UNITARY INTERPOLATION ON $AX = Y$ IN ALG$L$

JOO HO KANG

Abstract. Given operators $X$ and $Y$ acting on a Hilbert space $H$, an interpolating operator is a bounded operator $A$ such that $AX = Y$. In this paper, we showed the following: Let $L$ be a subspace lattice acting on a Hilbert space $H$ and let $X_i$ and $Y_i$ be operators in $B(H)$ for $i = 1, 2, \cdots$. Let $P_i$ be the projection onto $\text{range} X_i$ for all $i = 1, 2, \cdots$. If $P_k E = EP_k$ for some $k$ in $\mathbb{N}$ and all $E$ in $L$, then the following are equivalent:

1. $\sup \left\{ \frac{\|E + \sum_{i=1}^{n} Y_i f_i\|}{\|E + \sum_{i=1}^{n} X_i f_i\|} : f \in H, n \in \mathbb{N}, E \in L \right\} < \infty,$

2. $\text{range} Y_k = \text{range} X_k = H$, and $< X_k f, X_k g > = < Y_k f, Y_k g >$ for some $k$ in $\mathbb{N}$ and for all $f$ and $g$ in $H$.

(1) There exists an operator $A$ in $\text{Alg}L$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$ and $AA^* = I = A^*A$.

1. Introduction

Let $A$ be a sublagebra of the algebra $B(H)$ of all bounded operators acting on a Hilbert space $H$. An interpolating operator is a bounded operator $A$ in $A$ such that $AX = Y$. Unitary interpolation problem on $AX = Y$ in $A$ is the following: Given operators $X$ and $Y$ in $B(H)$, when is there an unitary operator $A$ in $A$ such that $AX = Y$? We investigate unitary interpolation problems in the subalgebra $\text{Alg}L$ of $B(H)$ when $L$ is a subspace lattice on $H$.

We establish notations and conventions. A subspace lattice $L$ is a strongly closed lattice of projections acting on a Hilbert space $H$. We assume that projections 0 and $I$ lie in $L$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. The symbol $\text{Alg}L$ is the algebra of all bounded linear operators on $H$ that leave

Received July 13, 2009. Revised August 27, 2009.
2000 Mathematics Subject Classification : 47L35

Key words and phrases : Interpolation Problem, Unitary Interpolation Problem, Subspace Lattice, Alg $L$. This research was supported by the Daegu University Research Grant, 2007.
invariant all the projections in $\mathcal{L}$. If each pairwise projections of a subspace lattice $\mathcal{L}$ is commutative, then $\mathcal{L}$ is called a *commutative subspace lattice* or *CSL*. Let $M$ be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of $M$ and $\overline{M}^\perp$ the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of all natural numbers and $\mathbb{C}$ be the set of all complex numbers.

An operator $A$ in $\mathcal{B}(\mathcal{H})$ is *unitary* if it is surjective isometry.

2. Results

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators acting on $\mathcal{H}$. Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. $\text{Alg}\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ which leave invariant each projection $E$ in $\mathcal{L}$.

Assume that $X$ and $Y$ are operators in $\mathcal{B}(\mathcal{H})$ and $A$ is an operator in $\text{Alg}\mathcal{L}$ such that $AX = Y$. Then $\|E^\perp Yf\| = \|E^\perp AXf\| = \|E^\perp AE^\perp Xf\| \leq \|A\|\|E^\perp Xf\|$ for all $E \in \mathcal{L}$. If, for convenience, we adopt the convention that a fraction whose numerator and denominator are both zero is equal to zero, then the inequality above may be stated in the form

$$\sup_{E \in \mathcal{L}} \frac{\|E^\perp Yf\|}{\|E^\perp Xf\|} \leq \|A\|.$$  

In [5], we showed the above fact is a necessary and sufficient condition for existence of interpolating operator in $\text{Alg}\mathcal{L}$.

**Theorem A[5].** Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$ and let $X$ and $Y$ be operators in $\mathcal{B}(\mathcal{H})$. Let $P$ be the projection onto $\text{range} X$. If $PE = EP$ for every $E \in \mathcal{L}$, then the following are equivalent:

1. There exists an operator $A$ in $\text{Alg}\mathcal{L}$ such that $AX = Y$.

2. \[ \sup \left\{ \frac{\|E^\perp Yf\|}{\|E^\perp Xf\|} : f \in \mathcal{H}, E \in \mathcal{L} \right\} = K < \infty. \]

Moreover, if condition (2) holds, we may choose an operator $A$ such that $\|A\| = K$.

We introduce Lemmas to investigate unitary interpolation problems.

**Lemma 2.1.** Let $A$, $X$ and $Y$ be operators in $\mathcal{B}(\mathcal{H})$. If $Y = AX$ and $A|_{\text{range} X^\perp} = 0$, then $\ker A^* = \ker Y^*$. 
PROOF. Let $f$ be a vector in Ker $A^*$. Then $A^*f = 0$. So $X^*A^*f = 0$. Hence $Y^*f = 0$. Therefore, $f$ is in Ker $Y^*$.

Conversely, if $f$ is in Ker $Y^*$, then for any $g$ in $H$,

$$< f, Ag > = < A^*f, g >$$

$$= < A^*f, Xg_1 + g_2 >$$

for some $g_1$ and $g_2 \in \text{range } X^\perp$

$$= < A^*f, Xg_1 > + < A^*f, g_2 >$$

$$= < f, AXg_1 > + < f, Ag_2 >$$

$$= < f, Yg_1 > + 0$$

$$= 0$$

Hence $f \in \text{range } A^\perp (=\text{Ker } A^*)$.

LEMMA 2.2. Let $A$, $X$ and $Y$ be operators in $B(\mathcal{H})$. If $Y = AX$, $A |_{\text{range } X^\perp} = 0$ and $AA^* = I = A^*A$, then $Y$ has dense range in $\mathcal{H}$.

PROOF. Suppose that $Y = AX$ and $AA^* = I = A^*A$. Then $A$ and $A^*$ are bijective operators on $\mathcal{H}$. So $\text{range } A^\perp = 0$. Since $Y = AX$, Ker $A^* = \text{Ker } Y^*$ by Lemma 2.1. Hence $0 = \text{range } A^\perp = \text{range } Y^\perp$. So $Y$ has dense range in $\mathcal{H}$.

LEMMA 2.3. Let $A, X$ and $Y$ be operators in $B(\mathcal{H})$. Assume $Y = AX$, $A |_{\text{range } X^\perp} = 0$ and $A^*A = I = AA^*$. If $\text{range } Y \subset \text{range } X$, then $X$ has dense range in $\mathcal{H}$.

PROOF. By Lemma 2.2, $Y$ has dense range in $\mathcal{H}$. Since $\text{range } Y \subset \text{range } X$, $X$ has dense range in $\mathcal{H}$.

LEMMA 2.4. Let $A, X$ and $Y$ be operators in $B(\mathcal{H})$. Assume $Y = AX$, $A |_{\text{range } X^\perp} = 0$ and $A^*A = AA^*$. If $f \in \text{range } X^\perp$, then $A^*f \in \text{range } X^\perp$. 


PROOF. Let \( f \in \overline{\text{range } X^\perp} \) and \( g \in \mathcal{H} \). Then \( Xg = A^*g_1 + g_2 \) for some \( g_2 \in \overline{\text{range } A^*^\perp} \). So

\[
< A^*f, Xg > = < A^*f, A^*g_1 + g_2 > \\
= < A^*f, A^*g_1 > + < A^*f, g_2 > \\
= < Af, A^*g_1 > + 0 \\
= 0
\]

Hence \( A^*f \in \overline{\text{range } X^\perp} \). \( \square \)

LEMMA 2.5. Let \( A, X \) and \( Y \) be operators in \( \mathcal{B}(\mathcal{H}) \). If \( Y = AX \) and \( Af = 0 \) for \( f \) in \( \overline{\text{range } X^\perp} \), then the following statements are equivalent.

(1) \( \overline{\text{range } Y} \subset \overline{\text{range } X} \)

(2) If \( f \in \overline{\text{range } X^\perp} \), then \( A^*f \in \overline{\text{range } X^\perp} \).

PROOF. Suppose that \( \overline{\text{range } Y} \subset \overline{\text{range } X} \) and \( f \in \overline{\text{range } X^\perp} \). Then \( \overline{\text{range } X^\perp} \subset \overline{\text{range } Y^\perp} \) and so \( < A^*f, Xg > = < f, AXg > = < f, Yg > = 0 \) for any \( g \) in \( \mathcal{H} \). Hence \( A^*f \in \overline{\text{range } X^\perp} \).

Conversely, assume that if \( f \in \overline{\text{range } X^\perp} \), then \( A^*f \in \overline{\text{range } X^\perp} \). Let \( f \in \overline{\text{range } X^\perp} \). Then for any \( g \) in \( \mathcal{H} \), \( 0 = < A^*f, Xg > = < f, AXg > = < f, Yg > \). So \( f \in \overline{\text{range } Y^\perp} \). Hence \( \overline{\text{range } Y} \subset \overline{\text{range } X} \). \( \square \)

THEOREM 2.6. Let \( \mathcal{L} \) be a subspace lattice acting on a Hilbert space \( \mathcal{H} \) and let \( X \) and \( Y \) be operators in \( \mathcal{B}(\mathcal{H}) \). Then the following are equivalent:

(1) \( \sup \left\{ \frac{\|E^\perp Y f\|}{\|E^\perp X f\|} : f \in \mathcal{H}, E \in \mathcal{L} \right\} < \infty, \overline{\text{range } Y} = \overline{\text{range } X}, \overline{\text{range } X} = \mathcal{H} \) and \( < Xf, Xg > = < Yf, Yg > \) for all \( f \) and \( g \) in \( \mathcal{H} \).

(2) There exists an operator \( A \) in \( \text{Alg}\mathcal{L} \) such that \( AX = Y \) and \( AA^* = I = A^*A \).

PROOF. Assume that \( \sup \left\{ \frac{\|E^\perp Y f\|}{\|E^\perp X f\|} : f \in \mathcal{H}, E \in \mathcal{L} \right\} < \infty \). Then since \( \overline{\text{range } X} = \mathcal{H} \), there exists an operator \( A \) in \( \text{Alg}\mathcal{L} \) such that \( AX = Y \) and \( A|_{\overline{\text{range } X^\perp}} = 0 \) by Theorem A. Since \( < Xf, Xg > = < Yf, Yg > \) for all
f and g in \( \mathcal{H} \), \( \langle Xf, Xg \rangle = \langle AXf, AXg \rangle \). So \( X^*X = X^* A^*AX \). Since \( X \) has dense range in \( \mathcal{H} \), \( I = A^*A \). Let \( f \) and \( g \) be vectors in \( \mathcal{H} \).

\[
\langle Y^*AA^*Yf, g \rangle = \langle AA^*Yf, yg \rangle \\
= \langle AA^*AXf, AXg \rangle \\
= \langle A^*AXf, A^*AXg \rangle \\
= \langle Xf, Xg \rangle \\
= \langle Yf, Yg \rangle .
\]

Hence \( Y^*AA^*Y = Y^*Y \). Since range \( Y = \text{range} \ X \), \( Y \) has dense range in \( \mathcal{H} \). So \( AA^* = I \). Hence \( A^*A = I = AA^* \).

Conversely, if there exists an operator \( A \) in \( \text{Alg} \mathcal{L} \) such that \( AX_i = Y_i \) for \( i = 1, 2, \cdots, n \) and \( AA^* = I = A^*A \), then

\[
\sup \left\{ \frac{\|E^\perp Xf_i\|}{\|E^\perp Xf_i\|} : f_i \in \mathcal{H}, \ E \in \mathcal{L} \right\} < \infty \text{ and range } Y \subset \text{range } X \text{ by Lemmas 2.4 and 2.5. By Lemma 2.1, Ker } A^* = \text{Ker } Y^* \text{. Since } A^* \text{ is one-to-one, range } Y^\perp = 0 \text{. So range } Y = \mathcal{H} \text{. Since range } Y \subset \text{range } X, \text{ range } X = \mathcal{H} \text{. Hence } X \text{ and } Y \text{ have dense ranges in } \mathcal{H} \text{. Also, for all } f \text{ and } g \text{ in } \mathcal{H}, \langle Yf, Yg \rangle = \langle AXf, AXg \rangle = \langle X^*A^*AXf, g \rangle = \langle X^*IXf, g \rangle = \langle Xf, Xg \rangle .
\]

\[\square\]

**THEOREM 2.7.** Let \( \mathcal{L} \) be a subspace lattice acting on a Hilbert space \( \mathcal{H} \) and let \( X_i \) and \( Y_i \) be operators in \( B(\mathcal{H}) \) for \( i = 1, 2, \cdots, n \). Let \( P_i \) be the projection onto \( \text{range } X_i \) for all \( i = 1, 2, \cdots, n \). If \( P_k E = EP_k \) for some \( k \) in \( \{1, 2, \cdots, n\} \) and all \( E \in \mathcal{L} \), then the following are equivalent:

1. \( \sup \left\{ \frac{\|E^\perp (\sum_{i=1}^{n} Y_i f_i)\|}{\|E^\perp (\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H}, \ E \in \mathcal{L} \right\} < \infty \text{, range } Y_k = \text{range } X_k = \mathcal{H} \text{, and} \)

\[< X_k f, X_k g >= < Y_k f, Y_k g > \text{ for some } k \in \{1, 2, \cdots, n\} \text{ and for all } f \text{ and } g \text{ in } \mathcal{H} .\]

2. There exists an operator \( A \) in \( \text{Alg} \mathcal{L} \) such that \( AX_i = Y_i \) for \( i = 1, 2, \cdots, n \) and \( AA^* = I = A^*A \).

**PROOF.** Assume that \( \sup \left\{ \frac{\|E^\perp (\sum_{i=1}^{n} Y_i f_i)\|}{\|E^\perp (\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H}, \ E \in \mathcal{L} \right\} < \infty \). Then there
exists an operator $A$ in $\text{Alg} \mathcal{L}$ such that $AX_i = Y_i$ and $A|_{\text{range } Y_i} = 0$ for $i = 1, 2, \cdots, n$, by Theorem 2.2[5]. Since $<X_k f, X_k g>=<Y_k f, Y_k g>$ for some $k = 1, 2, \cdots, n$ and for all $f$ and $g$ in $\mathcal{H}$, $<X_k f, X_k g>=<AX_k f, AX_k g>$ for all $f$ and $g$ in $\mathcal{H}$. So $X_k^* X_k = X_k^* A^* AX_k$. Since $\text{range } X_k = \mathcal{H}$, $I = A^* A$. Let $f$ and $g$ be vectors in $\mathcal{H}$.

$$<Y_k^* A A^* Y_k f, g> = <AA^* Y_k f, Y_k g>$$

$$= <AA^* X_k f, AX_k g>$$

$$= <A^* AX_k f, A^* AX_k g>$$

$$= <X_k f, X_k g>$$

$$= <Y_k f, Y_k g>.$$

Hence $Y_k^* A A^* Y_k = Y_k^* Y_k$. Since $Y_k$ has dense range in $\mathcal{H}$, $AA^* = I$. Hence $A^* A = I = A A^*$.

Conversely, if there exists an operator $A$ in $\text{Alg} \mathcal{L}$ such that $AX_i = Y_i$, $A|_{\text{range } X_i} = 0$ for all $i = 1, 2, \cdots, n$ and $AA^* = I = A^* A$, then

$$\sup \left\{ \frac{\|E^+(\sum_{i=1}^n Y_i f_i)\|}{\|E^+(\sum_{i=1}^n X_i f_i)\|} : f \in \mathcal{H}, E \in \mathcal{L} \right\} < \infty \text{ and } \text{range } Y_i \subset \text{range } X_i \text{ for all } i = 1, 2, \cdots, n \text{ by Lemmas 2.4 and 2.5. By Lemma 2.1, Ker } A^* = \text{Ker } Y_i^* \text{ for all } i = 1, 2, \cdots, n. \text{ Since } A^* \text{ is one-to-one, } \text{range } Y_i^* = 0. \text{ So range } Y_i = \mathcal{H}. \text{ Since } \text{range } Y_i \subset \text{range } X_i, \text{range } X_i = \mathcal{H}. \text{ Hence } X_i \text{ and } Y_i \text{ have dense ranges in } \mathcal{H} \text{ for all } i = 1, 2, \cdots, n. \text{ Also, for all } f \text{ and } g \in \mathcal{H}, <Y_i f, Y_i g>=<AX_i f, AX_i g>=<X_i^* AX_i f, g> = <X_i^* I X_i f, g> = <X_i f, X_i g>.$$

THEOREM 2.8. Let $\mathcal{L}$ be a subspace lattice acting on a Hilbert space $\mathcal{H}$ and let $X_i$ and $Y_i$ be operators in $B(\mathcal{H})$ for $i = 1, 2, \cdots$. Let $P_i$ be the projection onto $\text{range } X_i$ for all $i = 1, 2, \cdots$. If $P_k E = E P_k$ for some $k \in \mathbb{N}$ and all $E \in \mathcal{L}$, then the following are equivalent:

1. $\sup \left\{ \frac{\|E^+(\sum_{i=1}^n Y_i f_i)\|}{\|E^+(\sum_{i=1}^n X_i f_i)\|} : f \in \mathcal{H}, n \in \mathbb{N}, E \in \mathcal{L} \right\} < \infty$, $\text{range } Y_k = \text{range } X_k = \mathcal{H}$, and $<X_k f, X_k g>=<Y_k f, Y_k g>$ for some $k \in \mathbb{N}$ and for all $f$ and $g$ in $\mathcal{H}$.

2. There exists an operator $A$ in $\text{Alg} \mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$ and $AA^* = I = A^* A$. 

**Proof.** Assume that \( \sup \left\{ \frac{\| E^+ (\sum_{i=1}^n Y_i f_i) \|}{\| E^+ (\sum_{i=1}^n X_i f_i) \|} : f \in \mathcal{H}, n \in \mathbb{N}, E \in \mathcal{L} \right\} < \infty \).

Then there exists an operator \( A \) in \( \text{Alg} \mathcal{L} \) such that \( AX_i = Y_i \) and \( A \mid_{\text{range } X_i} = 0 \) for \( i = 1, 2, \cdots \), by Theorem 2.3[5]. Since \( < X_k f, X_k g > = < Y_k f, Y_k g > \) for some \( k \in \mathbb{N} \) and for all \( f \) and \( g \) in \( \mathcal{H} \), \(< X_k f, X_k g > = < AX_k f, AX_k g > \) for all \( f \) and \( g \) in \( \mathcal{H} \). So \( X_k^* X_k = X_k^* A^* AX_k \). Since \( X_k \) has dense range in \( \mathcal{H} \), \( I = A^* A \). Let \( f \) and \( g \) be vectors in \( \mathcal{H} \).

\[
< Y_k^* AA^* Y_k f, g > = < AA^* Y_k f, Y_k g > \\
= < AA^* X_k f, AX_k g > \\
= < A^* X_k f, A^* AX_k g > \\
= < X_k f, X_k g > \\
= < Y_k f, Y_k g > .
\]

Hence \( Y_k^* AA^* Y_k = Y_k^* Y_k \). Since \( Y_k \) has dense range in \( \mathcal{H} \), \( AA^* = I \). Hence \( A^* A = I = AA^* \).

Conversely, if there exists an operator \( A \) in \( \text{Alg} \mathcal{L} \) such that \( AX_i = Y_i \), \( A \mid_{\text{range } X_i} = 0 \) for all \( i = 1, 2, \cdots \) and \( AA^* = I = A^* A \), then

\[
\sup \left\{ \frac{\| E^+ (\sum_{i=1}^n Y_i f_i) \|}{\| E^+ (\sum_{i=1}^n X_i f_i) \|} : f \in \mathcal{H}, n \in \mathbb{N}, E \in \mathcal{L} \right\} < \infty \text{ and } \text{range } Y_i \subset \text{range } X_i \text{ for all } i \text{ in } \mathbb{N} \text{ by Lemmas 2.4 and 2.5. By Lemma 2.1, Ker } A^* = \text{Ker } Y_i^* \text{ for each } i \text{ in } \mathbb{N}. \text{ Since } A^* \text{ is one-to-one, } \text{range } Y_i = \mathcal{H}. \text{ Hence } X_i \text{ and } Y_i \text{ have dense ranges in } \mathcal{H} \text{ for all } i = 1, 2, \cdots. \text{ And } < Y_i f, Y_i g > = < AX_i f, AX_i g > = < X_i^* A^* AX_i f, g > = < X_i^* I X_i f, g > = < X_i f, X_i g > \text{ for all } f \text{ and } g \text{ in } \mathcal{H}. \qed
\]

**References**


Joo Ho Kang
Dept. of Math.,
Daegu University
Daegu, Korea
jhkang@daegu.ac.kr