A NOTE ON GENOCCHI-ZETA FUNCTIONS

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Abstract. In this paper, we study the Genocchi-zeta functions which are entire functions in whole complex s-plane these zeta functions have the values of the Genocchi numbers and the Genocchi polynomials at negative integers respectively. That is $\zeta_G(1-k) = \frac{G_k}{k}$ and $\zeta_G(1-k, x) = \frac{G_k(x)}{k}$.

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will respectively denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we always assumes $|q - 1|_p < 1$. For a fixed positive integer $d$ with $(p, d) = 1$, set

$$X = X_d = \lim_{N \to \infty} \mathbb{Z}/dP^NZ, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{0 < a < dp, (a,p) = 1} (a + dp\mathbb{Z}_p),$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$. We say that $f$ is uniformly differentiable function at $a \in \mathbb{Z}_p$, and we write $f \in UD(\mathbb{Z}_p)$.
if the difference quotients, 
\[ F_f(x, y) = \frac{f(x) - f(y)}{x - y} \]
have a finite limit \( f'(a) \) as \((x, y) \to (a, a)\). Throughout this paper, we use the following notation:

\[
[x]_q = \frac{1 - q^x}{1 - q} = 1 + q + q^2 + \cdots + q^{x-1}, \quad \text{and} \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.
\]

For a function \( f \in UD(\mathbb{Z}_p) \), the fermionic \( p \)-adic invariant \( q \)-integral on \( \mathbb{Z}_p \) is defined as

\[
I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x,
\]

see \([ 1-2, 4-31 ]\).

Note that
\[
I_{-1}(f) = \lim_{q \to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x).
\]

In this study, we investigate some interesting integral equations related to \( I_{-1}(f) \). From these integral equations related to \( I_{-1}(f) \), we can derive some properties of the Genocchi-zeta function and Hurwitz’s type Genocchi-zeta function at negative integers.

\[ \textbf{2. On Genocchi-zeta Functions} \]

For any complex number \( t \), it is known that the Genocchi polynomials \( G_n(x) \) are defined by means of the following generating function:

\[ \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi. \]

They satisfy that \( G_0 = 0, G_1 = 1, G_2 = -1, G_3 = 0, G_4 = 1, \cdots, G_2k+1 = 0 \), and even indexed coefficients are given by

\[ G_n = 2(1 - 2^n)B_n, \]

where \( B_n \) are Bernoulli numbers. The Genocchi polynomials \( G_n(x) \) are defined by means of the following function:

\[ F(t, x) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi. \]
Let \( f_1(x) \) be translation with \( f_1(x) = f(x + 1) \). Then we have

\[ I_{-1}(f_1) + I_{-1}(f) = 2f(0). \]

(3)

If we take \( f(x) = e^{xt} \), then we can derive the Euler numbers from the integral equation of \( I_{-1}(f) \) as follows:

\[ \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2}{e^t + 1}. \]

Then we can derive

\[ t \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = t \sum_{n=0}^{\infty} \frac{G_{n+1} t^n}{n+1 \cdot n!}, \]

and

\[ t \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = t \sum_{n=1}^{\infty} \frac{G_{n+1}(x) t^n}{n+1 \cdot n!}. \]

Thus, we obtain

\[ \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \frac{G_{n+1}}{n+1}, \quad \text{and} \quad \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = \frac{G_{n+1}(x)}{n+1}. \]

For \( n \in \mathbb{N} \), we have

\[ \int_{\mathbb{Z}_p} f(x+n) d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) + 2 \sum_{\ell=0}^{n-1} (-1)^{n-1-\ell} f(\ell), \]

see [19 - 31].

By (4) and (5), if we take \( f(x) = x^k (k \in \mathbb{Z}^+) \), then we easily see that

\[ \int_{\mathbb{Z}_p} (x+n)^k d\mu_{-1}(x) - \int_{\mathbb{Z}_p} x^k d\mu_{-1}(x) = 2 \sum_{\ell=0}^{n-1} (-1)^{\ell-1} \ell^k, \quad \text{if} \quad n \equiv 0 \pmod{2}, \]

and

\[ \int_{\mathbb{Z}_p} (x+n)^k d\mu_{-1}(x) + \int_{\mathbb{Z}_p} x^k d\mu_{-1}(x) = 2 \sum_{\ell=0}^{n-1} (-1)^{\ell} \ell^k, \quad \text{if} \quad n \equiv 1 \pmod{2}. \]

From (6) and (7), we have the following proposition.

**Proposition 1.** For \( n \in \mathbb{N} \), we have

\[ (i) \quad \frac{G_{k+1}(n)}{k+1} - \frac{G_{k+1}}{k+1} = 2 \sum_{\ell=0}^{n-1} (-1)^{\ell-1} \ell^k, \quad \text{if} \quad n \equiv 0 \pmod{2}, \]

\[ (ii) \quad \frac{G_{k+1}(n)}{k+1} - \frac{G_{k+1}}{k+1} = 2 \sum_{\ell=0}^{n-1} (-1)^{\ell} \ell^k, \quad \text{if} \quad n \equiv 1 \pmod{2}. \]
(ii) \( \frac{G_{k+1}(n)}{k+1} + \frac{G_{k+1}}{k+1} = 2 \sum_{\ell=0}^{n-1} (-1)^{\ell} k^{\ell}, \) if \( n \equiv 1 \pmod{2}. \)

For a \( s \in \mathbb{C}, \) the Genocchi-zeta function and Hurwitz’s type Genocchi-zeta function are defined by

\[
\zeta_G(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \quad \text{and} \quad \zeta_G(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}.
\]

Thus, we note that Genocchi-zeta functions which are entire functions in whole complex \( s \)-plane and these zeta functions have the values of the Genocchi numbers and Genocchi polynomials at negative integers respectively. The gamma function is defined by

\[
\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \quad \text{for } s \in \mathbb{C}.
\]

From (2) and (9), we can derive that

\[
\frac{-1}{\Gamma(s)} \int_0^\infty t^{s-2} P(-t, x) dt = \frac{-1}{\Gamma(s)} \int_0^\infty t^{s-2} \frac{-2te^{-xt}}{1 + e^{-t}} dt
\]

\[
= 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(n+x)t} dt
\]

\[
= 2 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+x)^s}
\]

\[
= \zeta_G(s, x).
\]

For \( s \in \mathbb{C}, \) by (2) and (10), we easily see that

\[
\zeta_G(s, x) = \frac{-1}{\Gamma(s)} \int_0^\infty t^{s-2} P(-t, x) dt
\]

\[
= \frac{-1}{\Gamma(s)} \int_0^\infty t^{s-2} \sum_{n=0}^{\infty} \frac{G_n(x)}{n!} \frac{-t^n}{n!} dt
\]

\[
= 2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} G_n(x) \frac{1}{\Gamma(s)} \int_0^\infty t^{s-2+n} dt.
\]

Thus we can derive the following theorem.

**Theorem 2.** For \( k \in \mathbb{N}, \) we have
(i) $\zeta_G(1-k, x) = \frac{G_k(x)}{k^k}$.

(ii) $\zeta_G(1-k, 1) = \frac{G_k(1)}{k} = -\frac{G_k}{k} = -\zeta_G(1-k)$.

Let $f(x) = \sin ax$. From (3), we see that

$$0 = \int_{\mathbb{Z}_p} \sin a(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} axd\mu_{-1}(x)$$

$$= (1 + \cos a) \int_{\mathbb{Z}_p} \sin axd\mu_{-1}(x) + \sin a \int_{\mathbb{Z}_p} \cos axd\mu_{-1}(x).$$

If we take $f(x) = \cos ax$, then we obtain

$$2 = (1 + \cos a) \int_{\mathbb{Z}_p} \cos axd\mu_{-1}(x) - \sin a \int_{\mathbb{Z}_p} \sin axd\mu_{-1}(x).$$

Then we can derive

$$\int_{\mathbb{Z}_p} \cos axd\mu_{-1}(x) = 1, \quad \text{and} \quad \int_{\mathbb{Z}_p} \sin axd\mu_{-1}(x) = -\frac{\sin a}{1 + \cos a}.$$

Thus we obtain the following theorem.

**Theorem 3.** Let the notation and assumptions be as above. Then we have

$$\tan \frac{a}{2} = -\int_{\mathbb{Z}_p} \sin axd\mu_{-1}(x) = \sum_{n=0}^{\infty} (-1)^{n+1}a^{2n+1}(2n+2)!G_{2n+2}.$$ 

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