INTERVAL-VALUED FUZZY SUBGROUPS AND RINGS

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Abstract. We introduce the concepts of interval-valued fuzzy subgroups [resp. normal subgroups, rings and ideals] and investigate some of it’s properties.

1. Introduction


2. Preliminaries
In this section, we list some concepts and results related to interval-valued fuzzy set theory and needed in next sections.

Let $D(I)$ be the set of all closed subintervals of the unit interval $[0, 1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where $M^L$ and $M^U$ are the lower and the upper end points respectively. Especially, we denoted $0 = [0, 0], 1 = [1, 1]$, and $a = [a, a]$ for every $a \in (0, 1)$. We also note that

(i) $(\forall M, N \in D(I)) (M = N \iff M^L = N^L, M^U = N^U)$,

(ii) $(\forall M, N \in D(I)) (M = N \leq M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the complement of $M$, denoted by $M^C$, is defined by $M^C = 1 - M = [1 - M^U, 1 - M^L]$ (See[12]).

**Definition 2.1**[7,14]. A mapping $A : X \rightarrow D(I)$ is called an interval-valued fuzzy set(is short, IVFS) in $X$, denoted by $A = [A^L, A^U]$, if $A^L, A^U \in I^X$ such that $A^L \leq A^U$, i.e., $A^L(x) \leq A^U(x)$ for each $x \in X$, where $A^L(x)[\text{resp } A^U(x)]$ is called the lower[resp upper] end point of $x$ to $A$. For any $[a, b] \in D(I)$, the interval-valued fuzzy $A$ in $X$ defined by $A(x) = [A^L(x), A^U(x)] = [a, b]$ for each $x \in X$ is denoted by $\tilde{a}$ and if $a = b$, then the IVF empty set and the interval-valued fuzzy whole set in $X$, respectively.

We will denote the set of all IVFSs in $X$ as $D(I)^X$. It is clear that set $A = [A, A] \in D(I)^X$ for each $A \in I^X$.

For sets $X, Y$ and $Z$, $f = (f_1, f_2) : X \rightarrow Y \times Z$ is called a complex mapping if $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow Z$ are mappings.

**Definition 2.1’**[1,9]. Let $X$ be a set. A complex mapping $A = (\mu_A, \nu_A) : X \rightarrow I \times I$ is called a intuitionistic fuzzy set(is short, IFS) in $X$ if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mappings $\mu_A : X \rightarrow I$ and $\nu_A : X \rightarrow I$ denote the degree of membership(namely $\mu_A(x)$) and the degree of nonmembership(namely $\nu_A(x)$) of each $x \in X$ to $A$, respectively. in particular, $0_\sim$ and $1_\sim$ denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in $X$ defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in X$, respectively.

We will denoted the set of all the IFSs in $X$ as IFS($X$).
**Result 2.** A[2, Lemma 1]. We define two mappings $f : D(I)^X \rightarrow \text{IFS}(X)$ and $g : \text{IFS}(X) \rightarrow D(I)^X$ as follows, respectively:

(i) $f(A) = (A^L , 1 - A^U)$, $\forall A \in D(I)^X$,

(ii) $g(B) = [\mu_B, 1 - \nu_B]$, $\forall B \in \text{IFS}(X)$.

In this case, we write as $f(A) = A^*$ and $g(B) = B^*$, respectively. Then

(a) $g \circ f = 1_{D(I)^X}$, i.e., $g(f(A)) = A$, $\forall A \in D(I)^X$.

(b) $f \circ g = 1_{\text{IFS}(X)}$, i.e., $f(g(B)) = B$, $\forall B \in \text{IFS}(X)$.

**Definition 2.2**[7]. An IVFS $A$ is called an *interval-valued fuzzy point* (in short, IVFP) in $X$ with the support $x \in X$ and the value $[a , b] \in D(I)$ with $b > 0$, denoted by $A = x_{[a,b]}$, if for each $y \in X$

$$A(y) = \begin{cases} 
[a,b] & \text{if } y = x, \\
0 & \text{otherwise}
\end{cases}$$

In particular, if $b = a$, then $x_{[a,b]}$ is denoted by $x_a$.

We will denote the set of all IVFPs in $X$ as $\text{IVFP}(X)$.

**Definition 2.3** [7]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then:

(i) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.


(iv) $A \cup B = [A^L \lor B^L , A^U \lor B^U]$.

(iv)$' \bigcup_{\alpha \in \Gamma} A_\alpha = [\bigvee_{\alpha \in \Gamma} A^L_\alpha , \bigvee_{\alpha \in \Gamma} A^U_\alpha]$.

(v) $A \cap B = [A^L \land B^L , A^U \land B^U]$.

(v)$' \bigcap_{\alpha \in \Gamma} A_\alpha = [\bigwedge_{\alpha \in \Gamma} A^L_\alpha , \bigwedge_{\alpha \in \Gamma} A^U_\alpha]$.
Result 2.B[7, Theorem 1]. Let $A, B, C \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then:

(a) $\tilde{0} \subset A \subset \tilde{1}$.
(b) $A \cup B = B \cup A$, $A \cap B = B \cap A$.
(c) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.
(d) $A, B \subset A \cup B$, $A \cap B \subset A, B$.
(e) $A \cap (\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} (A \cap A_\alpha)$.
(f) $A \cup (\bigcap_{\alpha \in \Gamma} A_\alpha) = \bigcap_{\alpha \in \Gamma} (A \cup A_\alpha)$.
(g) $(\tilde{0})^c = \tilde{1}$, $(\tilde{1})^c = \tilde{0}$.
(h) $(A^c)^c = A$.
(i) $(\bigcup_{\alpha \in \Gamma} A_\alpha)^c = \bigcap_{\alpha \in \Gamma} A_\alpha^c$, $(\bigcap_{\alpha \in \Gamma} A_\alpha)^c = \bigcup_{\alpha \in \Gamma} A_\alpha^c$.

Definition 2.4[7]. Let $A \in D(I)^X$ and let $x_M \in \text{IVF}_P(X)$. Then:

(i) The set $\{x \in X : A_U(x) > 0\}$ is called the support of $A$ and is denoted by $S(A)$.
(ii) $x_M$ said to belong to $A$, denoted by $x_M \in A$, if $M^L \leq A_U(x)$ and $M^U \leq A_L(x)$ for each $x \in X$.

It is obvious that $A = \bigcup_{x_M \in A} x_M$ and $x_M \in A$ if and only if $x_M^L \in A_L$ and $x_M^U \in A_U$.

Definition 2.5[7]. Let $f : X \to Y$ be a mapping, let $A \in D(I)^X$ and let $B \in D(I)^Y$. Then:

(i) the image of $A$ under $f$, denoted by $f(A)$, is an IVFS in $Y$ defined as follows: For each $y \in Y$,

$$f(A)_L(y) = \begin{cases} \bigvee_{y = f(x)} A_L(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

and

$$f(A)_U(y) = \begin{cases} \bigvee_{y = f(x)} A_U(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$
(ii) the preimage of $B$ under $f$, denoted by $f^{-1}(B)$, is an IVFS in $Y$ defined as follows: For each $y \in Y$,
\[
f^{-1}(B)^L(y) = (B^L \circ f)(x) = B^L(f(x))
\]
and
\[
f^{-1}(B)^U(y) = (B^U \circ f)(x) = B^U(f(x))
\].

It can be easily seen that $f(A) = [f(A^L), f(A^U)]$ and $f^{-1}(B) = [f^{-1}(B^L), f^{-1}(B^U)]$.

**Result 2.** [7, Theorem 2]. Let $f : X \to Y$ be a mapping and $g : Y \to Z$ be a mapping. Then:

(a) $f^{-1}(B^c)^c = [f^{-1}(B)^c]^c$, $\forall B \in D(I)^Y$.
(b) $[f(A)]^c \subset f(A^c)$, $\forall A \in D(I)^X$.
(c) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$, where $B_1, B_2 \in D(I)^Y$.
(d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$, where $A_1, A_2 \in D(I)^X$.
(e) $f(f^{-1}(B)) \subset B$, $\forall B \in D(I)^Y$.
(f) $A \subset f(f^{-1}(A))$, $\forall A \in D(I)^X$.
(g) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, $\forall C \in D(I)^Z$.
(h) $f^{-1}(\bigcup_{\alpha \in \Gamma} B_\alpha) = \bigcup_{\alpha \in \Gamma} f^{-1}B_\alpha$, where $\{B_\alpha\}_{\alpha \in \Gamma} \in D(I)^Y$.
(h) $f^{-1}(\bigcap_{\alpha \in \Gamma} B_\alpha) = \bigcap_{\alpha \in \Gamma} f^{-1}B_\alpha$, where $\{B_\alpha\}_{\alpha \in \Gamma} \in D(I)^Y$.

3. Interval-valued fuzzy subgroupoids

**Definition 3.1.** Let $(X, \cdot)$ be a groupoid and let $A, B \in D(I)^X$. Then the interval-valued fuzzy product of $A$ and $B$, denoted by $A \circ B$, is an IVFS in $X$ defined as follows: For each $x \in X$,
\[
(A \circ B)(x) = \begin{cases} 
\left[ \bigvee_{yz = x} [A^L(y) \land B^L(z)] \right. & \text{if } yz = x, \\
\left. \bigvee_{yz = x} [A^U(y) \land B^U(z)] \right] & \text{otherwise}.
\end{cases}
\]

**Definition 3.1** [8]. Let $X, \circ$ be groupoid and let $A, B \in IFS(X)$. Then the intuitionistic fuzzy product of $A$ and $B$, $A \circ B$, is defined as follows: For any $x \in X$,
\[\begin{align*}
\mu_{A \circ B}(x) &= \begin{cases} 
\bigvee_{yz=x} [\mu_A(y) \land \mu_B(z)] & \text{if } \exists (y, z) \in X \times X \text{ with } yz = x, \\
0 & \text{otherwise.}
\end{cases} \\
\nu_{A \circ B}(x) &= \begin{cases} 
\bigwedge_{yz=x} [\nu_A(y) \lor \nu_B(z)] & \text{if } \exists (y, z) \in X \times X \text{ with } yz = x, \\
1 & \text{otherwise.}
\end{cases}
\end{align*}\]

**Remark 3.1.** By Result 2.A, Definition 3.1 is reduced to Definition 3.1’ and the reverse holds.

**Proposition 3.2.** Let \( \circ \) be same as above, let \( x_M, y_N \in \text{IVFp}(X) \) and let \( A, B \in D(I)^X \). Then:

(a) \( x_M \circ y_N = (xy)_{M \cap N} \).
(b) \( A \circ B = \bigcup_{x_M \in A, y_N \in B} x_M \circ y_N \).

**Proof.** (a) Let \( z \in X \). Then

\[(x_M \circ y_N)(z) = \begin{cases} 
\bigvee_{x'y' = z} (x_M^L(x') \land y_N^L(y')) & \text{if } x'y' = z, \\
0 & \text{otherwise.}
\end{cases}\]

\[= \begin{cases} 
[M^L \land N^L, M^U \land N^U] & \text{if } z = xy, \\
0 & \text{otherwise.}
\end{cases}\]

\[= (xy)_{M \cap N}\]

(b) Let \( C = \bigcup_{x_M \in A, y_N \in B} x_M \circ y_N \), i.e.,

\[C = \bigcup_{x_M \in A, y_N \in B} x_M \circ y_N \bigcup_{x_M \in A, y_N \in B} (x_M^L \circ y_N^L), \bigcup_{x_M \in A, y_N \in B} (x_M^U \circ y_N^U)\].
For each \( z \in X \), we may assume that \( \exists u, v \in X \) such that \( uv = z \), \( x_M(u) \neq 0 \) and \( y_N(v) \neq 0 \), without loss of generality. Then

\[
(A \circ B)^L(z) = \bigvee_{z=uv} [A^L(u) \wedge B^L(v)]
\]

\[
\geq \bigvee_{z=uv} \left( \bigvee_{x_M \in A, y_N \in B} [x_M(u) \wedge y_N(v)] \right)
\]

\[
= \bigcup_{x_M \in A, y_N \in B} x_M \circ y_N
\]

\[
= C^L(z).
\]

Since \( u_{A(u)} \in A \) and \( v_{B(v)} \in B \),

\[
C^L(z) = \bigvee_{x_M \in A, y_N \in B} (\bigvee_{z=uv} [x_M(u) \wedge y_N(v)])
\]

\[
= \bigvee_{z=uv} \left( \bigvee_{x_M \in A, y_N \in B} [x_M(u) \wedge y_N(v)] \right)
\]

\[
\geq \bigvee_{z=uv} [u_{A(u)}(u) \wedge v_{B(v)}(v)]
\]

\[
= \bigvee_{z=uv} [A^L(u) \wedge B^L(v)]
\]

\[
= (A \circ B)^L(z).
\]

Thus \((A \circ B)^L = C^L\). By the similar arguments, we have \((A \circ B)^U = C^U\).

Hence

\[
A \circ B = \bigcup_{x_M \in A, y_N \in B} x_M \circ y_N.
\]

The following is the immediate result of Definition 3.1.

**Proposition 3.3.** Let \((X, \circ)\) be a groupoid, and let ”\(\circ\)” be same as above.

(a) if ”\(\circ\)” is associative[resp. commutative] in \(X\), the so is ”\(\circ\)” in \(D(I)^X\).

(b) if ”\(\circ\)” is has an identity \(e \in X\), then \(e_1 \in IVFp(X)\) is an identity of ”\(\circ\)” in \(D(I)^X\), i.e., \(A \circ e_1 = A = e_1 \circ A\) for each \(A \in D(I)^X\).

**Definition 3.4.** Let \((G, \cdot)\) be a groupoid and let \(\overline{0} = A \in D(I)^X\). Then \(A\) is called an *interval-valued fuzzy groupoid* (in short, *IVGP*) in \(G\) if
\[ A \circ A \subset A, \text{ i.e., } A^L \circ A^L \subset A^L \text{ and } A^U \circ A^U \subset A^U. \]

We will denote the IVGPs in \( G \) as IVGP(\( G \)).

**Remark 3.4.** (a) If \( A \) is a fuzzy groupoid in a group \( G \) in the sense of Liu[11], then \( A = [A, A] \in \text{IVGP}(G) \).

(b) If \( A \in \text{IVGP}(G) \), then \( A^L, A^U \in \text{FGP}(G) \) and \( A^* \in \text{IFGP}(G) \), where \( \text{FGP}(G) \) [resp. \( \text{IFGP}(G) \)] denoted the set of all fuzzy groupoids in the sense of Liu[resp. the set of all intuitionistic fuzzy groupoids in the sense of Hur et al.].

The followings are the immediate results of Definitions 3.1 and 3.4.

**Proposition 3.5.** Let \((G, \cdot)\) be a groupoid and let \( \tilde{0} \neq A \in D(I)^X \).

Then the followings are equivalent:

(a) \( A \in \text{IVGP}(G) \).

(b) For any \( x_M, y_N \in A \), \( x_M \circ y_N \in A \), i.e., \((A, \circ)\) is a groupoid.

(c) For any \( x, y \in G \), \( A^L(xy) \geq A^L(x) \land A^L(y) \) and \( A^U(xy) \geq A^U(x) \land A^U(y) \).

**Proposition 3.6.** Let \( \tilde{0} \neq A \in D(I)^X \). Then the followings are equivalent:

(a) If "\( \circ \)" is associative in \( G \), then so is "\( \circ \)" in \( A \), i.e., for any \( x_L, y_M, z_N \in A \), \( x_L \circ (y_M \circ z_N) = (x_L \circ y_M) \circ z_N. \)

(b) If "\( \circ \)" is commutative in \( G \), then so is "\( \circ \)" in \( A \), i.e., for any \( x_L, y_M \in A \), \( x_L \circ y_M = y_M \circ x_L. \)

(c) If "\( \circ \)" has an identity \( e \in G \), then \( e \circ x_L = x_L = x_L \circ e \) \( \forall x_L \in A. \)

From Proposition 3.5, we can define an IVGP in \( G \) as follows.

**Definition 3.4’.** An interval-valued fuzzy set \( A \) in \( G \) is called an interval-valued fuzzy subgroupoid (in short, IVGP) in \( G \) if

\[ A^L(xy) \geq A^L(x) \land A^L(y) \text{ and } A^U(xy) \geq A^U(x) \land A^U(y), \forall x, y \in G. \]

It is clear that \( \tilde{0}, \tilde{1} \in \text{IVGP}(G) \).
The following is the immediate result of Definition 3.4′.

**Proposition 3.7.** Let \( T \in \mathcal{P}(G) \), where \( \mathcal{P}(G) \) denoted the set of all subsets of \( G \). Then \( A = [\chi_T, \chi_T] \in \text{IVGP}(G) \) if and only if \( T \) is a subgroupoid of \( G \), where \( \chi_T \) is the characteristic function of \( T \).

**Proposition 3.8.** If \( \{A_\alpha\}_{\alpha \in \Gamma} \subset \text{IVGP}(G) \), then \( \bigcap_{\alpha \in \Gamma} A_\alpha \in \text{IVGP}(G) \).

**Proof.** Let \( A = \bigcap_{\alpha \in \Gamma} A_\alpha \) and let \( x, y \in G \). Then
\[
A_L(xy) = \bigwedge_{\alpha \in \Gamma} A_L(\alpha) \leq \bigwedge_{\alpha \in \Gamma} [A_\alpha(x) \wedge A_\alpha(y)], \quad \text{since } A_\alpha \in \text{IVGP}(G)
\]
\[
= \bigwedge_{\alpha \in \Gamma} A_\alpha(x) \wedge \bigwedge_{\alpha \in \Gamma} A_\alpha(y)
\]
\[
= A_L(x) \wedge A_L(y).
\]
Similarly, we can see that \( A_U(xy) \geq A_U(x) \wedge A_U(y) \). Hence \( \bigcap_{\alpha \in \Gamma} A_\alpha \in \text{IVGP}(G) \).

**Proposition 3.9.** Let \( f : G \to G' \) be a groupoid homomorphism, let \( A \in D(I)_X \) and let \( B \in D(I)_Y \).

(a) \( f(x_M \circ y_N) = f(x_M) \circ f(y_N), \forall x_M, y_N \in \text{IVFp}(G) \).

(b) If \( f \) is surjective and \( A \in \text{IVGP}(G) \), then \( f(A) \in \text{IVGP}(G') \).

(c) If \( B \in \text{IVGP}(G') \), then \( f^{-1}(B) \in \text{IVGP}(G) \).

**Proof.**

(a) Let \( x_M, y_N \in \text{IVP}(G) \) and let \( z \in G' \). Then
\[
f(x_M \circ y_N)^L(z) = f((xy)_{M \wedge N_L})^L(z) \quad \text{[By Proposition 3.2]}
\]
\[
= \bigvee_{z' = f(xy)} (xy)_{M \wedge N_L}(z')
\]
\[
= \begin{cases} 
M_L \wedge N_L & \text{if } z' = f(xy), \\
0 & \text{otherwise}.
\end{cases}
\]

On the other hand,
\[
(f(x)_M \circ f(y)_N)^L(z)
\]
This is a contradiction from the fact that or So or f Since or Thus f

\[
\begin{align*}
\{ & \bigvee_{z=uv} [f(x)_{M^L}(u) \land f(y)_{N^L}(v)] \quad \text{for } (u, v) \in G' \times G' \text{ with } z = \mu \nu, \\
& 0 \quad \text{otherwise.}
\end{align*}
\]

Thus \( f(x_M \circ y_N)^L(z) = (f(x)_M \circ f(y)_N)^L(z) \). Similarly, we can see that \( f(x_M \circ y_N)^U(z) = (f(x)_M \circ f(y)_N)^U(z), \forall z \in G' \). So \( f(x_M \circ y_N) = f(x_M) \circ f(y_N). \)

(b) Assume that \( f(A) \in \text{IVGP}(G') \). Then \( \exists y, y' \in G' \) such that \( f(A)^L(yy') < f(A)^L(y) \land f(A)^L(y') \) or \( f(A)^U(yy') < f(A)^U(y) \land f(A)^U(y') \).

Thus \[
\begin{align*}
\bigvee_{f(z)=yy'} A^L(z) &< \bigvee_{f(z)=y} A^L(x) \land \bigvee_{f(z)=y'} A^L(x') \\
or &\bigvee_{f(z)=yy'} A^U(z) < \bigvee_{f(z)=y} A^U(x) \land \bigvee_{f(z)=y'} A^U(x').
\end{align*}
\]

Since \( f \) is surjective, \( \exists x, x' \in G \) such that \( f(x) = y, f(x') = y' \), and

\[
\begin{align*}
\bigvee_{f(z)=yy'} A^L(z) &< A^L(x) \land A^L(x') \\
or &\bigvee_{f(z)=yy'} A^U(z) < A^U(x) \land A^U(x').
\end{align*}
\]

So

\[
A^L(xx') \leq \bigvee_{f(z)=yy'} A^L(z) < A^L(x) \land A^L(x')
\]

or

\[
A^U(xx') \leq \bigvee_{f(z)=yy'} A^U(z) < A^U(x) \land A^U(x')
\]

This is a contradiction from the fact that \( A \in \text{IVGP}(G) \).

(c) It can be easily seen that \( f^{-1}(B) \in \text{IVGP}(G) \)

\[\Box\]

**Definition 3.10**[2]. \( A \in D(I)^X \) is said to have the sup-property if for each \( T \in P(X), \exists t_0 \in T \) such that \( A(t_0) = \bigvee_{t \in T} A^L(t), \bigwedge_{t \in T} A^U(t) \).
Definition 3.10[8]. A ∈ IFS(X) is said to have the sup-property if each \( T \in P(X) \), \( \exists t_0 \in T \) such that \( A(t_0) = (\bigvee_{t \in T} \mu_A(t), \bigwedge_{t \in T} \nu_A(t)) \).

Remark 3.10. (a) If \( A \in I^X \) has the sup-property, \( A = [A, A] \in D(I)^X \) resp. \( A = (A, A^c) \in IFS(X) \) has the sup-property.

(b) If \( A = [A^L, A^U] \in D(I)^X \) resp. \( A = (\mu_A, \nu_A) \in IFS(X) \) has the sup-property, then \( A^L \) and \( A^U \in I^X \) resp. \( \mu_A \) and \( \nu_A^c \in I^X \) have the sup-property.

Proposition 3.11. Let \( f : G \rightarrow G' \) be a groupoid homomorphism and let \( A \in D(I)^X \) have the sup-property. If \( A \in IVGP(G) \), then \( f(A) \in IVGP(G') \).

proof. Let \( y, y' \in G' \). Then we can consider four cases:

\( i \) \( f^{-1}(y) \neq \emptyset \) and \( f^{-1}(y') \neq \emptyset \),

\( ii \) \( f^{-1}(y) \neq \emptyset \) and \( f^{-1}(y') = \emptyset \),

\( iii \) \( f^{-1}(y) = \emptyset \) and \( f^{-1}(y') \neq \emptyset \),

\( iv \) \( f^{-1}(y) = \emptyset \) and \( f^{-1}(y') = \emptyset \).

We prove only the case (i) and omit the remainders. Since \( A \) has the sup-property, \( \exists x_0 \in f^{-1}(y) \) and \( x'_0 \in f^{-1}(y') \) such that

\[
A(x_0) = \left[ \bigvee_{t \in f^{-1}(y)} A^L(t), \bigvee_{t \in f^{-1}(y)} A^U(t) \right]
\]

and

\[
A(x'_0) = \left[ \bigvee_{t' \in f^{-1}(y')} A^L(t'), \bigvee_{t' \in f^{-1}(y')} A^U(t') \right].
\]

Then

\[
f(A)^L(y) = \bigvee_{z \in f^{-1}(y')} A^L(z) \geq A^L(x_0 x'_0) \quad \text{[Since } f(x_0 x'_0) = f(x_0) f(x'_0) \]

\[
= y y' \]

\[
\geq A^L(x_0) \land A^L(x'_0) \quad \text{[Since } A \in IVGP(G) \],}

\[
= \left( \bigvee_{t \in f^{-1}(y)} A^L(t) \right) \land \left( \bigvee_{t' \in f^{-1}(y')} A^L(t') \right)
\]

\[
= f(A)^L(y) \land f(A)^L(y').
\]

Similarly, we have \( f(A)^U(y) \geq f(A)^U(y) \land f(A)^U(y') \) and \( f(A) \in IVGP(G') \).

Definition 3.12. Let \( f : X \rightarrow Y \) be a mapping and let \( A \in D(I)^X \). Then \( A \) is said to be interval-valued fuzzy invariant (in short, IVF-invariant) if \( f(x) = f(y) \) implies \( A(x) = A(y), \) i.e., \( A^L(x) = A^L(y) \)
and $A^U(x) = A^U(y)$. It is clear that if $A$ is IVF-invariant, i.e., $f^{-1}(f(A)) = A$.

The following is the immediate result of Definition 3.12.

**Proposition 3.13.** Let $f : X \to Y$ be a mapping and let $A = \{A \in D(I)^X : A$ is IVF-invariant and has the sup-property\}. Then there is a one-to-one correspondence between $A$ and $D(I)^{\text{Im} f}$, where $\text{Im} f$ denotes the image of $f$.

The following is the immediate result of Propositions 3.11 and 3.13.

**Corollary 3.13.** Let $f : G \to G'$ be a groupoid homomorphism and let $A = \{A \in \text{IVGP}(G) : A$ is IVF-invariant and has the sup-property\}. Then there is a one-to-one correspondence between $A$ and $\text{IVGP}(\text{Im} f)$.

### 4. Interval-value fuzzy subgroups

**Definition 4.1[4].** Let $A$ be an IVF's in a group $G$. Then $A$ is called an interval-valued fuzzy subgroup (in short, IVF) in $G$ if it satisfies the conditions: For any $x, y \in G$,

(i) $A^L(xy) \geq A^L(x) \land A^L(y)$ and $A^U(xy) \geq A^U(x) \land A^U(y)$

(ii) $A^L(x^{-1}) \geq A^L(x)$ and $A^U(x^{-1}) \geq A^U(x)$

We will denote the set of all IVGS of $G$ as IVG($G$).

**Example 4.1.** Consider the additive group $(\mathbb{Z}, +)$. We define a mapping $A = [A^L, A^U] : \mathbb{Z} \to D(I)$ as follows: For each $n \in \mathbb{Z}$,

$A(0) = [A^L(0), A^U(0)] = [1, 1]$

and

$A(n) = [A^L(n), A^U(n)] = \begin{cases} [\frac{1}{2}, \frac{2}{3}], & \text{if } n \text{ is odd;} \\ [\frac{1}{3}, \frac{4}{5}], & \text{if } n \text{ is even.} \end{cases}$

Then clearly $A \in D(I)^{\mathbb{Z}}$. Moreover, $A$ satisfies all the conditions of Definition 4.1. So $A \in \text{IVG}(\mathbb{Z})$. ■
Remark 4.1. (a) If $A \in \text{FG}(G)$, then $A = [A, A] \in \text{IVG}(G)$, where $\text{FG}(G)$ denotes the set of all fuzzy groups in $G$.

(b) If $A \in \text{IVG}(G)$, then $A^L, A^U \in \text{FG}(G)$ and $(A^L, A^{U^C}) \in \text{IFG}(G)$.

c) If $A \in \text{IFG}(G)$, then $[\mu_A, \nu^c_A] \in \text{IVG}(G)$.

The following two results can be easily proved from definition 4.1, Propositions 3.7 and 3.8.

Proposition 4.2. Let $G$ be a group and let $H \subset G$. Then $H$ is a subgroup of $G$ if and only if $[\chi_H, \chi_H] \in \text{IVG}(G)$.

Proposition 4.3. Let $\{A_\alpha\}_{\alpha \in \Gamma} \subset \text{IVG}(G)$. Then $\bigcap_{\alpha \in \Gamma} A_\alpha \in \text{IVG}(G)$.

The followings can be easily seen from Definitions 3.1 and 4.1.

Proposition 4.4. Let $G$ be group and let $A \in D(I)^G$. If $A \in \text{IVG}(G)$, then $A \circ A = A$.

Proposition 4.5. Let $A, B \in \text{IVG}(G)$. Then $A \circ B \in \text{IVG}(G)$ if and only if $A \circ B = B \circ A$.


(a) $A(x^{-1}) = A(x), \forall x \in G$.

(b) $A^L(e) \geq A^L(x) \text{ and } A^U(e) \geq A^U(x), \forall x \in G$, where $e$ is the identity of $G$.

Result 4.B [4, Proposition 3.2]. Let $A$ be an IVFS in a group $G$. Then $A$ is an IVG in $G$ if and only if $A^L(xy^{-1}) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy^{-1}) \geq A^U(x) \wedge A^U(y), \forall x, y \in G$.

Proposition 4.6. If $A \in \text{IVG}(G)$, then $G_A = \{x \in G : A(x) = A(e)\}$ is a subgroup of $G$.

Proof. let $x, y \in G_A$. Then

$A^L(xy^{-1}) \geq A^L(x) \wedge A^L(y)$

$= A^L(x) \wedge A^L(y) \text{ [ By Result 4.A ]}$

$= A^L(e) \wedge A^L(e) \text{ [ Since } x, y \in G_A \text{ ]}$

$= A^L(e)$. 

Similarly, we have $A^U(xy^{-1}) \geq A^U(e)$. On the other hand, by Result 4.A, it is clear that $A^L(xy^{-1}) \leq A^L(e)$ and $A^U(xy^{-1}) \leq A^U(e)$, thus
\[ A(xy^{-1}) = A(e) \text{.} \] So \( xy^{-1} \in G_A \). Hence \( G_A \) is a subgroup of \( G \). ■

**Proposition 4.7.** Let \( A \in \text{IVG}(G) \). If \( A(xy^{-1}) = A(e) \) for any \( x, y \in G \), then \( A(x) = A(y) \).

**Proof.** Let \( x, y \in G \). Then
\[
A^L(x) = A^L((xy^{-1})y)
\geq A^L(xy^{-1}) \land A^L(y) \quad \text{[Since \( A \in \text{IVG}(G) \)]}
= A^L(e) \land A^L(y) \quad \text{[By the hypothesis ]}
= A^L(y) \quad \text{[By Result 4.A. ]}
\]
On the other hand, by Result 4.A, \( A^L(x^{-1}) = A^L(x) \). Then
\[
A^L(y) = A^L((yx^{-1})x)
\geq A^L(yx^{-1}) \land A^L(x)
= A^L((yx^{-1})^{-1}) \land A^L(x) \quad \text{[By Result 4.A. ]}
= A^L(xy^{-1}) \land A^L(x)
= A^L(e) \land A^L(x)
= A^L(x).
\]
Similarly, we have \( A^U(x) = A^U(y) \). Hence \( A(x) = A(y) \). ■

**Corollary 4.7-1.** Let \( A \in \text{IVG}(G) \). If \( G_A \) is a normal subgroup of \( G \), then \( A \) is constant on each coset of \( G_A \).

**Proof.** Let \( a \in G \) and let \( x \in aG_A \). Then \( \exists y \in G_A \) such that \( x = ay \). Since \( G_A \) is normal, \( xa^{-1} \in G_A \). Thus, by the definition of \( G_A \), \( A(xa^{-1}) = A(e) \). By proposition 4.7, \( A(x) = A(a) \). So \( A \) is constant on \( aG_A \) \( \forall a \in G \). Similarly, we can see that \( A \) is constant on \( G_Aa \) \( \forall a \in G \). This completes the proof. ■

Let \( H \) be a subgroup of \( G \). Then the number of right [resp. left] cosets of \( H \) in \( G \) is called the index of \( H \) in \( G \) and denoted by \([G : H] \). If \( G \) is a finite group, then there can be only a finite number of distinct right [resp. left] cosets of \( H \); hence the index \([G : H] \) is finite. If \( G \) is an infinite group, then \([G : H] \) may be either finite or infinite.

**Corollary 4.7-2.** Let \( A \in \text{IVG}(G) \) and let \( G_A \) be normal. If \( G_A \) has a finite index, then \( A \) has the sup property.

**Proof.** Let \( T \subset G \). Since \( G_A \) has finite index, let the index \([G : G_A] = n \), say \( A = \{a_1G_A, \ldots, a_nG_A\} \), where \( a_i \in G(i = 1, \ldots, n) \) and \( a_iG_A \cap a_jG_A = \emptyset \) for any \( i \neq j \). Let \( t \in T \). Since \( G = \bigcup A = \bigcup_{i=1}^{n} a_iG_i \),
Hence there exists an $i \in \{1, \cdots, n\}$ such that $t \in a_i G_A$. Since $G_A$ is normal, by Corollary 4.7-1, $A(t) = A(a_i)$ on $a_i G_A$, say $A^L(t) = \alpha_i$ and $A^U(t) = \beta_i$, where $\alpha_i, \beta_i \in I$ and $\alpha_i \leq \beta_i$. Thus there exists a $t_0 \in T$ such that $A^L(t_0) = \bigvee_{t \in T} A^L(t) = \bigvee_{i=1}^n \alpha_i$ and $A^U(t_0) = \bigvee_{t \in T} A^U(t) = \bigvee_{i=1}^n \beta_i$. Hence $A$ has the sup property. ■

**Proposition 4.8.** A group $G$ cannot be the union of two proper IVGs.

**Proof.** Let $A$ and $B$ be proper IVGs of a group $G$ such that $A \cup B = 1$, $A \neq 1$ and $B \neq 1$. Since $A \cup B = (A^L \cup B^L, A^U \cup B^U)$, $A^L(x) \cup B^L(x) = 1$ and $A^U(x) \cup B^U(x) = 1$, $\forall x \in X$. Then $A^L(x) = 1$ or $B^L(x) = 1$ and $A^U(x) = 1$ or $B^U(x) = 1$. Since $A \neq 1$ and $B \neq 1$, $A^L(x) \neq 1$ or $A^U(x) \neq 1$ and $B^L(x) \neq 1$ or $B^U(x) \neq 1$. In either cases, this is a contradiction. This completes the proof. ■

**Proposition 4.9.** If $A$ is an IVGP of a finite group $G$, then $A \in \text{IVG}(G)$.

**Proof.** Let $x \in G$. Since $G$ is finite, $x$ has the finite order, say $n$. Then $x^n = e$, where $e$ is the identity of $G$. Thus $x^{-1} = x^{n-1}$. Since $A$ is an IVGP of $G$,

$$A^L(x^{-1}) = A^L(x^{n-1}) = A^L(x^{n-2}x) \geq A^L(x)$$

and

$$A^U(x^{-1}) = A^U(x^{n-1}) = A^U(x^{n-2}x) \geq A^U(x).$$

Hence $A \in \text{IVG}(G)$. ■

**Proposition 4.10.** Let $A$ be an IVG of a group $G$ and let $x \in G$. Then $A(xy) = A(y)$, for each $y \in G$ if and only if $A(x) = A(e)$.

**Proof.** ($\Rightarrow$): Suppose $A(xy) = A(y)$ for each $y \in G$. Then clearly $A(x) = A(e)$.

($\Leftarrow$): Suppose $A(x) = A(e)$. Then, by Result 4.5, $A^L(y) \leq A^L(x)$ and $A^U(y) \leq A^U(x)$ for each $y \in G$. Since $A$ is an IVG of $G$, $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \vee A^U(y)$. Thus $A^L(xy) \geq A^L(y)$ and $A^U(xy) \geq A^U(y)$ for each $y \in G$.

On the other hand, by Result 4.5,

$$A^L(y) = A^L(x^{-1}x) \geq A^L(x) \wedge A^L(xy)$$

and

$$A^U(y) = A^U(x^{-1}xy) \geq A^U(x) \wedge A^U(xy).$$

Since $A^L(x) \geq A^L(y)$ for each $y \in G$, $A^L(x) \wedge A^L(xy) = A^L(xy)$ and $A^U(x) \wedge A^U(xy) = A^U(xy)$. So $A^L(y) \geq A^L(xy)$ and $A^U(y) \geq A^U(xy)$.
for each \( y \in G \). Hence \( A(xy) = A(y) \) for each \( y \in G \). \( \blacksquare \)

**Proposition 4.11.** Let \( f : G \to G' \) be a group homomorphism, let \( A \in \text{IVG}(G) \) and let \( B \in \text{IVG}(G') \). Then the following hold:

(a) If \( A \) has the sup property, then \( f(A) \in \text{IVG}(G') \).

(b) \( f^{-1}(B) \in \text{IVG}(G) \).

**Proof.** (a) By Proposition 3.11, since \( f(A) \in \text{IVGP}(G) \), it is enough to show that \( f(A)^L(y^{-1}) \geq f(A)^L(y) \) and \( f(A)^U(y^{-1}) \geq f(A)^U(y) \) for each \( y \in f(G) \).

Let \( y \in f(G) \). Then \( \phi \neq f^{-1}(y) \subseteq G \). Since \( A \) has the sup property, there exists an \( x_0 \in f^{-1}(y) \) such that \( A^L(x_0) = \bigvee_{t \in f^{-1}(y)} A^L(t) \) and \( A^U(x_0) = \bigvee_{t \in f^{-1}(y)} A^U(t) \).

Thus

\[
f(A)^L(y^{-1}) = \bigvee_{t \in f^{-1}(y^{-1})} A^L(t) \geq A^L(x_0^{-1}) \geq A^L(x_0) = f(A)^L(y)
\]

and

\[
f(A)^U(y^{-1}) = \bigvee_{t \in f^{-1}(y^{-1})} A^U(t) \geq A^U(x_0^{-1}) \geq A^U(x_0) = f(A)^U(y).
\]

Hence \( f(A) \in \text{IVG}(G) \).

(b) By proposition 3.9, since \( f^{-1}(B) \in \text{IVGP}(G) \), it is enough to show that \( f^{-1}(B)^L(x^{-1}) \geq f^{-1}(B)^L(x) \) and \( f^{-1}(B)^U(x^{-1}) \geq f^{-1}(B)^U(x) \) for each \( x \in G \).

Let \( x \in G \). Then

\[
f^{-1}(B)^L(x^{-1}) = B^L(f(x^{-1})) = B^L(f(x)^{-1}) \geq B^L(f(x)) = f^{-1}(B)^L(x)
\]

and

\[
f^{-1}(B)^U(x^{-1}) = B^U(f(x^{-1})) = B^U(f(x)^{-1}) \geq B^U(f(x)) = f^{-1}(B)^U(x).
\]

Thus \( f^{-1}(B) \in \text{IVG}(G) \). This completes the proof. \( \blacksquare \)

**Proposition 4.12.** Let \( G_p \) be the cyclic group of prime order \( p \). Then \( A \in \text{IVG}(G_p) \) if and only if \( A^L(x) = A^L(1) \leq A^L(0) \) and \( A^U(x) = A^U(1) \leq A^U(0) \) for each \( 0 \neq x \in G_p \).
Proof. ($\Rightarrow$): Suppose $A \in \text{IVG}(G_p)$ and let $0 \neq x \in G_p$. Then $A^L(xy) \geq A^L(x) \land A^L(y)$ and $A^U(xy) \geq A^U(x) \land A^U(y)$ for any $x, y \in G_p$. Since $G_p$ is the cyclic group of prime order $p, G_p = \{0, 1, 2, \ldots, p - 1\}$. Since $x$ is the sum of 1's and 1 is the sum of $x$'s, $A^L(x) \geq A^L(1) \geq A^L(x)$ and $A^U(x) \geq A^U(1) \geq A^U(x)$. Thus $A^L(x) = A^L(1)$ and $A^U(x) = A^U(1)$. Since 0 is the identity element of $G_p$, $A^L(x) \leq A^L(0)$ and $A^U(x) \leq A^U(0)$. Hence the necessary conditions hold.

($\Leftarrow$): Suppose the necessary conditions hold and let $x, y \in G_p$. Then we have four cases: (i) $x \neq 0, y \neq 0$ and $x = y$, (ii) $x \neq 0, y = 0$, (iii) $x = 0, y \neq 0$, (iv) $x \neq 0, y \neq 0$ and $x \neq y$.

Case(i) Suppose $x \neq 0, y \neq 0$ and $x = y$. Then, by the hypothesis, $A^L(x) = A^L(y) = A^L(1) \leq A^L(0)$ and $A^U(x) = A^U(y) = A^U(1) \leq A^U(0)$. So $A^L(x - y) = A^L(0) \geq A^L(x) \land A^L(y)$ and $A^L(x - y) \geq A^U(x) \land A^U(y)$.

Case(ii) Suppose $x \neq 0$ and $y = 0$. Since $x - y \neq 0$, by the hypothesis, $A^L(x - y) = A^L(x) = A^L(1) \leq A^L(0) = A^L(y)$ and $A^U(x - y) = A^U(1) \leq A^U(0) = A^U(y)$. So $A^L(x - y) \geq A^L(x) \land A^L(y)$ and $A^L(x - y) \geq A^U(x) \land A^U(y)$.

Case(iii) Suppose $x \neq 0, y \neq 0$ and $x \neq y$. In all, $A^L(x - y) \geq A^L(x) \land A^L(y)$ and $A^U(x - y) \geq A^U(x) \land A^U(y)$. Hence, by Result 4.B, $A \in \text{IFG}(G_p)$.

**Definition 4.13.** Let $G$ be a groupoid and let $A \in \text{IVS}(G)$. Then $A$ is called an:

1. interval-valued fuzzy left ideal (in short, $\text{IVLI}$) of $G$ if for any $x, y \in G, A^L(xy) \geq A^L(x)$ and $A^U(xy) \geq A^U(y)$.

2. interval-valued fuzzy right ideal (in short, $\text{IVRI}$) of $G$ if for any $x, y \in G, A^L(xy) \geq A^L(x)$ and $A^U(xy) \geq A^U(x)$.

3. interval-valued fuzzy ideal (in short, $\text{IVI}$) of $G$ if it is both an IFLI and an IFRI.

We will denote the set of all IVLIs [resp. IVRIs and IVIs] of a groupoid $G$ as $\text{IVLI}(G)$ [resp. $\text{IVRI}(G)$ and $\text{IVI}(G)$].

It is clear that $A \in \text{IVI}(G)$ if and only if and only if for any $x, y \in G, A^L(xy) \geq A^L(x) \lor A^L(y)$ and $A^U(xy) \geq A^U(x) \lor A^U(y)$. Moreover, an IFLI (resp. IFLI, IFRI) is an IVGP of $G$. Note that for any $A \in \text{IVGP}(G)$,
we have $A^L(x^n) \geq A^L(x)$ and $A^U(x^n) \geq A^U(x)$ for each $x \in G$, where $x^n$ is any composite of $x$’s.

**Proposition 4.14.** The IVLIs (resp. IVLIs, IVRIs) in a group $G$ are just the constant mappings.

**Proof.** Suppose $A$ is an constant mapping and let $x, y \in G$. Then $A(xy) = A(x) = A(y)$. Thus $A \in IVI(G)$.

Now suppose $A \in IVLI(G)$. Then $A^L(xy) \geq A^L(y)$ and $A^U(xy) \geq A^U(y)$ for any $x, y \in G$. In particular, $A^L(x) \geq A^L(e)$ and $A^U(x) \geq A^U(e)$ for each $x \in G$. Moreover, $A^L(e) = A^L(x^{-1}x) \geq A^L(x)$ and $A^U(e) = A^U(x^{-1}x) \geq A^U(x)$ for each $x \in G$. So $A(x) = A(e)$ for each $x \in G$. Hence $A$ is a constant mapping.

**Definition 4.15.** Let $A$ be an IVS in a set $X$ and let $\lambda, \mu \in I$ with $\lambda \leq \mu$. Then the set $A^{[\lambda, \mu]} = \{x \in X : A^L(x) \geq \lambda \text{ and } A^U(x) \geq \mu\}$ is called a $[\lambda, \mu]$-level subset of $A$.

**Proposition 4.16.** Let $A$ be an IVG of a group $G$. Then, for each $(\lambda, \mu) \in I \times I$ such that $\lambda \leq \mu, A^{[\lambda, \mu]}$ is a subgroup of $G$.

**Proof.** Clearly, $A^{[\lambda, \mu]} \neq \emptyset$. Let $x, y \in A^{[\lambda, \mu]}$. Then $A^L(x) \geq \lambda, A^U(y) \geq \mu$ and $A^L(y) \geq \lambda, A^U(y) \geq \mu$. Since $A \in IVG(G)$, $A^L(xy) \geq A^L(x)$ and $A^U(xy) \geq A^U(x) \wedge A^L(y) \geq \lambda$ and $A^U(xy) \geq A^U(x) \wedge A^U(y) \geq \mu$. Thus $A^L(xy) \geq \lambda$ and $A^U(xy) \geq \mu$. So $xy \in A^{[\lambda, \mu]}$. On the other hand, $A^L(x^{-1}) \geq A^L(x) \geq \lambda$ and $A^U(x^{-1}) \geq A^U(x) \geq \mu$. Thus $A^L(x^{-1}) \lambda$ and $A^U(x^{-1}) \geq \mu$. So $x^{-1} \in A^{[\lambda, \mu]}$. Hence $A^{[\lambda, \mu]}$ is a subgroup of $G$.

**Proposition 4.16.** Let $A$ be an IVS in a group $G$ such that $A^{[\lambda, \mu]}$ is a subgroup of $G$ for each $(\lambda, \mu) \in I \times I$ such that $\lambda \leq A^L(e), \mu \leq A^U(e)$ and $\lambda \leq \mu$. Then $A$ is an IVG of $G$.

**Proof.** For any $x, y \in G$, let $A(x) = [t_1, s_1]$ and let $A(y) = [t_2, s_2]$. Then clearly, $\lambda \leq A^L(x) \wedge A^L(y)$ and $\mu \leq A^U(x) \wedge A^U(y)$. Since $A^{[\lambda, \mu]}$ is a subgroup of $G$, $xy \in A^{[\lambda, \mu]}$. Then $A^L(xy) \geq t_1$ and $A^U(xy) \geq s_1$. So $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$. For each $x \in G$, let $A(xy) = [\lambda, \mu]$. Then $x \in A^{[\lambda, \mu]}$. Since $A^{[\lambda, \mu]}$ is a subgroup of $G$, $x^{-1} \in A^{[\lambda, \mu]}$. So $A^L(x^{-1}) \geq A^L(x)$ and $A^U(x^{-1}) \geq A^U(x)$. Hence $A \in IVG(G)$.
5. Interval-value fuzzy normal subgroups

**Definition 5.1.** Let $A \in \text{IVG}(G)$. Then $A$ is called an *interval-valued fuzzy normal subgroup* (in short, *IVNG*) of $G$ if $A(xy) = A(yx)$, for any $x, y \in G$.

We will denote the set of all IVNGs of a group $G$ as $\text{IVNG}(G)$. It is clear that if $G$ is abelian, then $A \in \text{IVNG}(G), \forall A \in \text{IVG}(G)$.

**Example 5.1.** Consider the general linear group of degree $n$, $\text{GL}(n, \mathbb{R})$. Then clearly, $\text{GL}(n, \mathbb{R})$ is a non abelian group. Let us define a mapping $A : \text{GL}(n, \mathbb{R}) \rightarrow D(I)$ as follows: for any $I_n \neq M \in \text{GL}(n, \mathbb{R})$, where $I_n$ is the unit matrix, $A(I_n) = \overline{1}$, $A(U)(M) = \begin{cases} \frac{1}{5} & \text{if } M \text{ is not a triangular matrix} \\ \frac{1}{3} & \text{if } M \text{ is a triangular matrix} \end{cases}$

and

$A(L)(M) = \begin{cases} \frac{2}{3} & \text{if } M \text{ is not a triangular matrix} \\ \frac{1}{2} & \text{if } M \text{ is a triangular matrix} \end{cases}$

Then we can easily see that $A$ is an IVNG of $\text{GL}(n, \mathbb{R})$. ■

The following is the immediate result of Definitions 3.1 and 5.1.

**Proposition 5.2.** Let $A \in D(I)^G$ and let $B \in \text{IVNG}(G)$. Then $A \circ B = B \circ A$.

**Proposition 5.3.** Let $A \in \text{IVNG}(G)$. If $B \in \text{IVG}(G)$, then so is $B \circ A$.

**Proof.** By Definitions 3.1 and 3.4, it can be easily seen that $B \circ A \in \text{IVGP}(G)$. Thus it is sufficient to show that $(B \circ A)^L(x^{-1}) \geq (B \circ A)^L(x)$ and $(B \circ A)^U(x^{-1}) \geq (B \circ A)^U(x)$ for each $x \in G$. 

Let \( x \in G \). Then  
\[
(B \circ A)^L(x^{-1}) = \bigvee_{y = x^{-1}} [B^L(y) \wedge A^L(z)]
\]
= \[
\bigvee_{z^{-1}y^{-1} = x} [B^L((y^{-1})^{-1}) \wedge A^L((z^{-1})^{-1})]
\geq \bigvee_{z^{-1}y^{-1} = x} [B^L(y^{-1}) \wedge A^L(z^{-1})]
\]
= \((A \circ B)^L(x) = (B \circ A)^L(x)\).
Similarly, we have \((B \circ A)^U(x^{-1}) \geq (B \circ A)^U(x)\) for each \( x \in G \). Hence \( B \circ A \in \text{IVG}(G) \).

**Corollary 5.3.** Let \( A, B \in \text{IVNG}(G) \). Then \( A \circ B \in \text{IVNG}(G) \).

**Proof.** By Proposition 4.5, \( A \circ B \in \text{IVNG}(G) \). Let \( a, b \in G \). Then there exists \( x, y \in G \) such that \( ab = xy \). Since \( b = a^{-1}xy, ba = (a^{-1}xa)(a^{-1}ya) \). Since \( A, B \in \text{IVNG}(G) \),

\[
(A \circ B)(ab) = [(A \circ B)^L(ab), (A \circ B)^U(ab)]
\]
= \[
\bigvee_{ab = xy} (A^L(x) \wedge B^L(y)), \bigvee_{ab = xy} (A^U(x) \wedge B^U(y))
\]
= \[
\bigvee_{ba = (a^{-1}xa)(a^{-1}ya)} (A^L(a^{-1}xa) \wedge B^L(a^{-1}ya)), \bigvee_{ba = (a^{-1}xa)(a^{-1}ya)} (A^U(a^{-1}xa) \wedge B^U(a^{-1}ya))
\]
= \([(A \circ B)^L(ba), (A \circ B)^U(ba)]
\]
= \((A \circ B)(ba)\).
Hence \((A \circ B) \in \text{IFNG}(G)\).

**Proposition 5.4.** If \( A \in \text{IVNG}(G) \), then \( G_A \) is a normal subgroup of \( G \).

**Proof.** By Proposition 4.6, \( G_A \) is a subgroup of \( G \). Moreover \( G_A \neq \emptyset \). Let \( x \in G_A \) and let \( y \in G \). Then

\[
A^L(xy^{-1}) = A^L((yx)x^{-1}) = A^L(y^{-1}(yx)) = A^L(x) = A^L(e)
\]
and

\[
A^U(xy^{-1}) = A^U((yx)x^{-1}) = A^U(y^{-1}(yx)) = A^U(x) = A^U(e)
\]
Thus \( xy^{-1} \in G_A \). Hence \( G_A \) is a normal subgroup of \( G \).

It is clear that if \( A \) is a (usual) normal subgroup of \( G \), then \( A = [\chi_A, \chi_A] \in \text{IVNG}(G) \) and \( G_A = A \).
Definition 5.5. Let \( A \in \text{IVNG}(G) \). Then the quotient group \( G/G_A \) is called the \textit{interval-valued fuzzy quotient subgroup} (in short, \( \text{IVQG} \)) of \( X \) with respect to \( A \).

Now let \( \pi : G \rightarrow G/G_A \) be the natural projection.

Proposition 5.6. If \( A \in \text{IVNG}(G) \) and \( B \in D(I)^G \), then \( \pi^{-1}(\pi(B)) = G_A \circ B \).

Proof. Let \( x \in G \). then
\[
\pi^{-1}(\pi(B))^L = \bigvee_{\pi(y) = \pi(x)} B^L(y) = \bigvee_{xy^{-1} \in G_A} B^L(y)
\]
and
\[
\pi^{-1}(\pi(B))^U = \bigvee_{\pi(y) = \pi(x)} B^U(y) = \bigvee_{xy^{-1} \in G_A} B^U(y).
\]

On the other hand
\[
(G_A \circ B)^L(x) = \bigvee_{xy=x} [G_A(z) \land B^L(y)] = \bigvee_{z=xy^{-1} \in G_A} B^L(y)
\]
and
\[
(G_A \circ B)^U(x) = \bigvee_{xy=x} [G_A(z) \land B^U(y)] = \bigvee_{z=xy^{-1} \in G_A} B^U(y).
\]

Thus \( \pi^{-1}(\pi(b))(x) = (G_A \circ B)(x) \) for each \( x \in G \). Hence \( \pi^{-1}(\pi(B)) = G_A \circ B \).

6. Interval-valued fuzzy subrings and ideals

Definition 6.1. Let \((R, +, \cdot)\) be a ring and let \( \tilde{0} \neq A \in D(I)^R \). Then \( A \) is called an \textit{interval-valued fuzzy subring} (in short, \( \text{IVR} \)) in \( R \) if it satisfies the following conditions:

(i) \( A \) is an IVG in \( R \) with respect to the operation “+” (in the sense of Definition 4.1).

(ii) \( A \) is an IVGP in \( R \) with respect to the operation “\( \cdot \)” (in the sense of Definition 3.4 or Definition 3.4’).

We will denote the set of all IVRs of \( R \) as \( \text{IVR}(R) \).
Example 6.1. Consider the ring \((\mathbb{Z}_2, +, \cdot)\), where \(\mathbb{Z}_2 = \{0, 2\}\). We define the mapping \(A : \mathbb{Z}_2 \rightarrow D(I)\) as follows: \(A(0) = [0.2, 0.7]\) and \(A(1) = [0.5, 0.6]\). Then we can see that \(A \in \text{IVR}(\mathbb{Z}_2)\). \(\square\)

Remark 6.1. (1) If \(A\) is a fuzzy subring of a ring \(R\), then \([A, A] \in \text{IVR}(R)\)

(2) If \(A \in \text{IVR}(R)\), then \(A^L\) and \(A^U\) are fuzzy subrings of \(R\).

The following is the immediate result of Definition 3.4' and Result 4.B.

Proposition 6.2. Let \(R\) be a ring and let \(\tilde{0} \neq A \in D(I)\). Then \(A \in \text{IVR}(R)\) if and only if for any \(x, y \in R\),

(i) \(A^L(x - y) \geq A^L(x) \land A^L(y)\) and \(A^U(x - y) \geq A^U(x) \land A^U(y)\).

(ii) \(A^L(xy) \geq A^L(x) \land A^L(y)\) and \(A^U(xy) \geq A^U(x) \land A^U(y)\).

The following is easily seen.

Proposition 6.3. Let \(R\) be a ring. Then \(A\) is a subring of \(R\) if and only if \([\chi_A, \chi_A] \in \text{IVR}(R)\).

Definition 6.4. Let \(R\) be a ring and let \(\tilde{0} \neq A \in \text{IVR}(R)\). Then \(A\) is called an:

(1) interval-valued fuzzy left ideal (in short, \(\text{IVLI}\)) in \(R\) if \(A^L(xy) \geq A^L(y)\) and \(A^U(xy) \geq A^U(y)\) for any \(x, y \in R\).

(2) interval-valued fuzzy right ideal (in short, \(\text{IVRI}\)) in \(X\) if \(A^L(xy) \geq A^L(x)\) and \(A^U(xy) \geq A^U(x)\) for any \(x, y \in R\).

(3) interval-valued fuzzy ideal (in short, \(\text{IFI}\)) in \(X\) if it both an IVLI and an IVRI in \(R\).

We will denote the set of all IVLIs [resp. IVRIs and IVIs] of a ring \(R\) as \(\text{IVLI}(R)\) [resp. \(\text{IVRI}(R)\) and \(\text{IVI}(R)\)].

Example 6.4. Consider the ring \((\mathbb{Z}_4, +, \cdot)\), where \(\mathbb{Z}_4 = \{0, 1, 2, 3\}\). We define the mapping \(A : \mathbb{Z}_4 \rightarrow D(I)\) as follows: \(A(0) = [0.2, 0.8]\), \(A(1) = [0.3, 0.6] = A(3)\), and \(A(2) = [0.4, 0.5]\). Then we can easily see that \(A \in \text{IVI}(\mathbb{Z}_4)\). \(\square\)

Remark 6.4. (1) If \(A\) is a fuzzy [resp. left, right] ideal of a ring \(R\), then \([A, A^c] \in \text{IVI}(R)\) [resp. \(\text{IVLI}(R)\) and \(\text{IVRI}(R)\)].
(2) If $A \in \text{IVI}(R)$ [resp. IVLI($R$) and IVRI($R$)], then $A^L$ and $A^U$ are fuzzy [resp. left and right] ideals of $R$.

The following can be directly verified.

**Proposition 6.5.** Let $R$ be a ring and let $0 \neq A \in D(I)^R$. Then $A$ is an IVI [resp. IFLI and IFRI] of $R$ if and only if for any $x, y \in R$,

(i) $A^L(x - y) \geq A^L(x) \wedge A^L(y)$ and $A^U(x - y) \geq A^U(x) \vee A^U(y)$.

(ii) $A^L(xy) \geq A^L(x) \lor A^L(y)$ and $A^U(xy) \geq A^U(x) \lor A^U(y)$ [resp. $A^L(xy) \geq A^L(y)$ and $A^U(xy) \geq A^U(y)$, $A^L(xy) \geq A^L(x)$ and $A^U(xy) \geq A^U(x)$].

The following is easily seen.

**Proposition 6.6.** Let $R$ be a ring. Then $A$ is an ideal [resp. a left ideal and a right ideal] of $R$ if and only if $[\chi_A, \chi_A] \in \text{IVI}(R)$ [resp. IVLI($R$) and IVRI($R$)].

**Proposition 6.7.** Let $R$ be a skew field (also division ring) and let $0 \neq A \in D(I)^R$. Then $A$ is an IFI (IFLI, IFRI) of $R$ if and only if $A^L(0) = A^L(e) \leq A^L(0)$ and $A^U(0) = A^U(e) \geq A^U(0)$ for any $0 \neq x \in R$, where $0$ is the identity of $R$ for "+" and $e$ is the identity of $R$ for ".".

**Proof.** ($\Rightarrow$): Suppose $A \in \text{IVLI}(R)$ and let $0 \neq x \in R$. Then

$A^L(x) = A^L(xe) \geq A^L(e), A^L(e) = A^L(x^{-1}x) \geq A^L(x)$

and

$A^U(x) = A^U(xe) \geq A^U(e), A^U(e) = A^U(x^{-1}x) \geq A^U(x)$.

Thus $A(x) = A(e)$. On the other hand,

$A^L(0) = A^L(e - e) \geq A^L(e) \wedge A^L(e) = A^L(e)$

and

$A^U(0) = A^U(e - e) \geq A^U(e) \wedge A^U(e) = A^U(e)$.

So $A^L(e) \leq A^L(0)$ and $A^U(e) \leq A^U(0)$. Hence the necessary conditions hold.

($\Leftarrow$): Suppose the necessary conditions hold. Let $x \in R$. Then we have four cases:

(i) $x \neq 0, y \neq 0$ and $x \neq y$ (ii) $x \neq 0, y \neq 0$ and $x = y$

(iii) $x \neq 0, y = 0$ (iv) $x = 0, y \neq 0$. 
Case (i) Suppose $x \neq 0, y \neq 0$ and $x \neq y$. Then

\[ A^L(x - y) = A^L(e) \geq A^L(x) \land A^L(y), \]

\[ A^U(x - y) = A^U(e) \geq A^U(x) \land A^U(y) \]

and

\[ A^L(xy) = A^L(e) \geq A^L(x) \lor A^L(y), \]

\[ A^U(xy) = A^U(e) \geq A^U(x) \lor A^U(y). \]

Case (ii): Suppose $x \neq 0, y \neq 0$ and $x = y$. Then

\[ A^L(x - y) = A^L(0) \geq A^L(x) \land A^L(y), \]

\[ A^U(x - y) = A^U(0) \geq A^U(x) \land A^U(y) \]

and

\[ A^L(xy) = A^L(e) \geq A^L(x) \lor A^L(y), \]

\[ A^U(xy) = A^U(e) \geq A^U(x) \lor A^U(y). \]

Case (iii): Suppose $x \neq 0$ and $y = 0$. Then

\[ A^L(x - y) = A^L(x) = A^L(e) \geq A^L(x) \land A^L(y), \]

\[ A^U(x - y) = A^U(x) = A^U(0) \geq A^U(x) \land A^U(y) \]

and

\[ A^L(xy) = A^L(0) \geq A^L(x) \lor A^L(y), \]

\[ A^U(xy) = A^U(0) \geq A^U(x) \lor A^U(y). \]

Case (iv): It is similar to case (iii).

In all, $A \in\text{IVI}(R)$. This completes the proof.

\[ \square \]

Remark 6.8. Proposition 6.5 shows that an IVLI(IVRI) is an IVI in a skew field.

The following gives a characteristic of a (usual) field by an IVI.

**Proposition 6.9.** Let $R$ be a commutative ring with a unity $e$. If for $A \in\text{IVI}(R)$, $A^L(x) = A^L(e) \leq A^L(0)$ and $A^U(x) = A^U(e) \leq A^U(0)$ for each $0 \neq x \in R$, then $R$ is a field.

**Proof.** Let $A$ be an ideal of $R$ such that $A \neq R$. Then clearly $A = [\chi_A, \chi_A] \in\text{IVI}(R)$ such that $A \neq \bar{1}$. Thus there exists $y \in R$ such that $y \notin A$. Thus $\chi_A(y) = 0$. By the hypothesis, $\chi_A(x) = \chi_A(e) \leq \chi_A(0)$, for each $0 \neq x \in X$. So $\chi_A(0) = 1$, i.e., $A = \{0\}$. Hence $R$ is a field. $\square$

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