INTERVAL-VALUED FUZZY $\alpha$-IRRESOLUTE MAPPINGS ON INTERVAL-VALUED FUZZY TOPOLOGICAL SPACES

WON KEUN MIN

Abstract. In this paper, we introduce the concepts of IVF $\alpha$-irresolute mappings and IVF $\alpha$-irresolute open mappings and investigate some characterizations for them on the interval-valued fuzzy topological spaces.

1. Introduction

Zadeh [4] introduced the concept of fuzzy set and investigated basic properties. Gorzalczany [1] introduced the concept of interval-valued fuzzy set which is a generalization of fuzzy sets. In [3], Mondal and Samanta introduced the concepts of interval-valued fuzzy topology, continuity and compactness and studied some topological properties. In [2], Jun et al. introduced the concepts of IVF $\alpha$-open sets and IVF $\alpha$-open mappings and studied some results about them. In this paper, we introduce the concepts of IVF $\alpha$-irresolute mappings and IVF $\alpha$-irresolute open mappings and investigate some characterizations for them.

2. Preliminaries

Let $D[0,1]$ be the set of all closed subintervals of the interval $[0,1]$. The elements of $D[0,1]$ are generally denoted by capital letters $M, N, \cdots$ and note that $M = [M^L, M^U]$, where $M^L$ and $M^U$ are the lower and the upper end points respectively. Especially, we denote $0 = [0,0], 1 = [1,1]$, and $a = [a, a]$ for $a \in (0, 1)$. We also note that

1. $(\forall M, N \in D[0,1]) (M = N \iff M^L = N^L, M^U = N^U)$.
2. $(\forall M, N \in D[0,1]) (M \leq N \iff M^L \leq N^L, M^U \leq N^U)$.

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For every $M \in D[0, 1]$, the complement of $M$, denoted by $M^c$, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$. Let $X$ be a nonempty set. A mapping $A : X \rightarrow D[0, 1]$ is called an interval-valued fuzzy set (simply, IVF set) in $X$. For each $x \in X$, $A(x)$ is a closed interval whose lower and upper end points are denoted by $A(x)^L$ and $A(x)^U$, respectively. For any $[a, b] \in D[0, 1]$, the IVF set whose value is the interval $[a, b]$ for all $x \in X$ is denoted by $[a, b]$. In particular, for any $a \in [a, b]$, the IVF set whose value is $a = [a, a]$ for all $x \in X$ is denoted by simply $a$. For a point $p \in X$ and for $[a, b] \in D[0, 1]$ with $b > 0$, the IVF set which takes the value $[a, b]$ at $p$ and $0$ elsewhere in $X$ is called an interval-valued fuzzy point (simply, IVF point) and is denoted by $[a, b]_p$. In particular, if $b = a$, then it is also denoted by $a_p$. Denoted by $IVF(X)$ the set of all IVF sets in $X$.

For every $A, B \in IVF(X)$, we define

$$A = B \iff (\forall x \in X)([A(x)]^L = [B(x)]^L \text{ and } [A(x)]^U = [B(x)]^U),$$

$$A \subseteq B \iff (\forall x \in X)([A(x)]^L \subseteq [B(x)]^L \text{ and } [A(x)]^U \subseteq [B(x)]^U).$$

The complement $A^c$ of $A$ is defined by

$$[A^c(x)]^L = 1 - [A(x)]^U \text{ and } [A^c(x)]^U = 1 - [A(x)]^L$$

for all $x \in X$.

For a family of IVF sets $\{A_i : i \in J\}$ where $J$ is an index set, the union $G = \bigcup_{i \in J} A_i$ and $F = \bigcap_{i \in J} A_i$ are defined by

$$(\forall x \in X) \ ([G(x)]^L = \sup_{i \in J} [A_i(x)]^L, [G(x)]^U = \sup_{i \in J} [A_i(x)]^U),$$

$$(\forall x \in X) \ ([F(x)]^L = \inf_{i \in J} [A_i(x)]^L, [F(x)]^U = \inf_{i \in J} [A_i(x)]^U).$$

Let $f : X \rightarrow Y$ be a mapping and let $A$ be an IVF set in $X$. Then the image of $A$ under $f$, denoted by $f(A)$, is defined as follows

$$[f(A)(y)]^L = \begin{cases} 
\sup_{f(x)=y} [A(x)]^L, & \text{if } f^{-1}(y) \neq \emptyset, \\
0, & \text{otherwise,}
\end{cases}$$

$$[f(A)(y)]^U = \begin{cases} 
\sup_{f(x)=y} [A(x)]^U, & \text{if } f^{-1}(y) \neq \emptyset, \\
0, & \text{otherwise,}
\end{cases}$$

for all $y \in Y$.

Let $B$ be an IVF set in $Y$. Then the inverse image of $B$ under $f$, denoted by $f^{-1}(B)$, is defined as follows

$$(\forall x \in X)([f^{-1}(B)(x)]^L = [B(f(x))]^L, [f^{-1}(B)(x)]^U = [B(f(x))]^U).$$
Definition 2.1 ([3]). A family $\tau$ of IVF sets in $X$ is called an interval-valued fuzzy topology (simply, IVFT) on $X$ if it satisfies:

1. $0, 1 \in \tau$.
2. $A, B \in \tau \Rightarrow A \cap B \in \tau$.
3. For $i \in J$, $A_i \in \tau \Rightarrow \cup_{i \in J} A_i \in \tau$.

Every member of $\tau$ is called an IVF open set. An IVF set $A$ is called an IVF closed set if the complement of $A$ is IVF open. And $(X, \tau)$ is called an interval-valued fuzzy topological space (simply, IVFTS).

In an IVF topological space $(X, \tau)$, for an IVF set $A$ in $X$, the IVF closure and the IVF interior of $A$ [3], denoted by $cl(A)$ and $int(A)$, respectively, are defined as

$$cl(A) = \cap\{B \in IVF(X) : B^c \in \tau \text{ and } A \subseteq B\}$$

and

$$int(A) = \cup\{B \in IVF(X) : B \in \tau \text{ and } B \subseteq A\}.$$

An IVF set $A$ in an IVF topological space $X$ is said to be IVF compact [3] if every IVF open cover $A = \{A_i : i \in J\}$ of $B$ has a finite IVF subcover.

Theorem 2.2 ([3]). Let $A$ be an IVF set in an IVF topological space $(X, \tau)$. Then $1 - cl(1 - A) = int(A)$ and $1 - int(1 - A) = cl(A)$.

Definition 2.3 ([2]). Let $A$ be an IVF set in an IVFTS $(X, \tau)$. Then $A$ is said to be IVF $\alpha$-open if $A \subseteq int(cl(int(A)))$. We denote the set of all IVF $\alpha$-open sets by $IVF\alphaO(X)$.

Definition 2.4 ([2, 3]). Let $(X, \tau_1)$ and $(Y, \tau_2)$ be two IVFTS’s. Then $f : X \to Y$ is said to be continuous (resp., IVF $\alpha$-continuous) if for every IVF open set $B$ in $Y$, $f^{-1}(B)$ is IVF open (resp., IVF $\alpha$-open) in $X$.

Definition 2.5 ([2]). Let $(X, \tau_1)$ and $(Y, \tau_2)$ be two IVFTS’s. Then $f : X \to Y$ is said to be IVF open (resp., IVF $\alpha$-open) if for every IVF open set $A$ in $X$, $f(A)$ is IVF open (resp., IVF $\alpha$-open) in $Y$.

3. IVF $\alpha$-irresolute mappings and IVF $\alpha$-irresolute open mappings

Definition 3.1. Let $f : X \to Y$ be a mapping between IVFTS’s $(X, \tau_1)$ and $(Y, \tau_2)$. Then $f$ is said to be IVF $\alpha$-irresolute if for every IVF $\alpha$-open set $U$ in $Y$, $f^{-1}(U)$ is IVF $\alpha$-open in $X$. 
**Remark 3.2.** Let $f : X \to Y$ be a mapping between IVFTS’s $(X, \tau_1)$ and $(Y, \tau_2)$. Every IVF $\alpha$-irresolute mapping is IVF $\alpha$-continuous but the converse need not be true.

IVF continuous $\implies$ IVF $\alpha$-continuous $\iff$ IVF $\alpha$-irresolute

**Example 3.3.** Let $X = \{x, y, z\}$ and let $A$ and $B$ be IVF sets defined as follows

$A(x) = [0.7, 0.8], A(y) = [0.6, 0.8], A(z) = [0.6, 0.7]$ and

$B(x) = [0.1, 0.2], B(y) = [0.1, 0.2], B(z) = [0.2, 0.3].$

Define IVF topologies $T_1$ and $T_2$ on $X$ as follows

$T_1 = \{0, A, B, 1\}$ and

$T_2 = \{0, A, 1\}.$

Consider the identity mapping $f : (X, T_1) \to (X, T_2)$. Then $f$ is an IVF $\alpha$-continuous mapping but it is not IVF $\alpha$-irresolute.

**Theorem 3.4.** Let $f : X \to Y$ be a mapping between IVFTS’s $(X, \tau_1)$ and $(Y, \tau_2)$. Then the following statements are equivalent:

1. $f$ is IVF $\alpha$-irresolute.
2. $f^{-1}(B)$ is IVF $\alpha$-closed for each IVF $\alpha$-closed set $B$ of $Y$.
3. $f(cl_{\alpha}(A)) \subseteq cl_{\alpha}(f(A))$ for each IVF set $A$ in $X$.
4. $cl_{\alpha}(f^{-1}(B)) \subseteq f^{-1}(cl_{\alpha}(B))$ for an IVF fuzzy set $B$ in $Y$.
5. $f^{-1}(int_{\alpha}(B)) \subseteq int_{\alpha}(f^{-1}(B))$ for an IVF fuzzy set $B$ in $Y$.

**Proof.** (1) $\iff$ (2) It is obvious.

(2) $\implies$ (3) For any IVF set $A$ in $X$, since $cl_{\alpha}(f(A))$ is an IVF $\alpha$-closed set in $Y$, by (2), $f^{-1}(cl_{\alpha}(f(A)))$ is IVF $\alpha$-closed. Thus we have $cl_{\alpha}(A) \subseteq cl_{\alpha}(f^{-1}(f(A))) \subseteq f^{-1}(cl_{\alpha}(f(A)))$. It implies $f(cl_{\alpha}(A)) \subseteq cl_{\alpha}(f(A)).$

(3) $\implies$ (4) For any IVF set $B$ in $Y$, from (3), it follows that $f(cl_{\alpha}(f^{-1}(B))) \subseteq cl_{\alpha}(f(f^{-1}(B))) \subseteq cl_{\alpha}(B)$. Hence we have $cl_{\alpha}(f^{-1}(B)) \subseteq f^{-1}(cl_{\alpha}(B)).$

(4) $\implies$ (5) For any IVF set $B$ in $Y$, from (4), it follows

\[
\begin{align*}
    f^{-1}(int_{\alpha}(B)) &= 1 - (f^{-1}(cl_{\alpha}(1 - B))) \\
    &\subseteq 1 - cl_{\alpha}(f^{-1}(1 - B)) \\
    &= int_{\alpha}(f^{-1}(B)).
\end{align*}
\]
Hence, we have
\[
f^{-1}(\text{int}_\alpha(B)) \subseteq \text{int}_\alpha(f^{-1}(B)).
\]

(5) ⇒ (1) Let \( V \) be an IVF \( \alpha \)-open set of \( Y \). By (5),
\[
f^{-1}(V) = f^{-1}(\text{int}_\alpha(V)) \subseteq \text{int}_\alpha(f^{-1}(V)).
\]
This implies \( f^{-1}(V) \) is an IVF \( \alpha \)-open set.

**Theorem 3.5.** Let \( f : X \to Y \) be a mapping between IVFTS's \((X, \tau_1)\) and \((Y, \tau_2)\). Then the following statements are equivalent:
1. \( f \) is IVF \( \alpha \)-irresolute.
2. \( f : (X, \text{IVF}_\alpha O(X)) \to (Y, \text{IVF}_\alpha O(Y)) \) is IVF continuous.

**Proof.** Obvious.

**Theorem 3.6.** Let \( f : X \to Y \) be a bijective mapping between IVFTS's \((X, \tau_1)\) and \((Y, \tau_2)\). Then \( f \) is IVF \( \alpha \)-irresolute if and only if \( \text{int}_\alpha(f(A)) \subseteq f(\text{int}_\alpha(A)) \) for each IVF set \( A \) in \( X \).

**Proof.** Suppose that \( f \) is IVF \( \alpha \)-irresolute. For any IVF set \( A \) of \( X \), since \( f^{-1}(\text{int}_\alpha(f(A))) \) is IVF-\( \alpha \)-open, from Theorem 3.4 and injectivity, it follows
\[
f^{-1}(\text{int}_\alpha(f(A))) \subseteq \text{int}_\alpha(f^{-1}(f(A))) = \text{int}_\alpha(A).
\]
And from surjectivity of \( f \), it follows
\[
\text{int}_\alpha(f(A)) = f(f^{-1}(\text{int}_\alpha(f(A)))) \subseteq f(\text{int}_\alpha(A)).
\]
For the converse, let \( B \) be an IVF \( \alpha \)-open set of \( Y \). From the hypothesis and surjectivity, it follows
\[
f(\text{int}_\alpha(f^{-1}(B))) \supseteq \text{int}_\alpha(f(f^{-1}(B))) = \text{int}_\alpha(B) = B.
\]
Since \( f \) is injective, it is \( \text{int}_\alpha(f^{-1}(B)) \supseteq f^{-1}(B) \). It implies \( \text{int}_\alpha(f^{-1}(B)) = f^{-1}(B) \). Hence \( f \) is IVF \( \alpha \)-irresolute.

**Definition 3.7.** Let \((X, \tau)\) be an IVFTS. An IVF set \( A \) in \( X \) is said to be IVF \( \alpha \)-compact if for every IVF \( \alpha \)-open cover \( \mathcal{A} = \{A_i \in \text{IVF}(X) : i \in J\} \) of \( A \), there exists \( J_0 = \{1, 2, \ldots, n\} \subseteq J \) such that \( A \subseteq \bigcup_{i \in J_0} A_i \).

**Theorem 3.8.** Let \( f : (X, \tau_1) \to (Y, \tau_2) \) be IVF \( \alpha \)-irresolute on two IVFTS's. If \( A \) is an IVF \( \alpha \)-compact set in \( X \), then \( f(A) \) is also IVF \( \alpha \)-compact.
Proof. Let \( \{B_i \in IVF(Y) : i \in J\} \) be an IVF \( \alpha \)-open cover of \( f(A) \) in \( Y \). Then \( \{f^{-1}(B_i) : i \in J\} \) is an IVF \( \alpha \)-open cover of \( A \) in \( X \). By the definition of IVF \( \alpha \)-compactness, there exists \( J_0 = \{1, 2, \cdots, n\} \subseteq J \) such that \( A \subseteq \bigcup_{i \in J_0} (f^{-1}(B_i)) \). This implies

\[
    f(A) \subseteq f(\bigcup_{i \in J_0} (f^{-1}(B_i))) = \bigcup_{i \in J_0} f(f^{-1}(B_i)) \subseteq \bigcup_{i \in J_0} B_i.
\]

Hence \( f(A) \subseteq \bigcup_{i \in J_0} B_i \).

\[\square\]

**Theorem 3.9.** Let \( f : (X, \tau_1) \to (Y, \tau_2) \) be IVF \( \alpha \)-continuous on two IVFTS’s. If \( A \) is an IVF \( \alpha \)-compact set in \( X \), then \( f(A) \) is IVF compact.

Proof. It is similarly proved from the above Theorem 3.8. \[\square\]

**Definition 3.10.** Let \((X, \tau_1)\) and \((Y, \tau_2)\) be two IVFTS’s. Then \( f : X \to Y \) is called an IVF \( \alpha \)-irresolute open (resp., IVF \( \alpha \)-irresolute closed mapping if for every IVF \( \alpha \)-open (resp., IVF \( \alpha \)-closed) set \( A \) in \( X \), \( f(A) \) is IVF \( \alpha \)-open (resp., IVF \( \alpha \)-closed) in \( Y \).

**Remark 3.11.** Let \( f : X \to Y \) be a mapping between IVFTS’s \((X, \tau_1)\) and \((Y, \tau_2)\). Every IVF \( \alpha \)-irresolute open (resp., IVF \( \alpha \)-irresolute closed) mapping is IVF \( \alpha \)-open (resp., IVF \( \alpha \)-closed) but the converse need not be true.

\[
    \text{IVF open} \Rightarrow \text{IVF } \alpha \text{-open} \leftarrow \text{IVF } \alpha \text{-irresolute open}
\]

**Example 3.12.** In Example 3.3, consider the identity mapping \( f : (X, T_2) \to (X, T_1) \) is an IVF \( \alpha \)-open mapping but not IVF \( \alpha \)-irresolute open.

**Theorem 3.13.** Let \( f : X \to Y \) be a mapping on IVFTS’s \((X, \tau_1)\) and \((Y, \tau_2)\). The following are equivalent:

1. \( f \) is IVF \( \alpha \)-irresolute open.
2. \( f(int_{\alpha}(A)) \subseteq int_{\alpha}(f(A)) \) for \( A \in IVF(X) \).
3. \( int_{\alpha}(f^{-1}(B)) \subseteq f^{-1}(int_{\alpha}(B)) \) for \( B \in IVF(Y) \).
4. For \( B \in IVF(Y) \) and each IVF \( \alpha \)-closed set \( A \) of \( X \) with \( f^{-1}(B) \subseteq A \), there exists an IVF \( \alpha \)-closed set \( C \) of \( Y \) such that \( B \subseteq C \) and \( f^{-1}(C) \subseteq A \).
Proof. (1) ⇒ (2) For \( A \in \text{IVF}(X) \),

\[
f(\text{int}_\alpha(A)) = f(\cup\{B \in \text{IVF}(X) : B \subseteq A, B \in \text{IVFAO}(X)\})
\]

\[
= \cup\{f(B) \in \text{IVF}(Y) : f(B) \subseteq f(A), f(B) \in \text{IVFAO}(Y)\}
\]

\[
\subseteq \cup\{U \in \text{IVF}(Y) : U \subseteq f(A), U \in \text{IVFAO}(Y)\}
\]

\[
= \text{int}_\alpha(f(A)).
\]

Hence \( f(\text{int}_\alpha(A)) \subseteq \text{int}_\alpha(f(A)). \)

(2) ⇒ (3) For \( B \in \text{IVF}(Y) \), from (2) it follows that

\[
f(\text{int}_\alpha(f^{-1}(B))) \subseteq \text{int}_\alpha(f(f^{-1}(B))) \subseteq \text{int}_\alpha(B).
\]

Hence \( \text{int}_\alpha(f^{-1}(B)) \subseteq f^{-1}(\text{int}_\alpha(B)). \)

(3) ⇒ (4) Let \( A \) be an IVF \( \alpha \)-closed set of \( X \) with \( f^{-1}(B) \subseteq A \) for \( B \in \text{IVF}(Y) \). Since \( 1 - A \subseteq 1 - f^{-1}(B) = f^{-1}(1 - B) \),

\[
\text{int}_\alpha(1 - A) = 1 - A \subseteq \text{int}_\alpha(f^{-1}(1 - B)).
\]

By (3),

\[
1 - A \subseteq \text{int}_\alpha(f^{-1}(1 - B)) \subseteq f^{-1}(\text{int}_\alpha(1 - B)).
\]

Thus

\[
A \supseteq 1 - (f^{-1}(\text{int}_\alpha(1 - B)))
\]

\[
= f^{-1}(1 - \text{int}_\alpha(1 - B))
\]

\[
= f^{-1}(\text{cl}_\alpha(B)).
\]

Now set \( C = \text{cl}_\alpha(B) \). Then \( C \) is an IVF \( \alpha \)-closed set of \( Y \) such that \( B \subseteq C \) and \( f^{-1}(C) \subseteq A \).

(4) ⇒ (1) Let \( A \) be an IVF \( \alpha \)-open set of \( X \). Then

\[
f^{-1}(1 - f(A)) = 1 - f^{-1}(f(A)) \subseteq 1 - A
\]

and \( 1 - A \) is IVF \( \alpha \)-closed. By (4), there exists an IVF \( \alpha \)-closed set \( C \) such that \( 1 - f(A) \subseteq C \) and \( f^{-1}(C) \subseteq 1 - A \). It implies

\[
1 - C \subseteq f(A)
\]

and

\[
f(A) \subseteq f(1 - f^{-1}(C)) = f(f^{-1}(1 - C)) \subseteq 1 - C.
\]

Hence \( f(A) \) is an IVF \( \alpha \)-closed set in \( Y \). \( \square \)
**Theorem 3.14.** Let \( f : X \rightarrow Y \) be a mapping on IVFTS’s \((X, \tau_1)\) and \((Y, \tau_2)\). The following are equivalent:

1. \( f \) is IVF \( \alpha \)-irresolute closed.
2. \( \text{cl}_\alpha(f(A)) \subseteq f(\text{cl}_\alpha(A)) \) for \( A \in IVF(X) \).
3. For \( B \in IVF(Y) \) and each IVF \( \alpha \)-open set \( A \) of \( X \) with \( f^{-1}(B) \subseteq A \), there exists an IVF \( \alpha \)-open set \( C \) of \( Y \) such that \( B \subseteq C \) and \( f^{-1}(C) \subseteq A \).

**Proof.** It is similarly proved from Theorem 3.13. \( \Box \)

**Theorem 3.15.** Let \( f : X \rightarrow Y \) be a bijective mapping between IVFTS’s \((X, \tau_1)\) and \((Y, \tau_2)\). Then \( f \) is IVF \( \alpha \)-irresolute open if and only if \( \text{int}_\alpha(f^{-1}(A)) \subseteq f^{-1}(\text{int}_\alpha(A)) \) for each \( A \in IVF(Y) \).

**Proof.** Suppose that \( f \) is IVF \( \alpha \)-irresolute open. For any IVF set \( A \) of \( Y \), from Theorem 3.13 and surjectivity,

\[
\text{f}(\text{int}_\alpha(f^{-1}(A))) \subseteq \text{int}_\alpha(f(f^{-1}(A))) = \text{int}_\alpha(A).
\]

This implies

\[
f^{-1}(f(\text{int}_\alpha(f^{-1}(A)))) = \text{int}_\alpha(f^{-1}(A)) \subseteq f^{-1}(\text{int}_\alpha(A)).
\]

For the converse, let \( B \) be an IVF \( \alpha \)-open set in \( X \). Then from the hypothesis and injectivity, it follows

\[
\text{int}_\alpha(f^{-1}(f(A))) = \text{int}_\alpha(A) \subseteq f^{-1}(\text{int}_\alpha(f(A))).
\]

And from surjectivity, it follows

\[
f(\text{int}_\alpha(A)) \subseteq f(f^{-1}(\text{int}_\alpha(f(A)))) = \text{int}_\alpha(f(A)).
\]

From Theorem 3.13, \( f \) is IVF \( \alpha \)-irresolute open. \( \Box \)

**Theorem 3.16.** Let \( f : X \rightarrow Y \) be a bijective mapping between IVFTS’s \((X, \tau_1)\) and \((Y, \tau_2)\). Then \( f \) is IVF \( \alpha \)-irresolute closed if and only if \( f^{-1}(\text{cl}_\alpha(A)) \subseteq \text{cl}_\alpha(f^{-1}(A)) \) for each \( A \in IVF(Y) \).

**Proof.** It is similarly proved from Theorem 3.15. \( \Box \)

**References**

Won Keun Min
Department of Mathematics,
Kangwon National University,
Chuncheon, 200-701, Korea
E-mail: wkmin@kangwon.ac.kr