{L^p} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \cdots \|f_k\|_{L^{p_k}} \]

where \( (p, p_i) \) is related by \( \frac{1}{p} + 1 = \frac{2}{n} + \sum_{i=1}^k \frac{1}{p_i} \).
We first estimate a pointwise bound on the symbol

\[
\left| 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right| \leq B(N_2, N_3, N_4)
\]

Factoring \(B(N_1, N_2, N_3)\) out of the integral in \(E_1\), it leaves a symbol \(\sigma_1\), which satisfies the condition of Proposition 3.2, as the following:

\[
\sum_{N_1 : N_2, N_3, N_4} B(N_2, N_3, N_4) \int_0^T \int_{\mathbb{R}^n} [\Lambda(\Delta I\phi_1, I\phi_2, I\phi_3)](\xi_4) \hat{I}\phi_4(\xi_4) d\xi_4 dt
\]

\[
+ \sum_{N_1, N_2, N_3, N_4} B(N_2, N_3, N_4) \int_0^T \int_{\mathbb{R}^n} [\Lambda(I(|x|^{-2} * |\phi_1|^2\phi_1), I\phi_2, I\phi_3)](\xi_4) \hat{I}\phi_4(\xi_4) d\xi_4 dt
\]

where

\[
[\Lambda(f, g, h)](x) = \int_{\mathbb{R}^3} e^{ix(\xi_1 + \xi_2 + \xi_3)} \sigma_1(\xi_1, \xi_2, \xi_3) |\xi_2 + \xi_3|^{-(n-2)} \hat{f}(\xi_1) \hat{g}(\xi_2) \hat{h}(\xi_3) \, d\xi_1 \, d\xi_2 \, d\xi_3.
\]

We shall show that

\[
E_{1a} + E_{1b} \lesssim N^{-1+}(Z_1(T))^P
\]

for some \(P > 0\).

For this aim, we claim that

(3.17)

\[
\sum_{N_1 : N_2, N_3, N_4} \int_0^T \int_{\mathbb{R}^n} B(N_2, N_3, N_4) [\Lambda(\Delta I\phi_1, I\phi_2, I\phi_3)](x, t) I\phi_4(x, t) \, dx \, dt
\]

(3.18)

\[
+ \sum_{N_1, N_2, N_3, N_4} \int_0^T \int_{\mathbb{R}^n} B(N_2, N_3, N_4) [\Lambda(I(|x|^{-2} * |\phi_1|^2\phi_1), I\phi_2, I\phi_3)](x, t) I\phi_4(x, t) \, dx \, dt
\]

\[
\lesssim N^{-1+}(Z_1(T))^4 + Z_1(T)^6.
\]

From Proposition 3.2, we have

(3.19)

\[
\|\Lambda(f, g, h)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \|h\|_{L^{p_3}}
\]

where \(\frac{1}{p} = \frac{2}{n} - 1 + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}\).

For the first term \(E_{1a}\), we use (3.19) and Hölder inequality to get
\begin{equation}
\left| \int_0^T \int_{\mathbb{R}^n} B(N_2, N_3, N_4) \left[ \Lambda(\Delta I \phi_1, I \phi_2, I \phi_3) \right] (x, t) I \phi_4 (x, t) dx dt \right| \\
\lesssim \| \Delta I \phi_1 \|_{L_t^q L_x^p} \| I \phi_2 \|_{L_t^{q_2} L_x^{p_2}} \| I \phi_3 \|_{L_t^{q_3} L_x^{p_3}} \| I \phi_4 \|_{L_t^{q_4} L_x^{p_4}}
\end{equation}

where $\sum \frac{1}{p_i} + \frac{2}{n} - 1 = 1$, $\sum \frac{1}{q_i} = 1$. Choosing $\frac{1}{p_i} = \frac{n-1}{2n}$, $\frac{1}{q_i} = \frac{1}{4}$, and using Bernstein inequality, we obtain

\[ \text{LHS of } (3.20) \lesssim B(N_2, N_3, N_4) \frac{N_1}{N_2 N_3 N_4} (Z_1(T))^4. \]

We reduce to show

\begin{equation}
\sum_{N_1, N_2, N_3, N_4} B(N_2, N_3, N_4) \frac{N_1}{N_2 N_3 N_4} \lesssim N^{-1+\epsilon}
\end{equation}

By symmetry we may assume $N_2 \geq N_3 \geq N_4$. Then it suffices to consider the following three cases.

**Case 1:** $N \gg N_2$. We have $m(\xi_i) = 1$ since $\sum_i \xi_i = 0$. So, the symbol

\[ \left| 1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right| = 0. \]

**Case 2:** $N_2 \geq N \gg N_3 \geq N_4$. Since $\sum_i \xi_i = 0$, we have $N_1 \sim N_2$. By the mean value theorem,

\[ \left| 1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right| = \frac{|m(\xi_2) - m(\xi_2 + \xi_3 + \xi_4)|}{m(\xi_2)} \leq \frac{\|\nabla m(\xi_2) \cdot (\xi_3 + \xi_4)\|}{m(\xi_2)} \lesssim \frac{N_3}{N_2}. \]

Thus,

\[ \text{LHS of } (3.20) \lesssim \frac{1}{N_2 N_4} (Z_1(T))^4 \lesssim N^{-1+\epsilon} N_2^{-\epsilon} (Z_1(T))^4. \]

Summing up with $N_4, N_3, N_2$, we have (3.21).

**Case 3:** $N_2 \geq N_3 \geq N$. In this case we need to consider two subcases $N_1 \sim N_2$ and $N_2 \gg N_1$ since by $\sum_i \xi_i = 0$ the case $N_1 \gg N_2$ cannot happen.
For the first case, $N_1 \sim N_2$, we estimate
\[
\frac{1-m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)} \frac{N_1}{N_2N_3N_4} \lesssim \frac{1}{N_3m(\xi_3)N_4m(\xi_4)} \approx \frac{1}{N_3^{1-s}N_4^{1-s}} \frac{1}{N_3N_4m(\xi_4)} \lesssim N^{-1+\epsilon} N_3^{-\epsilon}
\]
since $xm(x) \geq 1$ for $x \geq 1$. We can sum up $N_3, N_4$ directly. But when summing up $N_2$, we use the Cauchy-Schwartz inequality with $\phi_i = P_N I \phi$ as follows:
\[
\sum_N P_N \nabla I \phi \cdot P_N \nabla I \phi \leq \left( \sum_N (P_N \nabla I \phi)^2 \right)^{1/2}.
\]
In the second case, $N_2 \gg N_1$, again by $\sum_i \xi_i = 0$, we have $N_2 \sim N_3$.
\[
\frac{1-m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \frac{N_1}{N_2N_3N_4} \lesssim \frac{m(\xi_1)}{m(\xi_2)^2m(\xi_4)} \frac{N_1}{N_2N_3N_4} \sim N_1 m(\xi_1) \frac{1}{N^2} \frac{N_2^{2s}}{N_3^{2s}} \frac{1}{N_4m(\xi_4)}
\]
For our purpose, we want to show
\[
\left( \frac{N}{N_2} \right)^{2s} N_1 m(\xi_1) \lesssim N.
\]
If $N_1 \leq N$, then $m(\xi_1) = 1$ and this is true. If $N_1 \gg N$, then
\[
\left( \frac{N}{N_2} \right)^{2s} N_1 m(\xi_1) = \frac{N_1^{1+s}}{N_2^s} \left( \frac{N_1}{N_2} \right)^s = N \frac{(NN_1)^s}{N_2^{2s}} \leq N.
\]
This conclude the proof of (3.17). Now we turn to the estimate of $E_{1b}$. The above analysis is applied to $E_{1b}$, once we show the following lemma.

**Lemma 3.1.**

(3.22) \[
\|PMI([|x|^{-2} * (\phi_1 \phi_2)]\phi_3)\|_{L^1_t L^{2n}_x} \lesssim M^{1/2}(Z_1(T))^3
\]

**Proof.** We divide $\phi$ into $\phi = \phi_{lo} + \phi_{hi}$ where

$\text{supp } \hat{\phi}_{lo}(\xi, t) \subseteq \{||| \xi | < 2\}$

$\text{supp } \hat{\phi}_{hi}(\xi, t) \subseteq \{|| | \xi | > 1\}$
In the case that all \( \phi \)'s are \( \phi_{lo} \) we simply estimate
\[
\left\| P_M \left( I \left[ |x|^{-2} \ast (\phi_{lo} \bar{\phi}_{lo}) \phi_{lo} \right] \right) \right\|_{L_t^{2n/3} L_x^{2n/3}} \lesssim \left\| \phi_{lo} \right\|_{L_t^{6n/3} L_x^{6n/3}}^{3} \\
\lesssim \left\| \langle \nabla \rangle I \phi_{lo} \right\|_{L_t^{12} L_x^{3n/4}}^{3} \lesssim (Z_1(T))^3.
\]
When all \( \phi \)'s are \( \phi_{hi} \), we use Bernstein inequality, Sobolev embedding and the Leibniz rule as following:
\[
\left\| M^{-\frac{1}{2}} P_M \left( I \left[ |x|^{-2} \ast (\phi_{hi} \bar{\phi}_{hi}) \phi_{hi} \right] \right) \right\|_{L_t^{2n/3} L_x^{2n/3}} \lesssim \left\| \nabla^{-1} P_M \left( I \left[ |x|^{-2} \ast (\phi_{hi} \bar{\phi}_{hi}) \phi_{hi} \right] \right) \right\|_{L_t^{6n/3} L_x^{6n/3}}^{3} \\
\lesssim \left\| \langle \nabla \rangle I \phi_{hi} \right\|_{L_t^{12} L_x^{3n/4}}^{3} \lesssim (Z_1(T))^3.
\]
where note that \( \frac{3n-1}{6n} - \frac{3}{4n} = -\frac{1}{p} \).
The remaining \( lo - hi \) cases are controlled in a similar manner to the \( hi - hi \) case. We omit the detail here.

Hence, we have shown (3.17), (3.18) and so conclude the proof.

From the proposition above we induce that \( H^s \) kinetic energy dominates modified total energy, which corresponds to Proposition 2.1 in [12]. Before we state the proposition, let us define the blowup parameter and blowup rate as follows:
\[
\Lambda(t) = \sup_{0 \leq \tau \leq t} \|u(t)\|_{H^s}, \\
\rho(t) = \frac{\|\nabla Q\|_{L^2}}{\|\nabla I N \phi(t, \cdot)\|_{L^2}}.
\]

**Proposition 3.3.** There exists \( s_Q \leq \frac{9+\sqrt{71}}{40} \) such that for all \( s_Q < s < 1 \) there exists \( p(s) < 2 \) with the following holding true: if \( H^s \in \)
\( \phi_0 \mapsto \phi(t) \) solves (1.1) on a forward maximal finite existence \([0,T^*)\), for all \( T < T^* \) there exists \( N = N(T) \) such that

\[
|E(I_N(T)\phi(T))| \leq C_0(\Lambda(T))^{p(s)}
\]

with \( C_0 = C_0(s,T^*,\|\phi_0\|_{H^s}) \). More precisely \( N(T) = C(\Lambda(T))^{\frac{p(s)}{2(1-s)(4+s)}} \),

\[
p(s) = \frac{6+2s}{2(1-s)(4+s)}2(1-s).
\]

The proof of the above proposition is identical with the proof of Proposition 2.1 in [12] once we have the modified local well posedness (2.12) and the almost conservation law (3.14). The choice of \( p(s) \) is relevant to how fast the increment of the modified energy approximates zero, which is calculated in Proposition 3.1. The number \( s_Q \) is a consequence of \( p(s) < 2 \).

4. Proof of Theorem 1.1

Let us now prove Theorem 1.1. The proof is parallel with that of the corollary 1.10 in [20]. There T. Himidi- S. Kerrani nicely combined ideas in [12] with a profile decomposition theorem following the work by P.Gerard which we state in the below.

**Proposition 4.1.** Let \( \{v_n\}_{n=1}^{\infty} \) be a bounded sequence in \( H^1(\mathbb{R}^n) \). Then there exist a subsequence of \( \{v_n\}_{n=1}^{\infty} \) (still denoted \( \{v_n\}_{n=1}^{\infty} \)), a family \( \{x^j\}_{j=1}^{\infty} \) of sequences in \( \mathbb{R}^d \), and a sequence \( \{V^j\}_{j=1}^{\infty} \) of \( H^1 \) functions such that

1. for every \( k \neq j \), \( |x^k_n - x^j_n| \to \infty \) as \( n \to \infty \);

2. for every \( l \geq 1 \) and every \( x \in \mathbb{R}^n \),

\[
v_n(x) = \sum_{j=1}^{l} V^j(x - x^j_n) + v^l_n(x),
\]

with

\[
\limsup_{n \to \infty} \|v^l_n\|_{L^p(\mathbb{R}^n)} = 0 \text{ as } l \to \infty,
\]

for every \( p \in (2,2^*) \).

Moreover, we have, as \( n \to \infty \),

\[
\|v_n\|_{L^2}^2 = \sum_{j=1}^{l} \|V^j\|_{L^2}^2 + \|v^l_n\|_{L^2}^2 + o(1)
\]
and
\[ \| \nabla v_n \|_{L^2}^2 = \sum_{j=1}^{l} \| \nabla V_j \|_{L^2}^2 + \| \nabla v_n \|_{L^2}^2 + o(1). \]

As in [12], we choose \( \{ t_n \}_{n=1}^{\infty} \) to be a sequence such that \( t_n \uparrow T^* \) and for each \( t_n \)
\[ \| \phi(t_n) \|_{H^s} = \Lambda(t_n). \]
We set
\[ \psi_n = \rho_n I_N \phi(t_n, \rho_n x), \]
where
\[ \rho_n = \frac{\| \nabla Q \|_{L^2}}{\| \nabla N \phi \|_{L^2}^2} \leq \frac{1}{\| \phi(t_n, \cdot) \|_{H^s}} = \frac{1}{\Lambda(t_n)} \]
by (2.10). Also from the modified local well posedness, it holds that
\[ \rho_n \leq A(T^* - t_n)^{\frac{s}{2}} \]
for some constant \( A > 0 \). Hence \( \| \psi_n \|_{H^1} \) is normalized as
\[ \| \psi_n \|_{L^2} \leq \| \phi_0 \|_{L^2}, \quad \| \nabla \psi_n \|_{L^2} = \| \nabla Q \|_{L^2}. \]
On the other hands, the energy of \( \psi_n \) satisfies
\[ E(\psi_n) = \rho_n^2 E(I_N(\phi(t_n))) \leq \rho_n^{p(s)-2} \]
by Proposition 3.3. Since \( \Lambda(t_n) \to \infty \) and \( p(s) < 2 \), it holds that
\[ E(\psi_n) \to 0 \quad \text{as } n \to \infty, \]
which yields
\[ \int |x|^{-2} * |\psi_n|^2(x)|\psi_n(x)|^2 dx \to 2\| \nabla Q \|_{L^2}. \]
Note that the sequence \( \{ \psi_n \} \) satisfies the condition of Proposition 4.1, thus it admits the decomposition as
\[ \psi_n(x) = \sum_{j=1}^{l} U^{(j)}(x - x_n^j) + r_n^l(x) \]
up to subsequence, where \( U^{(j)}, x_n^j, r_n^l(x) \) meet the conditions in Proposition 4.1. Under the circumstance it is shown in [23] that there exists \( j_0 \) such that
\[ \psi_n(\cdot + x_n) \to U^{j_0} \in H^1 \quad \text{with } \| U^{j_0} \|_{L^2} \geq \| Q \|_{L^2}. \]
This is a replacement of Theorem 5 in [20] for the $L^2$ critical Hartree equation. Let us denote $U^{j_0}_n$ by $V$. Coming back to $\{\phi_n\}_{n=1}^{\infty}$ one has
\[ \frac{\dot{\rho}_n^2}{\lambda(t_n)} I_N \phi(t_n, \rho_n x + x_n) \to V \text{ in } H^1. \]
It turns into
\[ (4.28) \quad \frac{\dot{\rho}_n^2}{\lambda(t_n)} \phi(t_n, \rho_n x + x_n) \to V \text{ in } H^{\tilde{s}-} \quad \text{for } \tilde{s} = \frac{s + 1}{4 - 2s}, \]
where
\[ \|\rho_n^2 (I_N \phi(t_n) - \phi(t_n)) (\rho \cdot + x_n)\|_{H^1}^2 \leq \rho_n^2 \int_{|\xi| \geq N} (N^{1-s} |\xi|^s - 1)^2 |\xi|^{2s} |\hat{\phi}(t_n)|^2 d\xi \]
\[ \leq \rho_n^2 N^{2(s-\tilde{s})} \|\phi(t_n)\|_{H^1}^2 \]
\[ \leq \left( \Lambda(t_n) \frac{p(s)(\tilde{s}-s)}{2(1-\tilde{s})} + 1 - \tilde{s} \right) \]
by using $\rho_n \lesssim \Lambda^{-1}(t_n)$ and $N(t_n) = C(\Lambda(t_n)) \frac{p(s)}{2(1-\tilde{s})}$ in Proposition 3.3.
Note that $\frac{p(s)(\tilde{s}-s)}{2(1-\tilde{s})} + 1 - \tilde{s} < 0$ if $\tilde{s} < \tilde{s}$. Thus we have
\[ \|\rho_n^2 (I_N \phi(t_n) - \phi(t_n)) (\rho \cdot + x_n)\|_{H^{\tilde{s}-}}^2 \to 0, \text{ as } n \to \infty, \]
which implies (4.28).

Now we are ready to prove Theorem 1.1. From (4.28) it follows that
\[ \lim_{n \to \infty} \frac{\rho_n^2}{\lambda(t_n)} \int_{|x| \leq R} |\phi(t_n, \rho_n x + x_n)|^2 dx \geq \int_{|x| \leq R} |V|^2 dx \]
for every $R > 0$. By change of variable,
\[ \lim_{n \to \infty} \sup_{y \in \mathbb{R}^n} \int_{|x-y| \leq R \rho_n} |\phi(t_n, x)|^2 dx \geq \int_{|x| \leq R} |V|^2 dx. \]
Combining the assumption $\frac{(T^* - t)^{\tilde{s}}}{\lambda(t)} \to 0$ as $t \to T^*$ and (4.26), we have
\[ \frac{\rho_n}{\lambda(t_n)} \to 0, \text{ and then} \]
\[ \lim_{n \to \infty} \sup_{y \in \mathbb{R}^n} \int_{|x-y| \leq \lambda(t_n)} |\phi(t_n, x)|^2 dx \geq \int_{|x| \leq R} |V|^2 dx \]
for every $R > 0$. Letting $R$ go to infinity we obtain
\[ \lim_{n \to \infty} \sup_{y \in \mathbb{R}^n} \int_{|x-y| \leq \lambda(t_n)} |\phi(t_n, x)|^2 dx \geq \int_{\mathbb{R}^n} |V|^2 dx \geq \|Q\|_{L^2}^2. \]
This proves
\[ \limsup_{t\to \infty} \sup_{y \in \mathbb{R}^n} \int_{|x-y| \leq \lambda(t)} |\phi(t,x)|^2 \, dx \geq \|Q\|_{L^2}^2. \]

Since the map \( y \to \int_{|x-y| \leq \lambda(t_n)} |\phi(t_n,x)|^2 \, dx \) is continuous and goes to 0 at infinity for each \( t \), there exists a family \( x(t) \) such that
\[ \sup_{y \in \mathbb{R}^n} \int_{|x-y| \leq \lambda(t)} |\phi(t,x)|^2 \, dx = \int_{|x-x(t)| \leq \lambda(t)} |\phi(t,x)|^2 \, dx, \]
which concludes the proof of Theorem 1.1.

References

Ground state mass concentration for Hartree below $H^3$


Myeongju Chae
Department of Applied Mathematics,
Hankyong National University,
Ansong 456-749, Korea
E-mail: mchae@hknu.ac.kr