ON SYMMETRIC GENERALIZED 3-DERIVATIONS AND COMMUTATIVITY IN PRIME NEAR-RINGS

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Abstract. In this note, we introduce a symmetric generalized 3-derivation in near-rings and investigate some conditions for a near-ring to be a commutative ring.

1. Introduction and preliminaries

A non-empty set $R$ with two binary operations $+$ (addition) and $\cdot$ (multiplication) is called a near-ring if it satisfies the following axioms:

i) $(R, +)$ is a group (not necessarily abelian),
ii) $(R, \cdot)$ is a semigroup,
iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Precisely speaking, it is a left near-ring because it satisfies the left distributive law. We will use the word near-ring to mean left near-ring and denote $xy$ instead of $x \cdot y$.

For a near-ring $R$, the set $R_0 = \{x \in R : 0x = 0\}$ is called the zero-symmetric part of $R$. A near-ring $R$ is said to be zero-symmetric if $R = R_0$. Throughout this note, $R$ will be a zero-symmetric near-ring and $R$ is called prime if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. Recall that $R$ is called $n$-torsion-free, where $n$ is a positive integer, if $nx = 0$ implies $x = 0$ for all $x \in R$. The symbol $C$ will represent the multiplicative center of $R$, that is, $C = \{x \in R : xy = yx \text{ for all } y \in R\}$. For $x \in R$, the symbol $C(x)$ will denote the centralizer of $x$ in $R$. As usual, for $x, y \in R$, $[x, y]$ will denote the commutator $xy - yx$, while $\langle x, y \rangle$ will indicate the additive-group commutator $x + y - x - y$. As for terminologies concerning near-rings used here without special mention, we refer to G. Pilz [10].

An additive map $d : R \rightarrow R$ is called a derivation if the Leibniz rule $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. By a bi-derivation we mean...
a bi-additive map $D : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (i.e., $D$ is additive in both arguments) which satisfies the relations

$$D(xy, z) = D(x, z)y + xD(y, z)$$

and

$$D(x, yz) = D(x, y)z + yD(x, z)$$

for all $x, y, z \in \mathbb{R}$. Let $D$ be symmetric, that is, $D(x, y) = D(y, x)$ for all $x, y \in \mathbb{R}$. The map $\tau : \mathbb{R} \to \mathbb{R}$ defined by $\tau(x) = D(x, x)$ for all $x \in \mathbb{R}$ is called the trace of $D$.

Derivations and bi-derivations in rings and near-rings have been studied by many mathematicians in several ways [1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 15, 16]. Furthermore, M. Uçkun and M.A. Öztürk [14] investigated symmetric bi-$\Gamma$-derivations and commutativity in $\Gamma$-near-rings.

A 3-additive map $\Delta : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a 3-derivation if the relations

$$\Delta(x_1x_2, y, z) = \Delta(x_1, y, z)x_2 + x_1\Delta(x_2, y, z),$$

$$\Delta(x, y_1y_2, z) = \Delta(x, y_1, z)y_2 + y_1\Delta(x, y_2, z)$$

and

$$\Delta(x, y, z_1z_2) = \Delta(x, y, z_1)z_2 + z_1\Delta(x, y, z_2)$$

are fulfilled for all $x, y, z, x_i, y_i, z_i \in \mathbb{R}, i = 1, 2$.

Let a map $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be symmetric, namely, the equation $F(x_1, x_2, x_3) = F(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$ holds for all $x_1, x_2, x_3 \in \mathbb{R}$ and for every permutation $\{\pi(1), \pi(2), \pi(3)\}$. A map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = F(x, x, x)$ for all $x \in \mathbb{R}$, is the trace of $F$. It is obvious that, in the case when $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a symmetric map which is also 3-additive (i.e., additive in each argument), the trace $f$ of $F$ satisfies the relation

$$f(x + y) = f(x) + 2F(x, x, y) + F(x, y, y) + F(x, x, y) + 2F(x, y, y) + f(y)$$

for all $x, y \in \mathbb{R}$.

Since we have

$$F(0, y, z) = F(0 + 0, y, z) = F(0, y, z) + F(0, y, z)$$

for all $y, z \in \mathbb{R}$, we obtain $F(0, y, z) = 0$ for all $y, z \in \mathbb{R}$. Hence we get

$$0 = F(0, y, z) = F(x - x, y, z) = F(x, y, z) + F(-x, y, z)$$

and so we see that $F(-x, y, z) = -F(x, y, z)$ for all $x, y, z \in \mathbb{R}$. This tells us that $f$ is an odd function. If $\Delta$ is symmetric, then the above three relations are equivalent to each other.
An additive map $h : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $h(xy) = d(x)y + xh(y)$ for all $x, y \in R$. M. Brešar [4] defined this concept in rings. This notion is found in [13] and other properties of generalized derivations in rings and near-rings were given by B. Hvala [9], H.E. Bell [2] and Ö. Gölbasi [8], respectively.

A symmetric bi-additive map $H : R \times R \to R$ is said to be a symmetric generalized bi-left derivation (resp. symmetric generalized bi-right derivation) associated with $D : R \times R \to R$ such that $H(xy, z) = D(x, z)y + xH(y, z)$ (resp. $H(xy, z) = H(x, z)y + xD(y, z)$) for all $x, y, z \in R$. $H$ is called a symmetric generalized bi-derivation associated with $D$ if it is both a symmetric generalized bi-left derivation and a symmetric generalized bi-right derivation associated with $D$ (see [14]).

Here we introduce the following map:

By a symmetric generalized 3-left derivation (resp. symmetric generalized 3-right derivation) associated with $\Delta$, we mean a symmetric 3-additive map $G : R \times R \times R \to R$ such that there exists a symmetric 3-derivation $\Delta : R \times R \times R \to R$ satisfying $G(xy, z, w) = \Delta(x, z, w)y + xG(y, z, w)$ (resp. $G(xy, z, w) = G(x, z, w)y + x\Delta(y, z, w)$) for all $w, x, y, z \in R$. The map $G$ is called a symmetric generalized 3-derivation associated with $\Delta$ if it is both a symmetric generalized 3-left derivation and a symmetric generalized 3-right derivation associated with $\Delta$. We here will denote the generalized 3-(left or right) derivation $G$ associated with the symmetric 3-derivation $\Delta$ by the tuple $(G, \Delta)$. Note that the symmetric generalized 3-derivation $(\Delta, \Delta)$ is the symmetric 3-derivation $\Delta$.

As a simple example, let $N$ be a commutative near-ring and let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in N \right\}.$$ 

Then it is easy to see that $R$ is a near-ring under matrix addition and matrix multiplication and that the map $\Delta : R \times R \times R \to R$ defined by

$$\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & a_1a_2a_3 \\ 0 & 0 \end{pmatrix}$$

is a symmetric 3-derivation.

Furthermore, we define a map $G : R \times R \times R \to R$ by

$$\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & a_1a_2a_3 + b_1b_2b_3 \\ 0 & 0 \end{pmatrix}.$$
Then \((G, \Delta)\) is a symmetric generalized 3-left derivation.

In this note, we investigate the conditions for a near-ring with symmetric generalized 3-derivations to be a commutative ring.

2. Lemmata

We need the following lemmas to obtain our main results.

**Lemma 2.1** ([3, Lemma 3]). Let \(R\) be a prime near-ring. If \(C \setminus \{0\}\) contains an element \(z\) for which \(z + z \in C\), then \((R, +)\) is abelian.

**Lemma 2.2.** Let \(R\) be a \(2\) and \(3\)-torsion free near-ring. Suppose that there exists a symmetric 3-additive map \(F : R \times R \times R \to R\) such that \(f(x) = 0\) for all \(x \in R\), where \(f\) is the trace of \(F\). Then we have \(F = 0\).

**Proof.** For any \(x, y \in R\),
\[
f(x+y) = f(x) + 2F(x, x, y) + F(x, y, y) + F(x, x, y) + 2F(x, y, y) + f(y)
\]
and so, by the hypothesis, we get
\[
(2.1) \quad 2F(x, x, y) + F(x, y, y) + F(x, x, y) + 2F(x, y, y) = 0
\]
for all \(x, y \in R\). Putting \(-x\) instead of \(x\) in (2.1), we obtain
\[
(2.2) \quad 2F(x, x, y) - F(x, y, y) + F(x, x, y) - 2F(x, y, y) = 0
\]
for all \(x, y \in R\).

On the other hand, for any \(x, y \in R\),
\[
f(y+x) = f(y) + 2F(y, y, x) + F(y, x, x) + F(y, y, x) + 2F(y, x, x) + f(x)
\]
and thus, by the hypothesis, we have
\[
(2.3) \quad 2F(x, y, y) + F(x, x, y) + F(x, y, y) + 2F(x, x, y) = 0
\]
for all \(x, y \in R\) since \(F\) is symmetric. Comparing (2.1) with (2.2), we get
\[
2F(x, y, y) + F(x, x, y) + F(x, y, y) = F(x, x, y) - 3F(x, y, y)
\]
which implies that
\[
2F(x, y, y) + F(x, x, y) + F(x, y, y) + 2F(x, x, y) = F(x, x, y) - 3F(x, y, y) + 2F(x, x, y)
\]
for all \(x, y \in R\). Hence it follows from (2.3) that
\[
(2.4) \quad F(x, x, y) - 3F(x, y, y) + 2F(x, x, y) = 0
\]
for all \(x, y \in R\). The substitution \(x = -x\) in (2.4) leads to
\[
(2.5) \quad F(x, x, y) + 3F(x, y, y) + 2F(x, x, y) = 0
\]
for all $x, y \in R$. Combining (2.4) and (2.5), we obtain
\begin{equation}
F(x, y, y) = 0
\end{equation}
for all $x, y \in R$ since $R$ is 6-torsion free. The replacement $y = y + z$ to linearize (2.6) yields
\begin{equation*}
F(x, y, z) = 0
\end{equation*}
for all $x, y, z \in R$, i.e., $F = 0$ which completes the proof. \hfill \Box

**Lemma 2.3.** Let $R$ be a 2 and 3-torsion free prime near-ring and let $x \in R$. Suppose that there exists a nonzero symmetric generalized 3-derivation $(\mathcal{G}, \Delta) : R \times R \times R \to R$ such that $x\mathcal{G}(y) = 0$ for all $y \in R$, where $\mathcal{G}$ is the trace of $\mathcal{G}$. Then we have $x = 0$.

**Proof.** Since we have
\begin{equation}
g(y + z) = g(y) + 2\mathcal{G}(y, y, z) + \mathcal{G}(y, z, z) + \mathcal{G}(y, y, z) + 2\mathcal{G}(y, z, z) + g(z)
\end{equation}
for all $y, z \in R$, the hypothesis gives
\begin{equation}
2x\mathcal{G}(y, y, z) + x\mathcal{G}(y, z, z) + x\mathcal{G}(y, y, z) + 2x\mathcal{G}(y, z, z) = 0
\end{equation}
for all $y, z \in R$. Setting $y = -y$ in (2.7), it follows that
\begin{equation}
2x\mathcal{G}(y, y, z) - x\mathcal{G}(y, z, z) + x\mathcal{G}(y, y, z) - 2x\mathcal{G}(y, z, z) = 0
\end{equation}
for all $y, z \in R$.

On the other hand, for any $y, z \in R$,
\begin{equation*}
g(z + y) = g(z) + 2\mathcal{G}(z, z, y) + \mathcal{G}(z, y, y) + \mathcal{G}(z, z, y) + 2\mathcal{G}(z, y, y) + g(y)
\end{equation*}
and so, by the hypothesis, we have
\begin{equation}
2x\mathcal{G}(y, z, z) + x\mathcal{G}(y, y, z) + x\mathcal{G}(y, z, z) + 2x\mathcal{G}(y, y, z) = 0
\end{equation}
for all $y, z \in R$ since $\mathcal{G}$ is symmetric. Comparing (2.8) with (2.9), we get
\begin{equation*}
2x\mathcal{G}(y, z, z) + x\mathcal{G}(y, y, z) + x\mathcal{G}(y, z, z) = x\mathcal{G}(y, y, z) - 3x\mathcal{G}(y, z, z)
\end{equation*}
which means that
\begin{equation*}
2x\mathcal{G}(y, z, z) + x\mathcal{G}(y, y, z) + x\mathcal{G}(y, z, z) + 2x\mathcal{G}(y, y, z) = x\mathcal{G}(y, y, z) - 3x\mathcal{G}(y, z, z) + 2x\mathcal{G}(y, z, z)
\end{equation*}
for all $y, z \in R$. Now, from (2.9), we obtain
\begin{equation}
x\mathcal{G}(y, y, z) - 3x\mathcal{G}(y, z, z) + 2x\mathcal{G}(y, y, z) = 0
\end{equation}
for all $y, z \in R$. Taking $y = -y$ in (2.10) leads to
\begin{equation}
x\mathcal{G}(y, y, z) + 3x\mathcal{G}(y, z, z) + 2x\mathcal{G}(y, y, z) = 0
\end{equation}
for all \(y, z \in R\). Combining (2.10) and (2.11), we obtain
\[
(2.12) \quad x\mathcal{G}(y, z, z) = 0
\]
for all \(y \in R\) since \(R\) is 6-torsion free. Replacing \(z = z + w\) to linearize (2.12) and using the conditions show that
\[
(2.13) \quad x\mathcal{G}(w, y, z) = 0
\]
for all \(w, y, z \in R\). Assume first that \(\Delta = 0\). Since \((\mathcal{G}, \Delta)\) is a nonzero symmetric generalized 3-left derivation, substituting \(wv\) for \(w\) in (2.13) implies that
\[
xw\mathcal{G}(v, y, z) = 0
\]
for all \(v, w, y, z \in R\). By the primeness of \(R\), we get \(x = 0\). On the other hand, assume that \(\Delta \neq 0\). Since \((\mathcal{G}, \Delta)\) is also a symmetric generalized 3-right derivation, again substituting \(wv\) for \(w\) in (2.13) implies that
\[
xw\Delta(v, y, z) = 0
\]
for all \(v, w, y, z \in R\). From the primeness of \(R\), we have \(x = 0\) which gives the conclusion.

**Lemma 2.4.** Let \(R\) be a near-ring and let \((\mathcal{G}, \Delta) : R \times R \times R \to R\) be a symmetric generalized 3-derivation. Then for all \(v, w, x, y, z \in R\), we have, respectively,

(i) \([\mathcal{G}(x, z, w)y + x\Delta(y, z, w)]v = \mathcal{G}(x, z, w)yv + x\Delta(y, z, w)v\);

(ii) \([\Delta(x, z, w)y + x\mathcal{G}(y, z, w)]v = \Delta(x, z, w)yv + x\mathcal{G}(y, z, w)v\).

**Proof.** (i) Since we have
\[
\mathcal{G}(xy, z, w) = \mathcal{G}(x, z, w)y + x\Delta(y, z, w)
\]
for all \(w, x, y, z \in R\), the associative law gives
\[
\mathcal{G}((xy)v, z, w) = \mathcal{G}(xy, z, w)v + xy\Delta(v, z, w)
\]
(2.14) \[\mathcal{G}(x(yv), z, w) = \mathcal{G}(x, z, w)yv + x\Delta(yv, z, w)
\]
for all \(v, w, x, y, z \in R\) and
\[
\mathcal{G}(x(yv), z, w) = \mathcal{G}(x, z, w)yv + x\Delta(yv, z, w)
\]
(2.15)
for all \(v, w, x, y, z \in R\). Comparing (2.14) and (2.15), we see that
\[
[\mathcal{G}(x, z, w)y + x\Delta(y, z, w)]v = \mathcal{G}(x, z, w)yv + x\Delta(y, z, w)v
\]
for all \(v, w, x, y, z \in R\).

(ii) It is proved by the similar way to (i).  \(\Box\)
3. Main results

Now we are ready to prove our main results.

**Theorem 3.1.** Let $R$ be a 2 and 3-torsion free prime near-ring. Suppose that there exists a nonzero symmetric generalized 3-derivation $(\mathcal{G}, \Delta) : R \times R \times R \rightarrow R$ such that

$$\mathcal{G}(x, y, z) \in C$$

for all $x, y, z \in R$. Then $R$ is a commutative ring.

**Proof.** Assume that $\mathcal{G}(x, y, z) \in C$ for all $x, y, z \in R$. Since $(\mathcal{G}, \Delta)$ is nonzero, there exist $x_0, y_0, z_0 \in R$ such that $\mathcal{G}(x_0, y_0, z_0) \in C \setminus \{0\}$ and

$$\mathcal{G}(x_0, y_0, z_0) + \mathcal{G}(x_0, y_0, z_0) = \mathcal{G}(x_0, y_0, z_0 + z_0) \in C.$$

Hence $(R, +)$ is abelian by Lemma 2.1.

Let $\Delta = 0$. Then $\mathcal{G}(x, y, z) = \mathcal{G}(x, z, w)y \in C$ holds for all $w, x, y \in R$ and hence we have $\mathcal{G}(x, z, w)y = v\mathcal{G}(x, z, w)y$ for all $v, w, x, y \in R$ which implies that $[y, v]g(x) = 0$ for all $v, x, y \in R$. Reminding of Lemma 2.3, we get $[y, v] = 0$ for all $v, y \in R$, i.e., $R$ is a commutative ring.

Let $\Delta \neq 0$. Since the hypothesis implies that

$$w\mathcal{G}(x, y, z) = \mathcal{G}(x, y, z)w$$

for all $w, x, y, z \in R$, we replace $x$ by $xv$ in (3.1) to get

$$w[\mathcal{G}(x, y, z)v + x\Delta(v, y, z)] = [\mathcal{G}(x, y, z)v + x\Delta(v, y, z)]w$$

and thus, from Lemma 2.4(i) and the hypothesis, it follows that

$$\mathcal{G}(x, y, z)vw + wx\Delta(v, y, z) = \mathcal{G}(x, y, z)vw + x\Delta(v, y, z)w$$

which means that

$$\mathcal{G}(x, y, z)[w, v] = [x\Delta(v, y, z), w]$$

for all $v, w, x, y, z \in R$. Since $(\mathcal{G}, \Delta)$ is a symmetric generalized 3-left derivation and $g(x) \in C$, setting $g(x)u$ in place of $x$ in (3.2) and using Lemma 2.4(ii) yield

$$\Delta(g(x), y, z)u[w, v] + g(x)\mathcal{G}(u, y, z)[w, v] = g(x)[u\Delta(v, y, z), w]$$

for all $u, w, x, y, z \in R$. So the relation (3.2) gives

$$\Delta(g(x), y, z)u[w, v] = 0$$

for all $u, w, x, y, z \in R$. 


Suppose that $R$ is not commutative. Since $R$ is prime, we get, from (3.3),

\[(3.4) \quad \Delta(g(x), y, z) = 0\]

for all $x, y, z \in R$. Let us $x = x + w$ in (3.4). Then we have

\[
0 = \Delta(g(x + w), y, z) = \Delta(g(x) + g(w) + 3G(x, x, w) + 3G(w, w, x), y, z) = 3\Delta(G(x, x, w), y, z) + 3\Delta(G(x, w, w), y, z),
\]

that is,

\[(3.5) \quad \Delta(G(x, x, w), y, z) + \Delta(G(x, w, w), y, z) = 0\]

for all $w, x, y, z \in R$. Letting $x = -x$ in (3.5) and then comparing the result with (3.5), we see that

\[(3.6) \quad \Delta(G(x, x, w), y, z) = 0\]

for all $w, x, y, z \in R$. We replace $x$ by $x + v$ in (3.6) to obtain

\[(3.7) \quad \Delta(G(x, v, w), y, z) = 0\]

for all $v, w, x, y, z \in R$. Substituting $xu$ for $x$ in (3.7), we get

\[
0 = \Delta(G((xu, v, w), y, z)) = \Delta(G(x, v, w)u + x\Delta(u, v, w), y, z) = \Delta(G(x, v, w)u, y, z) + \Delta(G(x, v, w)\Delta(u, v, w), y, z) + \Delta(x, y, z)\Delta(u, v, w) + x\Delta(\Delta(u, v, w), y, z)
\]

and so the equality (3.7) gives

\[(3.8) \quad G(x, v, w)\Delta(u, y, z) + \Delta(x, y, z)\Delta(u, v, w) + x\Delta(\Delta(u, v, w), y, z) = 0\]

for all $u, v, w, x, y, z \in R$. Let us write $g(w)$ in place of $w$ in (3.8). Then it follows from (3.4) that

\[(3.9) \quad G(g(w), x, v)\Delta(u, y, z) = 0\]

for all $u, v, w, x, y, z \in R$. Putting $tu$ in substitute for $u$ in (3.9) yields

\[
G(g(w), x, v)t\Delta(u, y, z) = 0
\]

for all $t, u, v, w, x, y, z \in R$ which implies that

\[(3.10) \quad G(g(w), x, v) = 0\]
for all $v, x, w \in R$ by the primeness of $R$ and $\Delta \neq 0$. Taking $y + w$ instead of $w$ in (3.10), we get

$$0 = G(g(y + w), x, v) = G(g(y) + g(w) + 3G(y, y, w), x, v) = 3G(G(y, y, w), x, v) + 3G(G(y, w, w), x, v),$$

that is,

(3.11) \[ G(G(y, y, w), x, v) + G(G(y, w, w), x, v) = 0 \]

for all $v, w, x, y \in R$. Setting $y = -y$ in (3.11) and then combining the result with (3.11), we see that

(3.12) \[ G(G(y, y, w), x, v) = 0 \]

for all $v, w, x, y \in R$. We replace $y$ by $y + z$ in (3.12) to obtain

(3.13) \[ G(G(y, z, w), x, v) = 0 \]

for all $v, w, x, y, z \in R$. Substituting $yu$ for $y$ in (3.13) gives the relation

$$0 = G(G((yu, z, w), x, v) = G(\Delta(y, z, w)u + yG(u, z, w), x, v))$$

$$= G(\Delta(y, z, w)u, x, v) + G(yG(u, z, w), x, v)$$

$$= G(\Delta(y, z, w), x, v)u + \Delta(y, z, w)\Delta(u, x, v) + \Delta(y, x, v)G(u, z, w) + yG(G(u, z, w), x, v)$$

from which the equality (3.13) leads to the relation

(3.14) \[ G(\Delta(y, z, w), x, v)u + \Delta(y, z, w)\Delta(u, x, v) + \Delta(y, x, v)G(u, z, w) = 0 \]

for all $u, v, w, x, y, z \in R$. Letting $u = g(u)$ in (3.14), we have, by (3.4) and (3.10),

(3.15) \[ G(\Delta(y, z, w), x, v)g(u) = 0 \]

for all $u, v, w, x, y, z \in R$. Now, from Lemma 2.3 and $G \neq 0$, the equality (3.15) implies that

(3.16) \[ G(\Delta(y, z, w), x, v) = 0 \]

for all $v, w, x, y, z \in R$ and putting $y = z = w$ in (3.16) yields

(3.17) \[ G(\delta(y), x, v) = 0 \]

for all $v, x, y \in R$. Hence, by (3.16), the equality (3.14) reduces to the relation

(3.18) \[ \Delta(y, z, w)\Delta(u, x, v) + \Delta(y, x, v)G(u, z, w) = 0 \]
for all $u, v, w, x, y, z \in R$. Replacing $u$ by $\delta(u)$ in (3.18), the equality (3.17) tells us that

\[(3.19) \quad \Delta(y, z, w)\Delta(\delta(u), x, v) = 0\]

for all $u, v, w, x, y, z \in R$ and so taking $zt$ instead of $z$ in (3.19) gives

\[(3.20) \quad \Delta(y, z, w)t\Delta(\delta(u), x, v) = 0\]

for all $t, u, v, w, x, y, z \in R$. The primeness of $R$ now leads to

\[(3.21) \quad \Delta(\delta(u), x, v) = 0\]

for all $u, v, x \in R$ in view of $\Delta \neq 0$. By (3.20), we see that

\[\Delta(\delta(u + x), u + x, u + x) + \Delta(\delta(-u + x), -u + x, -u + x) = 0\]

for all $u, x \in R$ from which we use the relation (3.20) to get

\[(3.22) \quad 0 = 6\Delta(\Delta(u, u, x), u, u) + 6\Delta(\Delta(u, u, x), u, x) + \Delta(\Delta(u, u, x), x, x)\]

\[+ 2\Delta(\Delta(u, x, x), u, u) + 2\Delta(\Delta(u, x, x), u, x)\]

for all $u, x \in R$ and so substituting $-u$ for $u$ in (3.22) and combining the result with (3.23) yield

\[(3.23) \quad \Delta(\Delta(u, x, x), u, u) + 3\Delta(\Delta(u, u, x), u, x) = 0\]

for all $u, x \in R$. Taking $u = u + x$ in (3.23) and using (3.20), we have

\[0 = \Delta(\Delta(u, x, x), u, u) + 3\Delta(\Delta(u, u, x), u, x) + 3\Delta(\Delta(u, u, x), x, x)\]

\[+ 3\Delta(\Delta(u, x, x), u, x) + 3\Delta(\Delta(u, x, x), x, x)\]

for all $u, x \in R$. Let us replace $x$ by $-x$ in (3.24) and compare the result with (3.24). Then we obtain

\[(3.25) \quad \Delta(\Delta(u, x, x), x, x) + \Delta(\Delta(u, x, x), u, x) = 0\]

for all $u, x \in R$. Putting $u = u + x$ in (3.25) and using (3.20), the relation (3.25) reduces to

\[(3.26) \quad \Delta(\Delta(u, u, x), x, x) + 3\Delta(\Delta(u, x, x), x, x) + \Delta(\Delta(u, x, x), u, x) = 0\]

for all $u, x \in R$. Replacing $x$ by $-x$ in (3.26) and combining the result with (3.26), we have

\[(3.27) \quad \Delta(\Delta(u, x, x), x, x) = 0\]
for all $u, x \in R$. We substitute $ux$ for $u$ in (3.27) and then employ (3.20) and (3.27) to arrive at

$$
\Delta(u, x, x)\delta(x) = 0
$$

for all $u, x \in R$. Finally, taking $xy$ instead of $u$ in (3.28), it follows that

$$
\delta(x)y\delta(x) = 0
$$

for all $x, y \in R$. Since $R$ is prime, we see that $\delta = 0$ which implies $\Delta = 0$ by Lemma 2.2. This is a contradiction because of $\Delta \neq 0$. Hence $R$ is commutative. This completes the proof. \qed

**Theorem 3.2.** Let $R$ be a 2 and 3-torsion free prime near-ring. Suppose that there exists a nonzero symmetric generalized 3-derivation $(\mathcal{G}, \Delta) : R \times R \times R \to R$ such that

$$
g(x), g(x) + g(x) \in C(\mathcal{G}(u, v, w))
$$

for all $u, v, w, x, y \in R$, where $g$ is the trace of $\mathcal{G}$. Then $R$ is a commutative ring.

**Proof.** Assume that

$$
g(x), g(x) + g(x) \in C(\mathcal{G}(u, v, w))
$$

for all $u, v, w, x \in R$. Thus we get, from (3.29),

$$
\mathcal{G}(u + p, v, w)(g(x) + g(x)) = (g(x) + g(x))\mathcal{G}(u + p, v, w)
$$

$$
= (g(x) + g(x))[\mathcal{G}(u, v, w) + \mathcal{G}(p, v, w)]
$$

$$
= (g(x) + g(x))\mathcal{G}(u, v, w) + (g(x) + g(x))\mathcal{G}(p, v, w)
$$

$$
= g(x)\mathcal{G}(u, v, w) + g(x)\mathcal{G}(u, v, w) + g(x)\mathcal{G}(p, v, w) + g(x)\mathcal{G}(p, v, w)
$$

$$
= g(x)[\mathcal{G}(u, v, w) + \mathcal{G}(u, v, w) + \mathcal{G}(p, v, w) + \mathcal{G}(p, v, w)]
$$

(3.30)

$$
= [\mathcal{G}(u, v, w) + \mathcal{G}(u, v, w) + \mathcal{G}(p, v, w) + \mathcal{G}(p, v, w)]g(x)
$$

for all $p, u, v, w, x \in R$ and

$$
\mathcal{G}(u + p, v, w)(g(x) + g(x)) = \mathcal{G}(u + p, v, w)g(x) + \mathcal{G}(u + p, v, w)g(x)
$$

$$
= g(x)\mathcal{G}(u, v, w) + g(x)\mathcal{G}(u, v, w)
$$

$$
= g(x)[\mathcal{G}(u, v, w) + \mathcal{G}(p, v, w)] + g(x)[\mathcal{G}(u, v, w) + \mathcal{G}(p, v, w)]
$$

$$
= g(x)[\mathcal{G}(u, v, w) + \mathcal{G}(p, v, w) + \mathcal{G}(u, v, w) + \mathcal{G}(p, v, w)]
$$

(3.31)

$$
= [\mathcal{G}(u, v, w) + \mathcal{G}(p, v, w) + \mathcal{G}(u, v, w) + \mathcal{G}(p, v, w)]g(x)
$$
for all \( p, u, v, w, x \in R \). Comparing (3.30) and (3.31), we obtain
\[
\mathcal{G}(\langle u, p \rangle, v, w)g(x) = 0
\]
for all \( p, u, v, w, x \in R \). From Lemma 2.3, it follows that
\[
\mathcal{G}(\langle u, p \rangle, v, w) = 0
\]
for all \( p, u, v, w \in R \).

First, let us consider the case \( \Delta = 0 \). Since \( (\mathcal{G}, \Delta) \) is a symmetric generalized 3-right derivation, we substitute \( uz \) for \( u \) and \( up \) for \( p \) in (3.32) to get
\[
0 = \mathcal{G}(u\langle z, p \rangle, v, w)
= \mathcal{G}(u, v, w)\langle z, p \rangle + u\Delta(\langle z, p \rangle, v, w)
= \mathcal{G}(u, v, w)\langle z, p \rangle
\]
and so we see that
\[
\mathcal{G}(u)\langle z, p \rangle = 0
\]
for all \( p, u, z \in R \). Letting \( z = wz \) and \( p = wp \) in (3.33) yields
\[
\mathcal{G}(u)w\langle z, p \rangle = 0
\]
for all \( p, u, z, w \in R \). Since Lemma 2.2 tells us that \( \mathcal{G} \neq 0 \) implies \( g \neq 0 \), we conclude, from (3.34) and the primeness of \( R \), that \( \langle z, p \rangle = 0 \) is fulfilled for all \( p, z \in R \). Therefore \( (R, +) \) is abelian.

Since we have \( \Delta = 0 \) and \( (\mathcal{G}, \Delta) \) is a symmetric generalized 3-right derivation, we obtain
\[
\mathcal{G}(xy, z, t) = \mathcal{G}(x, z, t)y \in C(\mathcal{G}(u, v, w))
\]
for all \( t, u, v, w, x, y, z \in R \). Thus we have
\[
\mathcal{G}(x, z, t)y\mathcal{G}(u, v, w) = \mathcal{G}(u, v, w)\mathcal{G}(x, z, t)y
\]
for all \( t, u, v, w, x, y, z \in R \) which implies that
\[
\mathcal{G}(x, z, t)[y, \mathcal{G}(u, v, w)] = 0
\]
for all \( t, u, v, w, x, y, z \in R \). Taking \( xs \) in place of \( x \) in this equation, we get
\[
\mathcal{G}(x, z, t)s[y, \mathcal{G}(u, v, w)] = 0
\]
for all \( s, t, u, v, w, x, y \in R \) since \( (\mathcal{G}, \Delta) \) is a symmetric generalized 3-right derivation. Since \( R \) is prime and \( \mathcal{G} \neq 0 \), we obtain \([y, \mathcal{G}(u, v, w)] = 0\) for all \( u, v, w, y \in R \), i.e. \( \mathcal{G}(u, v, w) \in C \) for all \( u, v, w \in R \). Invoking Theorem 3.1, \( R \) is a commutative ring.
Let $\Delta \neq 0$ and let $\delta$ be the trace of $\Delta$. Since $(\mathcal{G}, \Delta)$ is a symmetric generalized 3-left derivation, replacing $u$ by $uz$ and $p$ by $up$ in (3.32) yields that

$$0 = \mathcal{G}(u(z,p), v, w)$$

$$= \Delta(u, v, w)(z, p) + u\mathcal{G}((z, p), v, w)$$

$$= \Delta(u, v, w)(z, p)$$

and so we obtain

(3.35) $\Delta(u, v, w)(z, p) = 0$

for all $p, u, z \in R$. Letting $z = sz$ and $p = sp$ in (3.35) yields

(3.36) $\Delta(u, v, w)s(z, p) = 0$

for all $p, s, u, v, w, z \in R$. Since $\Delta \neq 0$, we conclude, from (3.36) and the primeness of $R$, that $(z, p) = 0$ is fulfilled for all $p, z \in R$. Hence $(R, +)$ is abelian.

We claim that $R$ is a commutative ring. Indeed, from the hypothesis, we see that

(3.37) $[g(x), \mathcal{G}(u, v, w)] = 0$

for all $u, v, w, x \in R$. Hence if we let $x = x + y$ in (3.37), then we deduce from (3.37) that

(3.38) $[\mathcal{G}(x, x, y), \mathcal{G}(u, v, w)] + [\mathcal{G}(x, y, y), \mathcal{G}(u, v, w)] = 0$

for all $u, v, w, x, y \in R$. Setting $y = -y$ into (3.38) and comparing the result with (3.38), we obtain

(3.39) $[\mathcal{G}(x, y, y), \mathcal{G}(u, v, w)] = 0$

for all $u, v, w, x, y \in R$. Replacing $y$ by $y + z$ in (3.39) and using (3.39), we have

$[\mathcal{G}(x, y, z), \mathcal{G}(u, v, w)] = 0$

since $\mathcal{G}$ is permuting, i.e.,

(3.40) $\mathcal{G}(x, y, z)\mathcal{G}(u, v, w) = \mathcal{G}(u, v, w)\mathcal{G}(x, y, z)$

for all $u, v, w, x, y, z \in R$. Taking $ut$ instead of $u$ in (3.40) and applying Lemma 2.4(ii), we obtain

(3.41) $\Delta(u, v, w)t\mathcal{G}(x, y, z) - \mathcal{G}(x, y, z)\Delta(u, v, w)t$

$$+ u\mathcal{G}(t, v, w)\mathcal{G}(x, y, z) - \mathcal{G}(x, y, z)u\mathcal{G}(t, v, w) = 0$$
for all $t, u, v, w, x, y, z \in R$. Substituting $g(u)$ for $u$ in (3.41) and then utilizing the hypothesis and (3.40), we get

\begin{equation}
\Delta(g(u), v, w)[t, G(x, y, z)] = 0
\end{equation}

for all $t, u, v, w, x, y, z \in R$. Let us write in (3.42) $w_s$ instead of $w$. Then we have

\begin{equation}
\Delta(g(u), v, w)s[t, G(x, y, z)] = 0
\end{equation}

for all $s, t, u, v, w, x, y, z \in R$. Since $R$ is prime, we arrive at either $\Delta(g(u), v, w) = 0$ or $[t, G(x, y, z)] = 0$ for all $t, u, v, w, x, y, z \in R$.

Following the same process among (3.4)~(3.28) in the proof of Theorem 3.1, the case when $\Delta(g(u), v, w) = 0$ holds for all $u, v, w \in R$ leads to the contradiction.

Consequently, we obtain

\begin{equation}
[t, G(x, y, z)] = 0
\end{equation}

for all $t, x, y, z \in R$, i.e, $G(x, y, z) \in C$ for all $x, y, z \in R$. Therefore, Theorem 3.1 tells us that $R$ is a commutative ring which is complete the proof.

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