COMPATIBLE MAPS OF TWO TYPES AND COMMON FIXED POINT THEOREMS ON INTUITIONISTIC FUZZY METRIC SPACE

JONG SEO PARK*

Abstract. In this paper, we introduce the concept of compatible mapping of type(1) and type(2), prove the some properties and common fixed point theorem for such maps in intuitionistic fuzzy metric space. Also, we give the example. Our research are an extension for the results of Kutukcu and Sharma[3] and Park et.al.[11].

1. Introduction

Cho et.al.[1] and Sharma[13] gave fuzzy version of compatible maps and proved common fixed point theorems for compatible maps in fuzzy metric spaces. Also, Kutukcu and Sharma[3] introduced two types compatible maps and proved a common fixed point theorem for such maps in Menger probabilistic metric spaces.

Recently, Park et.al.[10] defined the intuitionistic fuzzy metric space in which it is a little revised in Park[4], and Park et.al.[6] studied a fixed point theorem of Banach for the contractive mapping of a complete intuitionistic fuzzy metric space. Also, Park et al.[11] introduced the notion of compatible mapping and compatible mapping of type(α) in intuitionistic fuzzy metric spaces, and proved common fixed point theorem for five mappings in this space.

In this paper, we introduce the concept of compatible mapping of type(1) and type(2), and prove the some properties and common fixed point theorem for such maps on intuitionistic fuzzy metric space.

*Corresponding author.

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2. Preliminaries

We recall some definitions, properties and known results in the intuitionistic fuzzy metric space as following:

Let us recall(see [12]) that a continuous $t-$norm is a operation $\ast : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions: (a)$\ast$ is commutative and associative, (b)$\ast$ is continuous, (c)$a \ast 1 = a$ for all $a \in [0, 1]$, (d)$a \ast b \leq c \ast d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0, 1])$.

Also, a continuous $t-$conorm is a operation $\circ : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions: (a)$\circ$ is commutative and associative, (b)$\circ$ is continuous, (c)$a \circ 0 = a$ for all $a \in [0, 1]$, (d)$a \circ b \geq c \circ d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0, 1])$.

**Definition 2.1.** ([5])The 5-tuple $(X, M, N, \ast, \circ)$ is said to be an intuitionistic fuzzy metric space if $X$ is an arbitrary set, $\ast$ is a continuous $t-$norm, $\circ$ is a continuous $t-$conorm and $M, N$ are fuzzy sets on $X$ satisfying the following conditions; for all $x, y, z \in X$, such that (a)$M(x, y, t) > 0$, (b)$M(x, y, t) = 1 \iff x = y$, (c)$M(x, y, t) = M(y, x, t)$, (d)$M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$, (e)$M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous, (f)$N(x, y, t) > 0$, (g)$N(x, y, t) = 0 \iff x = y$, (h)$N(x, y, t) = N(y, x, t)$, (i)$N(x, y, t) \circ N(y, z, s) \geq N(x, z, t + s)$, (j)$N(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous.

Note that $(M, N)$ is called an intuitionistic fuzzy metric on $X$. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ with respect to $t$, respectively.

**Definition 2.2.** ([9]) Let $X$ be an intuitionistic fuzzy metric space. (a) $\{x_n\}$ is said to be convergent to a point $x \in X$ (written $x_n \to x$) if and only if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer $n_0 = n_0(\epsilon, \lambda)$ such that $M(x_n, x, \epsilon) > 1 - \lambda$, $N(x_n, x, \epsilon) < \lambda$ for all $n \geq n_0$. 

(b) \( \{x_n\} \) is called a Cauchy sequence if for every \( \epsilon > 0 \) and \( \lambda \in (0, 1) \), there exists an integer \( n_0 = n_0(\epsilon, \lambda) \) such that \( M(x_n, x_{n+p}, \epsilon) > 1 - \lambda \), \( N(x_n, x_{n+p}, \epsilon) < \lambda \) for all \( n \geq n_0 \) and \( p > 0 \).

(c) \( X \) is complete if every Cauchy sequence converges in \( X \).

**Lemma 2.3.** ([6]) Let \( \{x_n\} \) be a sequence in an intuitionistic fuzzy metric space \( X \) with \( t * t \geq t \) and \( t \circ t \leq t \). If there exist a number \( k \in (0, 1) \) such that
\[
M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t), \quad N(x_{n+2}, x_{n+1}, kt) \leq N(x_{n+1}, x_n, t)
\]
for all \( x, y \in X \), \( t > 0 \) and \( n = 1, 2, \ldots \), then \( \{x_n\} \) is a Cauchy sequence in \( X \).

**Lemma 2.4.** ([8]) Let \( X \) be an intuitionistic fuzzy metric space. If there exists a number \( k \in (0, 1) \) such that for all \( x, y \in X \) and \( t > 0 \),
\[
M(x, y, kt) \geq M(x, y, t), \quad N(x, y, kt) \leq N(x, y, t),
\]
then \( x = y \).

3. Compatible mappings of type\((\alpha - 1)\) and type\((\alpha - 2)\)

In this section, we introduce the concepts of compatible mappings of type\((\alpha - 1)\) and type\((\alpha - 1)\), and give some properties of these mappings for our main results.

**Definition 3.1.** ([7]) Let \( A \) and \( B \) be mappings from intuitionistic fuzzy metric space \( X \) into itself. The mappings are said to be compatible if
\[
\lim_{n \to \infty} M(ABx_n, BAx_n, t) = 1, \quad \lim_{n \to \infty} N(ABx_n, BAx_n, t) = 0
\]
for all \( t > 0 \), whenever \( \{x_n\} \subset X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \) for some \( z \in X \).

**Definition 3.2.** Let \( A \) and \( B \) be mappings from intuitionistic fuzzy metric space \( X \) into itself. The mappings are said to be compatible of type\((\alpha)\) if
\[
\lim_{n \to \infty} M(ABx_n, BBx_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} M(BAx_n, AAx_n, t) = 1,
\]
\[
\lim_{n \to \infty} N(ABx_n, BBx_n, t) = 0 \quad \text{and} \quad \lim_{n \to \infty} N(BAx_n, AAx_n, t) = 0
\]
for all \( t > 0 \), whenever \( \{x_n\} \subset X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \) for some \( z \in X \).
\textbf{Definition 3.3.} Let $A$ and $B$ be mappings from intuitionistic fuzzy metric space $X$ into itself. The mappings are said to be compatible of type$(\alpha - 1)$ if
\[
\lim_{n \to \infty} M(ABx_n, BBx_n, t) = 1, \quad \lim_{n \to \infty} N(ABx_n, BBx_n, t) = 0
\]
for all $t > 0$, whenever $\{x_n\} \subset X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z$ for some $z \in X$.

\textbf{Definition 3.4.} Let $A$ and $B$ be mappings from intuitionistic fuzzy metric space $X$ into itself. The mappings are said to be compatible of type$(\alpha - 2)$ if
\[
\lim_{n \to \infty} M(BAx_n, AAx_n, t) = 1, \quad \lim_{n \to \infty} N(BAx_n, AAx_n, t) = 0
\]
for all $t > 0$, whenever $\{x_n\} \subset X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z$ for some $z \in X$.

\textbf{Remark 3.5.} Clearly, if pair of mappings $(A, B)$ is compatible of type$(\alpha - 1)$, then the pair $(B, A)$ is compatible of type$(\alpha - 2)$. Furthermore, if $A$ and $B$ are compatible mappings of type$(\alpha)$, then the pair $(A, B)$ is compatible of type$(\alpha - 1)$ as well as type$(\alpha - 2)$.

The following is example of pair of self maps in an intuitionistic fuzzy metric space which are compatible of type$(\alpha - 1)$ and type$(\alpha - 2)$ but not compatible.

\textbf{Example 3.6.} Let $X$ be an intuitionistic fuzzy metric space. Define self maps $A$ and $B$ as follows
\[
Ax = \begin{cases} 
2 - x & \text{if } 0 \leq x < 1 \\
2 & \text{if } 1 \leq x \leq 2 
\end{cases}, \quad Bx = \begin{cases} 
x & \text{if } 0 \leq x < 1 \\
2 & \text{if } 1 \leq x \leq 2 
\end{cases}.
\]
If we define $\{x_n\} \subset X$ by $x_n = 1 - \frac{1}{n}$, then we have
\[
\lim_{n \to \infty} M(Ax_n, 1, t) = 1, \quad \lim_{n \to \infty} N(Ax_n, 1, t) = 0.
\]
Hence $\lim_{n \to \infty} Ax_n = 1$. Similarly, $\lim_{n \to \infty} Bx_n = 1$. Also,
\[
\lim_{n \to \infty} M(ABx_n, BAx_n, t) = M(1, 2, t) \neq 1,
\]
\[
\lim_{n \to \infty} N(ABx_n, BAx_n, t) = N(1, 2, t) \neq 0.
\]
Thus the pair $(A, B)$ is not compatible. But
\[
\lim_{n \to \infty} M(ABx_n, BBx_n, t) = M(1, 1, t) = 1,
\]
\[
\lim_{n \to \infty} N(ABx_n, BBx_n, t) = N(1, 1, t) = 0.
\]
Hence \((A, B)\) is compatible of type\((\alpha - 1)\). Also,
\[
\lim_{n \to \infty} M(BAx_n, AAx_n, t) = 1, \quad \lim_{n \to \infty} N(BAx_n, AAx_n, t) = 0.
\]
Therefore \((A, B)\) is compatible of type\((\alpha - 2)\).

**Proposition 3.7.** Let \(A\) and \(B\) be self maps of an intuitionistic fuzzy metric space \(X\) with 
\[t \ast t \geq t\quad \text{and}\quad t \ast t \geq t\] 
for all \(t \in [0, 1]\).

(a) If \(B\) is continuous, then the pair \((A, B)\) is compatible of type\((\alpha - 1)\) if and only if \(A\) and \(B\) are compatible,

(b) If \(A\) is continuous, then the pair \((A, B)\) is compatible of type\((\alpha - 2)\) if and only if \(A\) and \(B\) are compatible.

**Proof.** (a) Let \(\{x_n\} \subset X\) such that \(\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z\) for some \(z \in X\) and let \((A, B)\) be compatible of type\((\alpha - 1)\). Since \(B\) is continuous, we have \(\lim_{n \to \infty} BAx_n = Bz, \lim_{n \to \infty} BBx_n = Bz\). Therefore
\[
M(ABx_n, BAx_n, t) \geq M(ABx_n, BBx_n, \frac{t}{2}) \ast M(BAx_n, BAx_n, \frac{t}{2})
\]
\[
N(ABx_n, BAx_n, t) \leq N(ABx_n, BBx_n, \frac{t}{2}) \ast N(BAx_n, BAx_n, \frac{t}{2}).
\]
Therefore
\[
\lim_{n \to \infty} M(ABx_n, BAx_n, t) \geq 1 \ast 1 \geq 1,
\]
\[
\lim_{n \to \infty} N(ABx_n, BAx_n, t) \leq 0 \ast 0 \leq 0.
\]
Hence \(A\) and \(B\) are compatible.

Conversely, let the maps \(A\) and \(B\) are compatible, then using the continuity of \(B\), we have
\[
M(ABx_n, BBx_n, t) \geq M(ABx_n, BAx_n, \frac{t}{2}) \ast M(BAx_n, BBx_n, \frac{t}{2})
\]
\[
N(ABx_n, BBx_n, t) \leq N(ABx_n, BAx_n, \frac{t}{2}) \ast N(BAx_n, BBx_n, \frac{t}{2}).
\]
Therefore \(\lim_{n \to \infty} M(ABx_n, BBx_n, t) = 1, \lim_{n \to \infty} N(ABx_n, BBx_n, t) = 0\). Hence \(A\) and \(B\) are compatible of type\((\alpha - 1)\).

(b) It is similar to the proof of (a).

**Proposition 3.8.** Let \(A\) and \(B\) be self maps of an intuitionistic fuzzy metric space \(X\). If the pair \((A, B)\) is compatible of type\((\alpha - 1)\) and \(Az = Bz\) for some \(z \in X\), then \(ABz = BBz\).
Proof. Let \( \{x_n\} \subset X \) defined by \( \lim_{n \to \infty} x_n = z \) for \( n \in \mathbb{N} \) and let \( Az = Bz \). Then we have \( \lim_{n \to \infty} Ax_n = A z \), \( \lim_{n \to \infty} Bx_n = B z \). Since the pair \((A, B)\) is compatible of type \((\alpha - 1)\), we obtain

\[
M(Az, BBz, t) = \lim_{n \to \infty} M(Ax_n, BBx_n, t) = 1,
\]

\[
N(Az, BBz, t) = \lim_{n \to \infty} N(Ax_n, BBx_n, t) = 0.
\]

Therefore \( ABz = BBz \).

**Proposition 3.9.** Let \( A \) and \( B \) be self mappings of an intuitionistic fuzzy metric space \( X \). If the pair \((A, B)\) is compatible of type \((\alpha - 2)\) and \( Az = Bz \) for some \( z \in X \), then \( BAz = AAz \).

**Proof.** It is similar to the proof of Proposition 3.8.

**Proposition 3.10.** Let \( A \) and \( B \) be self maps of an intuitionistic fuzzy metric space \( X \). If the pair \((A, B)\) is compatible of type \((\alpha - 1)\) and \( \{x_n\} \subset X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \) for some \( z \in X \), then \( \lim_{n \to \infty} BBx_n = Az \) if \( A \) is continuous at \( z \).

**Proof.** Since \((A, B)\) is compatible of type \((\alpha - 1)\) and \( A \) is continuous at \( z \), we have \( \lim_{n \to \infty} ABx_n = Az \) and

\[
\lim_{n \to \infty} M(Ax_n, BBx_n, t) = 1, \quad \lim_{n \to \infty} N(Ax_n, BBx_n, t) = 0.
\]

Therefore

\[
M(Az, BBx_n, t) \geq M(Az, ABx_n, \frac{t}{2}) \cdot M(ABx_n, BBx_n, \frac{t}{2}),
\]

\[
N(Az, BBx_n, t) \leq N(Az, ABx_n, \frac{t}{2}) \cdot N(ABx_n, BBx_n, \frac{t}{2}).
\]

Therefore

\[
\lim_{n \to \infty} M(Az, BBx_n, t) \geq 1 \cdot 1 \geq 1, \quad \lim_{n \to \infty} N(Az, BBx_n, t) \leq 0 \cdot 0 \leq 0.
\]

Hence \( \lim_{n \to \infty} BBx_n = Az \).

**Proposition 3.11.** Let \( A \) and \( B \) be self maps of an intuitionistic fuzzy metric space \( X \). If the pair \((A, B)\) is compatible of type \((\alpha - 2)\) and \( \{x_n\} \subset X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \) for some \( z \in X \), then \( \lim_{n \to \infty} AAx_n = Bz \) if \( B \) is continuous at \( z \).

**Proof.** It is similar to the proof of Proposition 3.10.
4. Main results using compatibility of type$(\alpha - 1)$ and type$(\alpha - 2)$

Now, we prove the common fixed point theorems for six mappings satisfying some conditions.

**Theorem 4.1.** Let $A, B, P, Q, S$ and $T$ be self maps on a complete intuitionistic fuzzy metric space $X$ with $t \ast t \geq t, t \circ t \leq t$ for all $t \in [0, 1]$ and satisfy the condition:

(a) $P(X) \subset ST(X), Q(X) \subset AB(X)$,
(b) There exist $k \in (0, 1)$ such that for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$,

\[
M(Px, Qy, kt) \geq M(ABx, STy, t) \ast M(Px, ABx, t) \ast M(Qy, STy, t) \\
\ast M(Px, STy, \alpha t) \ast M(Qy, ABx, (2 - \alpha)t),
\]

\[
N(Px, Qy, kt) \leq N(ABx, STy, t) \circ N(Px, ABx, t) \circ N(Qy, STy, t) \\
\circ N(Px, STy, \alpha t) \circ N(Qy, ABx, (2 - \alpha)t),
\]

(c) $PB = BP, ST = TS, AB = BA$ and $QT = TQ$,
(d) Either $P$ or $AB$ are continuous,
(e) The pairs $(P, AB)$ and $(Q, ST)$ are compatible of type$(\alpha - 1)$ or type$(\alpha - 2)$.

Then $A, B, P, Q, S$ and $T$ have a unique common fixed point in $X$.

**Proof.** Let $x_0$ be an arbitrary point in $X$. From (a), there exists $x_1, x_2 \in X$ such that $Px_0 = STx_1 = y_0$ and $Qx_1 = ABx_2 = y_1$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $y_{2n} = Px_{2n} = STx_{2n+1}, y_{2n+1} = Qx_{2n+1} = ABx_{2n+2}$ for $n = 0, 1, 2, \ldots$.

First, by taking $x = x_{2n}, y = x_{2n+1}$ for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$ in (b), we have

\[
M(Px_{2n}, Qx_{2n+1}, kt) = M(y_{2n}, y_{2n+1}, kt) \\
\geq M(y_{2n-1}, y_{2n}, t) \ast M(y_{2n}, y_{2n-1}, t) \ast M(y_{2n+1}, y_{2n}, t) \\
\ast M(y_{2n}, y_{2n+1}, (1 - q)t) \ast M(y_{2n+1}, y_{2n-1}, (1 + q)t) \\
\geq M(y_{2n-1}, y_{2n}, t) \ast M(y_{2n-1}, y_{2n+1}, t) \ast M(y_{2n+1}, y_{2n+1}, qt),
\]

\[
N(Px_{2n}, Qx_{2n+1}, kt) = N(y_{2n}, y_{2n+1}, kt) \\
\leq N(y_{2n-1}, y_{2n}, t) \circ N(y_{2n}, y_{2n-1}, t) \circ N(y_{2n+1}, y_{2n}, t) \\
\circ N(y_{2n}, y_{2n}, (1 - q)t) \circ N(y_{2n+1}, y_{2n-1}, (1 + q)t) \\
\leq N(y_{2n-1}, y_{2n}, t) \circ N(y_{2n}, y_{2n+1}, t) \circ N(y_{2n}, y_{2n+1}, qt).
\]
Letting \( q \to 1 \), we have
\[
M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t) \ast M(y_{2n}, y_{2n+1}, t),
\]
\[
N(y_{2n}, y_{2n+1}, kt) \leq N(y_{2n-1}, y_{2n}, t) \circ N(y_{2n}, y_{2n+1}, t).
\]

Also, we have
\[
M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) \ast M(y_{2n+1}, y_{2n+2}, t),
\]
\[
N(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n}, y_{2n+1}, t) \circ N(y_{2n+1}, y_{2n+2}, t).
\]

Therefore, for all \( n \) even or odd,
\[
M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t) \ast M(y_n, y_{n+1}, t),
\]
\[
N(y_n, y_{n+1}, kt) \leq N(y_{n-1}, y_n, t) \circ N(y_n, y_{n+1}, t).
\]

Consequently, for positive integers \( n \) and \( p \), it follows that
\[
M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t) \ast M(y_n, y_{n+1}, \frac{t}{k^p}),
\]
\[
N(y_n, y_{n+1}, kt) \leq N(y_{n-1}, y_n, t) \circ N(y_n, y_{n+1}, \frac{t}{k^p}).
\]

Since \( \lim_{p \to \infty} M(y_n, y_{n+1}, \frac{t}{k^p}) = 1 \) and \( \lim_{p \to \infty} N(y_n, y_{n+1}, \frac{t}{k^p}) = 0 \), we have
\[
M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t), \quad N(y_n, y_{n+1}, kt) \leq N(y_{n-1}, y_n, t).
\]

Hence by Lemma 2.3, \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, \( \{y_n\} \) converges to a point \( z \in X \). Also, since subsequences of \( \{y_n\} \) converges to this point \( z \), \( \lim_{n \to \infty} P x_{2n} = z = \lim_{n \to \infty} AB x_{2n} = \lim_{n \to \infty} ST x_{2n+1} = \lim_{n \to \infty} Q x_{2n+1} = \lim_{n \to \infty} ST x_{2n+1}.

Now, we prove the case for continuity of \( AB \), and compatible of type \((\alpha-1)\) of \((P, AB)\) and \((Q, ST)\). Since \( AB \) is continuous, \( \lim_{n \to \infty} AB x_{2n} = AB z \) and \( \lim_{n \to \infty} AB P x_{2n} = AB z \). Also, since \((P, AB)\) is compatible of type \((\alpha-1)\), we have \( \lim_{n \to \infty} PP x_{2n} = AB z \).

Second, by taking \( x = P x_{2n}, y = x_{2n+1} \) with \( \alpha = 1 \) in (b), we obtain
\[
M(P P x_{2n}, Q x_{2n+1}, kt)
\geq M(A B P x_{2n}, S T x_{2n+1}, t) \ast M(P P x_{2n}, A B P x_{2n}, t) \ast M(Q x_{2n+1}, S T x_{2n+1}, t) \ast M(Q x_{2n+1}, A B P x_{2n}, t),
\]
\[
N(P P x_{2n}, Q x_{2n+1}, kt)
\leq N(A B P x_{2n}, S T x_{2n+1}, t) \circ N(P P x_{2n}, A B P x_{2n}, t) \circ N(Q x_{2n+1}, S T x_{2n+1}, t) \circ N(Q x_{2n+1}, A B P x_{2n}, t)
which implies that

\[
M(z, ABz, kt) \\
\geq M(z, ABz, t) \ast M(ABz, ABz, t) \ast M(z, z, t) \ast M(z, ABz, t) \\
\ast M(z, ABz, t) \\
\geq M(z, ABz, t), \\
N(z, ABz, kt) \\
\leq N(z, ABz, t) \circ N(ABz, ABz, t) \circ N(z, z, t) \circ N(z, ABz, t) \\
\circ N(z, ABz, t) \\
\leq N(z, ABz, t)
\]

Thus, by Lemma 2.3, \( z = ABz \).

Third, by taking \( x = z, y = x_{2n+1} \) with \( \alpha = 1 \) in (b), we obtain

\[
M(Pz, Qx_{2n+1}, kt) \\
\geq M(ABz, STx_{2n+1}, t) \ast M(Pz, ABz, t) \ast M(Qx_{2n+1}, STx_{2n+1}, t) \\
\ast M(Pz, STx_{2n+1}, t) \ast M(Qx_{2n+1}, ABz, t), \\
N(Pz, Qx_{2n+1}, kt) \\
\leq N(ABz, STx_{2n+1}, t) \circ N(Pz, ABz, t) \circ N(Qx_{2n+1}, STx_{2n+1}, t) \\
\circ N(Pz, STx_{2n+1}, t) \circ N(Qx_{2n+1}, ABz, t).
\]

Letting \( n \to \infty \), we have

\[
M(z, Pz, kt) \\
\geq M(z, z, t) \ast M(z, Pz, t) \ast M(z, z, t) \ast M(z, Pz, t) \ast M(z, z, t) \\
\geq M(z, Pz, t), \\
N(z, Pz, kt) \\
\leq N(z, z, t) \circ N(z, Pz, t) \circ N(z, z, t) \circ N(z, Pz, t) \circ N(z, z, t) \\
\leq N(z, Pz, t).
\]

Therefore \( z = Pz \) by Lemma 2.4.
Fourth, by taking \( x = z, y = x_{2n+1} \) with \( \alpha = 1 \) in (b) and using (c), we obtain
\[
M(PBz, Qx_{2n+1}, kt) \\
\geq M(ABBz, STx_{2n+1}, t) * M(PBz, ABBz, t) * M(Qx_{2n+1}, \\
STx_{2n+1}, t) * M(PBz, STx_{2n+1}, t) * M(Qx_{2n+1}, ABBz, t), \\
N(PBz, Qx_{2n+1}, kt) \\
\leq N(ABBz, STx_{2n+1}, t) \circ N(PBz, ABBz, t) \circ N(Qx_{2n+1}, \\
STx_{2n+1}, t) \circ N(PBz, STx_{2n+1}, t) \circ N(Qx_{2n+1}, ABBz, t).
\]

Letting \( n \to \infty \), we get
\[
M(z, Bz, kt) \\
\geq M(z, Bz, t) * M(Bz, Bz, t) * M(z, z, t) * M(z, Bz, t) * M(z, Bz, t) \\
\geq M(z, Bz, t), \\
N(z, Bz, kt) \\
\leq N(z, Bz, t) \circ N(Bz, Bz, t) \circ N(z, z, t) \circ N(z, Bz, t) \circ N(z, Bz, t) \\
\leq N(z, Bz, t).
\]

Thus, \( z = Bz \) from Lemma 2.4. Since \( z = ABz \), we have \( z = Az \).
Therefore \( z = Az = Bz = Pz \).

Fifth, since there exists \( w \in X \) such that \( z = Pz = STw \) from (a),
taking \( x = x_{2n}, y = w \) with \( \alpha = 1 \) in (b), we get
\[
M(Px_{2n}, Qw, kt) \\
\geq M(ABx_{2n}, STw, t) * M(Px_{2n}, ABx_{2n}, t) * M(Qw, STw, t) * M(Px_{2n}, ABx_{2n}, t) * M(Qw, STw, t), \\
N(Px_{2n}, Qw, kt) \\
\leq N(ABx_{2n}, STw, t) \circ N(Px_{2n}, ABx_{2n}, t) \circ N(Qw, STw, t) \circ N(Px_{2n}, STw, t) \circ N(Qw, ABx_{2n}, t).
\]

Letting \( n \to \infty \), we have
\[
M(z, Qw, kt) \\
\geq M(z, z, t) * M(z, z, t) * M(z, Qw, t) * M(z, z, t) * M(z, Qw, t) \\
\geq M(z, Qw, t), \\
N(z, Qw, kt) \\
\leq N(z, z, t) \circ N(z, Qw, t) \circ N(z, z, t) \circ N(z, Qw, t) \\
\leq N(z, Qw, t).
\]
Thus, $z = Qw$ from Lemma 2.4. Hence $STw = z = Qw$. Since $Q$ and $ST$ are compatible of type($\alpha - 1$), we have $Q(ST)w = ST(ST)w$. Hence $Qz = STz$.

Sixth, by taking $x = x_{2n}$, $y = z$ with $\alpha = 1$ in (b) and using fifth, we get

$$M(Px_{2n}, Qz, kt) \geq M(ABx_{2n}, STz, t) * M(Px_{2n}, ABx_{2n}, t) * M(Qz, STz, t) * M(Px_{2n}, STz, t) * M(Qz, ABx_{2n}, t),$$

$$N(Px_{2n}, Qz, kt) \leq N(ABx_{2n}, STz, t) \circ N(Px_{2n}, ABx_{2n}, t) \circ N(Qz, STz, t) \circ N(Px_{2n}, STz, t) \circ N(Qz, ABx_{2n}, t).$$

Letting $n \to \infty$, we obtain

$$M(z, Qz, kt) \geq M(z, Qz, t) * M(z, z, t) * M(Qz, Qz, t) * M(z, Qz, t) \geq M(z, Qz, t),$$

$$N(z, Qz, kt) \leq N(z, Qz, t) \circ N(z, z, t) \circ N(Qz, Qz, t) \circ N(z, Qz, t) \circ N(z, Qz, t).$$

Thus $z = Qz$ from Lemma 2.4. Since $STz = Qz$, we have $z = STz$. Therefore $z = A_z = B_z = P_z = Qz = STz$.

Seventh, by taking $x = x_{2n}$, $y = Tz$ with $\alpha = 1$ in (b) and using (c), we get

$$M(Px_{2n}, QTz, kt) \geq M(ABx_{2n}, STTz, t) * M(Px_{2n}, ABx_{2n}, t) * M(QTz, STTz, t) \circ M(Px_{2n}, STTz, t) * M(QTz, ABx_{2n}, t),$$

$$N(Px_{2n}, QTz, kt) \leq N(ABx_{2n}, STTz, t) \circ N(Px_{2n}, ABx_{2n}, t) \circ N(QTz, STTz, t) \circ N(Px_{2n}, STTz, t) \circ N(QTz, ABx_{2n}, t).$$
Letting \( n \to \infty \), we get
\[
M(z, Tz, kt) \\
\geq M(z, Tz, t) \ast M(z, z, t) \ast M(Tz, Tz, t) \ast M(z, Tz, t) \\
\geq M(z, Tz, t), \\
N(z, Tz, kt) \\
\leq N(z, Tz, t) \circ N(z, z, t) \circ N(Tz, Tz, t) \circ N(z, Tz, t) \circ N(z, Tz, t) \\
\leq N(z, Tz, t).
\]
Thus \( z = Tz \) from Lemma 2.4. Since \( z = STz \), we have \( z = STz \).

Therefore, \( z = Tz \).

Letting \( n \to \infty \), we get
\[
M(z, Pz, kt) \\
\geq M(z, Pz, t) \ast M(Pz, Pz, t) \ast M(z, Pz, t) \ast M(z, Pz, t) \\
\geq M(z, Pz, t), \\
N(z, Pz, kt) \\
\leq N(z, Pz, t) \circ N(Pz, Pz, t) \circ N(z, z, t) \circ N(z, Pz, t) \circ N(z, Pz, t) \\
\leq N(z, Pz, t). \]
Thus $z = Pz$ from Lemma 2.4. Using from fifth to seventh, we have $z = Qz = STz = Sz = Tz$.

Ninth, since there exists $w \in X$ such that $z = Qz = ABw$ from (a), taking $x = w$, $y = x_{2n+1}$ with $\alpha = 1$ in (b), we have

$$M(Pw, Qx_{2n+1}, kt) \geq M(ABw, STx_{2n+1}, t) * M(Pw, ABw, t) * M(Qx_{2n+1}, STx_{2n+1}, t)$$

$$N(Pw, Qx_{2n+1}, kt) \leq N(AB, STx_{2n+1}, t) \diamond N(Pw, ABw, t) \diamond N(Qx_{2n+1}, STx_{2n+1}, t)$$

Letting $n \to \infty$, we have

$$M(z, Pw, kt) \geq M(z, z, t) * M(z, Pw, t) * M(z, z, t) * M(z, Pw, t) * M(z, z, t)$$

$$N(z, Pw, kt) \leq N(z, z, t) \diamond N(z, Pw, t) \diamond N(z, z, t) \diamond N(z, Pw, t) \diamond N(z, z, t)$$

Thus $z = Pw$ from Lemma 2.4. Since $z = Qz = ABw$, we have $Pw = ABw$. Since $(P, AB)$ is compatible or type $(\alpha - 1)$, we have $Pz = ABz$.

Also, $z = Bz$ from fourth. Thus $z = Az = Bz = Pz$. Hence $z$ is the common fixed point of $A, B, S, T, P$ and $Q$.

Similarly, it is clear that $z$ is also the common fixed point of the six maps in the case $P$ is continuous, and $(P, AB)$ and $(Q, ST)$ are compatible of type $(\alpha - 2)$.

Tenth, let $u$ be another common fixed point of the six maps, taking $x = z$, $y = u$ with $\alpha = 1$ in (b), we have

$$M(Pz, Qu, kt) \geq M(ABz, STu, t) * M(Pz, ABz, t) * M(Qu, STu, t)$$

$$N(Pz, Qu, kt) \leq N(ABz, STu, t) \diamond N(Pz, ABz, t) \diamond N(Qu, STu, t) \diamond N(Pz, STu, t) \diamond N(Qu, ABz, t).$$
Therefore
\[ M(z, u, kt) \geq M(z, u, t) \ast M(z, z, t) \ast M(u, u, t) \ast M(u, z, t) \]
\[ \geq M(z, u, t), \quad N(z, u, kt) \leq N(z, u, t) \diamond N(z, z, t) \diamond N(u, u, t) \diamond N(u, z, t) \]
\[ \leq N(z, u, t). \]

Thus \( z = u \) from Lemma 2.4.

\[ \square \]

**Corollary 4.2.** Let \( P \) and \( Q \) be self maps on a complete intuitionistic fuzzy metric space \( X \) with \( t \ast t \geq t, \ t \diamond t \leq t \) for all \( t \in [0, 1] \). If there exists a constant \( k \in (0, 1) \) such that
\[ M(Px, Qy, kt) \geq M(x, y, t) \ast M(x, Px, t) \ast M(y, Qy, (2 - \alpha)t), \]
\[ N(Px, Qy, kt) \leq N(x, y, t) \diamond N(x, Px, t) \diamond N(y, Qy, (2 - \alpha)t) \]
for all \( x, y \in X, \ \alpha \in (0, 2) \) and \( t > 0 \), then \( P \) and \( Q \) have a unique common fixed point in \( X \).

**Proof.** The proof follows from Theorem 4.1 taking \( A = B = S = T = I_X \) (the identity map on \( X \)).

\[ \square \]

**Example 4.3.** Let \( (X, d) \) be the metric space with \( X = [0, 1] \). Denote \( a \ast b = \min\{a, b\} \) and \( a \diamond b = \max\{a, b\} \) for all \( a, b \in [0, 1] \) and let \( M_d, N_d \) be fuzzy sets on \( X^2 \times (0, \infty) \) defined as follows:
\[ M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}. \]

Then \( (M_d, N_d) \) is an intuitionistic fuzzy metric on \( X \) and \( X \) is an intuitionistic fuzzy metric space. Let self mappings \( A, B, P, Q, S \) and \( T \) be maps from \( X \) into itself defined as
\[ Ax = \frac{x}{5}, \quad Bx = \frac{x}{7}, \quad Px = \frac{x}{5}, \quad Qx = 0, \quad Sx = x, \quad Tx = \frac{x}{2} \]
for all \( x \in X \). Then \( P(X) = [0, \frac{1}{5}] \subset [0, \frac{1}{2}] = ST(X) \) and \( Q(X) = \{0\} \subset [0, \frac{1}{15}] = AB(X) \). Clearly, (b), (c) and (d) of Theorem 4.1 are satisfied. Moreover, \( (P, AB) \) and \( (Q, ST) \) are compatible of type \((\alpha - 1)\). In fact, if \( \lim_{n \to \infty} x_n = 0 \) where \( \{x_n\} \subset X \) such that \( \lim_{n \to \infty} Px_n = \)
\[
\lim_{n \to \infty} ABx_n = 0 \quad \text{and} \quad \lim_{n \to \infty} Qx_n = \lim_{n \to \infty} STx_n = 0 \quad \text{for some} \quad 0 \in X,
\]
then
\[
\lim_{n \to \infty} M(PABx_n, ABABx_n, t) = 1,
\]
\[
\lim_{n \to \infty} N(PABx_n, ABABx_n, t) = 0.
\]

and
\[
\lim_{n \to \infty} M(ABPx_n, PPx_n, t) = 1, \quad \lim_{n \to \infty} N(ABPx_n, PPx_n, t) = 0.
\]

Similarly, \((P, AB)\) and \((Q, ST)\) are also compatible of type\((\alpha - 2)\). Thus all conditions of Theorem 4.1 are satisfied and 0 is a unique common fixed point of \(A, B, P, Q, S\) and \(T\) on \(X\).

References

*Department of Mathematics Education, Chinju National University of Education, Jinju 660-756, Korea
E-mail: *parkjs@cue.ac.kr