

**RULED SUBMANIFOLDS OF FINITE TYPE IN LORENTZIAN SPACE-TIMES**

**Dong-Soo Kim**

**Abstract.** In this article, we study ruled submanifolds with nondegenerate rulings in a Lorentzian space-time, which have finite type immersion. We give a condition for \( k \)-finite type submanifolds to be of finite type.

1. Introduction

In late 1970's B.-Y. Chen ([1,2]) introduced the notion of finite type immersion into a Euclidean space. A lot of works have been done in this field of study since then. He also extended the notion of finite type immersion of submanifolds into a pseudo-Euclidean space in 1980's. It can be defined formally in the following: A pseudo-Riemannian submanifold \( M \) of an \( m \)-dimensional pseudo-Euclidean space \( E^m_s \) with signature \((s, m-s)\) is said to be of finite type if its position vector field \( x \) can be expressed as a finite sum of eigenvectors of the Laplacian \( \Delta \) of \( M \), that is, \( x = x_0 + \sum_{i=1}^{k} x_i \), where \( x_0 \) is a constant map, \( x_1, \cdots, x_k \) non-constant maps such that \( \Delta x_i = \lambda_i x_i, \lambda_i \in \mathbb{R}, i = 1, 2, \cdots, k \). If \( \lambda_1, \lambda_2, \cdots, \lambda_k \) are different, then \( M \) is said to be of \( k \)-type with eigenvalues \( \lambda_1, \lambda_2, \cdots, \lambda_k \).

Ruled surfaces in a Euclidean space of finite type were studied by B.-Y. Chen et al. ([3]). On the other hand, F. Dillen et al. ([4]) classified ruled surfaces of finite type in 3-dimensional Lorentzian space-time as an open portion of minimal, circular or hyperbolic cylinders and isoparametric surfaces with null rulings. Recently, the author et al. classified ruled surfaces of finite type in Lorentzian space-time([10]),

---

Received May 25, 2010. Accepted June 7, 2010.

2000 Mathematics Subject Classification. 53B25, 53C40

*Key words and phrases.* ruled submanifold, finite type submanifold, minimal submanifold.

This study was financially supported by Chonnam National University, 2009.
and classified ruled hypersurfaces of finite type in Lorentzian space-time with nondegenerate rulings([7]). More recently, the author et al. classified ruled submanifolds of finite type in Lorentzian space-time with arbitrary rulings([8]).

In this article, we study ruled submanifolds with nondegenerate rulings in an \( m \)-dimensional Lorentzian space-time \( \mathbb{L}^m \), and give a sufficient condition for \( k \)-finite type submanifolds to be of finite type.

Throughout this paper, we assume that all objects are smooth and all surfaces are connected unless otherwise mentioned.

### 2. Preliminaries

Let \( \mathbb{E}^m_1 \) be an \( m \)-dimensional pseudo-Euclidean space of signature \((1, m-1)\) with the metric \( ds^2 = -dx_1^2 + dx_2^2 + \cdots + dx_m^2 \), where \((x_1, x_2, \cdots, x_m)\) denotes the standard coordinate system in \( \mathbb{E}^m_1 \). In particular, for \( m \geq 2 \), \( \mathbb{E}^m_1 \) is called an \( m \)-dimensional Lorentzian space-time \( \mathbb{L}^m \).

Let \( x: M^n \rightarrow \mathbb{L}^m \) be an isometric immersion of an \( n \)-dimensional pseudo-Riemannian submanifold \( M^n \) into \( \mathbb{L}^m \). From now on, a submanifold in \( \mathbb{L}^m \) always means pseudo-Riemannian, that is, the induced metric on the submanifold is non-degenerate.

For the components \( g_{ij} \) of the induced pseudo-Riemannian metric \( \langle \cdot, \cdot \rangle \) on \( M \) from that of \( \mathbb{L}^m \) we denote by \( (g_{ij}) \) (resp. \( G \)) the inverse matrix (resp. the determinant) of the matrix \( (g_{ij}) \). Then, the Laplacian \( \Delta \) on \( M \) is given by

\[
\Delta = -\frac{1}{\sqrt{|G|}} \sum_{i,j} \frac{\partial}{\partial x^i} (\sqrt{|G|} g^{ij} \frac{\partial}{\partial x^j}).
\]

Let \( M^n \) be a ruled submanifold in \( \mathbb{L}^m \) with nondegenerate rulings. Let \( x(s) \) be an orthogonal trajectory of the rulings in \( M^n \). Then we may assume that \( x(s) \) is parametrized by arc length. Let \( \{e_2(s), \cdots, e_n(s)\} \) be a set of orthonormal vector fields along \( x \) such that \( \{e_2(s), \cdots, e_n(s)\} \) spans the nondegenerate ruling of \( M^n \) through \( x(s) \). As in the Euclidean case, the set \( \{e_2(s), \cdots, e_n(s)\} \) can be chosen such that for all \( i \) and \( j \)

\[
\langle e_i'(s), e_j(s) \rangle = 0.
\]

Hence we can give a parametrization of \( M^n \) by

\[
X(s, t_2, \cdots, t_n) = x(s) + \sum_{i=2}^{n} t_i e_i(s).
\]
Definition. ([8]) Let a ruled submanifold $M^n$ with nondegenerate rulings over a non-null base curve $x(s)$ be given by (2.3). Then $M^n$ is said to be of $k$-finite if the base curve $x(s)$ and all the generators $e_2(s), \cdots, e_k(s)$ of the rulings $E(s, r)$ are of finite type with the same eigenvalues, and other generators $e_{k+1}(s), \cdots, e_n(s)$ are constant vector fields.

In this definition, every constant function is regarded as an eigenfunction with eigenvalue 0.

3. Ruled submanifolds of finite type

In this section, first of all we state some classification results for finite type ruled submanifolds in $\mathbb{L}^m$ with nondegenerate rulings ([8]).

Proposition 3.1. Let $M^n$ be a cylindrical ruled submanifold with nondegenerate rulings in $\mathbb{L}^m$. Then, $M^n$ is of finite type if and only if it is a cylinder over a curve of finite type.

Proposition 3.2. Let $M^n$ be a non-cylindrical ruled submanifold of finite type in $\mathbb{L}^m$ with nondegenerate rulings over a non-null base curve. Then, it is minimal or of $k$-finite type for some $k$ ($2 \leq k \leq n$).

For later use, we give a proof of Proposition 3.2 as follows.

Suppose that $M^n$ is a ruled submanifold in $\mathbb{L}^m$ with nondegenerate rulings of which parametrization is given by (2.3). Furthermore we may assume that the parametrization $X(s, t_2, \cdots, t_n)$ of $M^n$ satisfies

\begin{equation}
\langle x'(s), e_i(s) \rangle = 0, \langle e'_i(s), e_j(s) \rangle = 0.
\end{equation}

If we define $\epsilon_1$ and $Q$ by

\begin{align*}
\epsilon_1 &= \langle x'(s), x'(s) \rangle = \pm 1, \\
Q &= |\langle X_s, X_s \rangle|,
\end{align*}

respectively, then for sufficiently small $t_i$ we have

\begin{equation}
Q = 1 + 2\epsilon_1 \sum_{i=2}^{n} U_i(s) t_i + \epsilon_1 \sum_{i=2}^{n} V_{ij}(s) t_i t_j,
\end{equation}

where

\begin{align*}
U_i(s) &= \langle x'(s), e'_i(s) \rangle, \\
V_{ij}(s) &= \langle e'_i(s), e'_j(s) \rangle.
\end{align*}

Note that $Q$ is a polynomial in $t = (t_2, \cdots, t_n)$ with functions in $s$ as coefficients. The Laplacian $\Delta$ of $M^n$ can be expressed as follows:

\begin{equation}
\Delta = -\frac{\epsilon_1}{Q} \frac{\partial^2}{\partial s^2} - \sum_{i=2}^{n} \epsilon_i \frac{\partial^2}{\partial t_i^2} + \frac{\epsilon_1}{2Q^2} \frac{\partial Q}{\partial s} \frac{\partial}{\partial s} - \frac{1}{2Q} \sum_{i=2}^{n} \epsilon_i \frac{\partial Q}{\partial t_i} \frac{\partial}{\partial t_i},
\end{equation}
where for each $i$, $\epsilon_i$ denotes $\langle e_i, e_i \rangle = \pm 1$. The degree of $Q$ is at most 2. We divide by three cases according to the degree of $Q$.

**Case 1.** Suppose that $\deg(Q) = 0$. Then it follows from (3.2) and (3.3) that for each $j$, $e'_j(s)$ is orthogonal to itself. Suppose that $e'_j(s) = 0$ for all $j = 2, 3, \cdots, n$, then $M^n$ is cylindrical, which is a contradiction. Hence we may assume that there exists some $k(2 \leq k \leq n)$ which satisfies if $j = 2, 3, \cdots, k$, $e'_j(s) \neq 0$, otherwise $e'_j(s) = 0$. Since $x'(s)$ and $e_i(s)$ are perpendicular to a null vector $e'_2(s)$, we see that $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_n = 1$. Therefore we get

\[
\Delta = -\{ \frac{\partial^2}{\partial s^2} + \sum_{i=2}^{n} \frac{\partial^2}{\partial t_i^2} \},
\]

and hence for $j = 1, 2, 3, \cdots$

\[
\Delta^jX = -\{x^{(2j)}(s) + \sum_{i=2}^{k} t_i e_i^{(2j)}(s) \}.
\]

Hence, $M^n$ is of $k$-finite type. Note that the meaning of finiteness of the null curve $e_i(s)$ is formally defined.

**Case 2.** Suppose that $\deg(Q) = 1$. Then (3.2) shows that $\langle e'_i, e'_j \rangle = 0$ for all $i, j$. In particular, $\langle e'_i, e'_i \rangle = 0$ for all $i$. If $e'_i = 0$ for all $i$, then it follows from (3.2) that $\deg(Q) = 0$, which is a contradiction. Hence we may assume that there exists some $k(2 \leq k \leq n)$ which satisfies if $j = 2, 3, \cdots, k$, $e'_j(s) \neq 0$, otherwise $e'_j(s) = 0$. Since $x'(s)$ and $e_i(s)$ are perpendicular to a null vector $e'_2(s)$, we see that $\epsilon = \epsilon_2 = \cdots = \epsilon_n = 1$. Then for each $i = 3, \cdots, k$, $e'_i$ is a null vector which is orthogonal to $e'_2$. Thus for some function $a_i(s)$, we have $e'_i(s) = a_i(s)e'_2(s)$. Since for each $i = 2, \cdots, n$, $e_i$ is orthogonal to the null vector $e'_2$, we see that $\epsilon_i = 1$. It follows from (3.2) and (3.4) that

\[
\Delta X = \frac{P_1(t)}{Q^2(t)}.
\]

where $P_1(t)$ is a polynomial of degree $\leq 2$. The proof of the following lemma is straightforward.

**Lemma 1.** If $P$ is a polynomial in $t = (t_2, \cdots, t_n)$ with functions in $s$ as coefficients and $\deg(P) = d$, then we have for $l = 1, 2, \cdots$

\[
\Delta(\frac{P(t)}{Q^l}) = \frac{\tilde{P}(t)}{Q^{l+3}},
\]
where $\tilde{P}$ is a polynomial in $t$ with functions in $s$ as coefficients and $\deg(\tilde{P}) \leq d + 2$.

Suppose that $M^n$ is of $k$-type. Then there exist constants $c_1, \cdots, c_k$ such that ([2, p.256])

$$\Delta^k X + c_1 \Delta^{k-1} X + \cdots + c_{k-1} \Delta X + c_k (X - X_0) = 0.$$  \hfill (3.9)

We know that $X - X_0$ is a linear function in $t$ with functions in $s$ as coefficients. By applying Lemma 1, for $r = 1, 2, \cdots, k$ we get

$$\Delta^r X = \frac{P_r(t)}{Q^{r-1}}, \quad \deg(P_r) \leq 2r.$$ \hfill (3.10)

Hence, by counting the degree of each term in (3.9), we see that $c_k = 0$. Since the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$ are different, we get $c_{k-1} \neq 0$. Thus the sum in (3.9) never vanish, unless $\Delta X = 0$. Therefore $M^n$ is a minimal submanifold.

**Case 3.** Finally, suppose that $\deg(Q) = 2$. It follows from (3.2) and (3.4) that

$$\Delta X = \frac{P_1(t)}{Q^2(t)},$$ \hfill (3.11)

where $P_1(t)$ is a polynomial of degree $\leq 3$. The proof of the following lemma is also straightforward.

**Lemma 2.** If $P$ is a polynomial in $t = (t_2, \cdots, t_m)$ with functions in $s$ as coefficients and $\deg(P) = d$, then we have for $l = 1, 2, \cdots$

$$\Delta(P(t)/Q^l) = \frac{\tilde{P}(t)}{Q^{l+3}},$$ \hfill (3.12)

where $\tilde{P}$ is a polynomial in $t$ with functions in $s$ as coefficients and $\deg(\tilde{P}) \leq d + 4$.

Suppose that $M^n$ is of $k$-type. Then there exist constants $c_1, \cdots, c_k$ such that

$$\Delta^k X + c_1 \Delta^{k-1} X + \cdots + c_{k-1} \Delta X + c_k (X - X_0) = 0.$$ \hfill (3.13)

By applying Lemma 2, for $r = 1, 2, \cdots, k$ we get

$$\Delta^r X = \frac{P_r(t)}{Q^{r-1}}, \quad \deg(P_r) \leq 4r - 1.$$ \hfill (3.14)
Hence, as in Case 2, we see that $c_k$ must vanish and $c_{k-1} \neq 0$. Thus the sum in (3.13) never be zero, unless $\Delta X = 0$. Therefore $M^n$ is a minimal submanifold.

This completes the proof of Proposition 3.2.

4. Ruled submanifolds of $k$-finite type

Now, we may raise a natural question: Does the converse of Proposition 3.2 hold?

First, it follows immediately from the proof of Proposition 3.2 that

**Proposition 4.1.** Let $M^n$ be a non-cylindrical ruled submanifold in $L^m$ with nondegenerate rulings over a unit speed curve $x(s)$ which is parametrized by (2.3) satisfying (3.1). Then the following hold:

1) If $M^n$ is of finite type with $\deg Q = 0$, then $M^n$ is of $k$-finite type for some $2 \leq k \leq n$.

2) If $M^n$ is of $k$-finite type for some $2 \leq k \leq n$ with $\deg Q = 0$, then $M^n$ is of finite type.

In the proof of Proposition 3.2, we see that if $\deg Q \geq 1$, then the rulings are spacelike. Conversely, for spacelike rulings we prove

**Theorem 4.2.** Let $M^n$ be a non-cylindrical ruled submanifold in $L^m$ with spacelike rulings over a unit speed curve $x(s)$ which is parametrized by (2.3) satisfying (3.1). If $M^n$ is of finite type and is of $k$-finite type for some $2 \leq k \leq n$, then we have $\deg Q = 0$.

**Proof.** Suppose that $\deg Q \geq 1$. Then the proof of Proposition 3.2 shows that $\Delta X = 0$. By a straightforward computation, we we obtain (4.1)

$$2Q^2 \Delta X = P_0(s) + \sum_{i=2}^{n} P_i(s)t_i + \sum_{i,j=2}^{n} P_{ij}(s)t_it_j + \sum_{i,j,k=2}^{n} P_{ijk}(s)t_it_jt_k,$$
where

(4.2)
\[ P_0(s) = -2\epsilon_1 \{ x''(s) + \sum_i \epsilon_i U_i e_i \}, \]

\[ P_i(s) = 2x'(s)U'_i - 2\epsilon_1 e''_i - 4x''(s)U_i - 4U_i \sum_j \epsilon_j U_j e_j - 2\epsilon_1 \sum_j \epsilon_j V_{ij} e_j, \]

\[ P_{ij}(s) = V'_{ij} x'(s) + 2U'_i e'_j - 2x''(s)V_{ij} - 4e''_i U_j - 2V_{ij} \sum_k \epsilon_k U_k e_k \]

\[ - 4U_i \sum_k \epsilon_k V_{jk} e_k, \]

\[ P_{ijk}(s) = V'_{ij} e'_k - 2V_{ij} e''_k - 2V_{ij} \sum_l \epsilon_l V_{kl} e_l. \]

Since \( P_0(s) = 0 \), we have

(4.3)
\[ x''(s) + \sum_i \epsilon_i U_i e_i = 0. \]

Since \( P_i(s) = 0 \), (4.3) shows that

(4.4)
\[ e''_i(s) = \epsilon_1 U'_i x'(s) - \sum_j \epsilon_j V_{ij} e_j. \]

Together with (4.3) and (4.4), it follows from \( \sum_{i,j=2}^n P_{ij}(s) t_i t_j = 0 \) that

(4.5)
\[ V'_{ij} = \epsilon_1 (U_i U_j)' \]

and

(4.6)
\[ V'_{ij} x'(s) = U'_i e'_j + U'_j e'_i. \]

Now suppose that \( U'_i \neq 0 \) for some \( i \). Since the coefficient of \( t_i^3 \) vanishes, (4.5) shows that

(4.7)
\[ e'_i(s) = \epsilon_1 U_i x'(s), \]

In case \( U'_j = 0 \) for some \( j \neq i \), (4.5), (4.6), and (4.7) imply

(4.8)
\[ e'_j(s) = \epsilon_1 U_j x'(s). \]

Thus it follows from (4.7) and (4.8) that for all \( j = 2, \ldots, n \)

(4.9)
\[ e'_j(s) = \epsilon_1 U_j x'(s), \]

and hence for all \( j, k = 2, \ldots, n \)

(4.10)
\[ V_{jk} = \epsilon_1 U_j U_k. \]
Since $x(s)$ is of finite type, it is a vector valued finite linear combination of functions of the form:

\[(4.11) \quad 1, s, \cos \alpha s, \sin \alpha s, \cosh \beta s, \sinh \beta s.\]

From the hypothesis that $e_i(s)$ is also of finite type with the same eigenvalues as $x(s)$, (4.7) yields $U_i$ must be a constant. This contradiction shows that $U_i$ are constant for all $i = 2, \cdots, n$. Hence it follows from (4.4) and (4.5) that $V_{ij}$ are constants and

\[(4.12) \quad e''_i(s) = -\sum_j \epsilon_j V_{ij} e_j.\]

Since the rulings are spacelike we see that

\[(4.13) \quad \epsilon_2 = \cdots = \epsilon_n = 1.\]

For the symmetric matrix $V = (V_{ij})$, there exists an orthogonal matrix $R = (R_{ij})$ which satisfies $R^tVR = D$, where $D = D(\lambda_2, \cdots, \lambda_n)$ is a diagonal matrix. Letting $\bar{e}_i = \sum_j R_{ji}e_j$, $\{\bar{e}_i\}$ is an orthonormal basis of the rulings satisfying (3.1). Furthermore, $\{\bar{e}_i\}$ satisfies for all $i = 2, \cdots, n$

\[(4.14) \quad \bar{e}''_i = -\lambda_i \bar{e}_i.\]

Hence, without loss of generality we may assume that the orthonormal basis $\{e_i\}$ already satisfies (4.14), that is,

\[(4.15) \quad V_{ij} = \delta_{ij}\lambda_i.\]

Since $M$ is of $k$-finite type for some $2 \leq k \leq n$, we see that

\[(4.16) \quad \lambda_2 = \cdots = \lambda_k = \lambda, e'_{k+1} = \cdots = e'_n = 0.\]

We divide by two cases according as $\lambda = 0$ or not.

**Case 1.** Suppose that $\lambda = 0$. Then it follows from (4.12) and (4.15) that $e''_i(s) = 0$. Since $x(s)$ is of the same type as $e_i(s)$, (4.3) shows that $x''(s) = 0$, that is $U_i = 0$. This implies that $\deg Q = 0$, which is a contradiction.

**Case 2.** Suppose that $\lambda \neq 0$. Consider a curve $\bar{x}(s)$ defined by

\[(4.17) \quad \bar{x}(s) = x(s) - \sum_i \frac{U_i}{\lambda} e_i.\]

Then $\bar{x}(s)$ is orthogonal to the rulings and furthermore it satisfies $\bar{x}''(s) = 0$. Therefore we may assume that the curve $x(s)$ is already a straight
line. Thus $x(s)$ is of null 1-type, but $e_i(s)$ is of non-null 1-type. This
contradicts to the $k$-finiteness of $X$.

This completes the proof of Theorem 4.2.

References


Department of Mathematics, Chonnam National University, Kwangju 500-757, Korea

E-mail: dosokim@chonnam.chonnam.ac.kr