INTERVAL-VALUED SMOOTH TOPOLOGICAL SPACES

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Abstract. We list two kinds of gradation of openness and we study in the sense of the followings:
(i) We give the definition of IVGO of fuzzy sets and obtain some basic results.
(ii) We give the definition of interval-valued gradation of clopeness and obtain some properties.
(iii) We give the definition of a subspace of an interval-valued smooth topological space and obtain some properties.
(iv) We investigate some properties of gradation preserving (in short, IVGP) mappings.

1. Introduction

In 1965, Zadeh [19] introduced the concept of fuzzy sets as a generalization of (ordinary) subsets. Soon after, Chang [6] was the first to introduce the notion of a fuzzy topology \(T\) on a set \(X\) by axiomatizing a collection \(T\) of fuzzy sets in \(X\) as follows:
(i) \(\emptyset, X \in T\),
(ii) \(A, B \in T \Rightarrow A \cap B \in T\),
(iii) \(\{A_\alpha\}_{\alpha \in \Gamma} \subset T \Rightarrow \bigcup_{\alpha \in \Gamma} A_\alpha \in T\),

where he referred to each member of \(T\) as an open set.

Some authors [7,9,10,18] noted that fuzziness in it was absent, and Šostak[18] began the study of fuzzy structures of the topological type and called a function \(\tau : I^X \to I\), satisfying the following conditions:
(i) \(\tau(\emptyset) = \tau(X) = 1\),
(ii) \(\tau(A \cap B) \geq \tau(A) \land \tau(B), \ \forall A, B \in I^X\),
(iii) \( \tau(\bigcup_{a \in \Gamma} A_a) \geq \bigwedge_{a \in \Gamma} \tau(A_a), \forall \{A_a\}_{a \in \Gamma} \subset I^X, \)

as a fuzzy topology on \( X \). In this case, the pair \((X, \tau)\) was called a fuzzy topological space (in short, \( FTS \)) and \( \tau(A) \) was called the degree of openness of the fuzzy set \( A \).

On the other hand, various generalizations of the notion of fuzzy set have been done by many authors. Zadeh\[20\] introduced the idea of interval-valued fuzzy sets. Later, Atanassov\[1\] introduced the concept of intuitionistic fuzzy set. Moreover, Atanassov and Gargov\[2\] introduced the notion of interval-valued intuitionistic fuzzy sets as the generalization of both interval-valued fuzzy sets and intuitionistic fuzzy sets. Some researchers \[1,2,3,4,5\] have worked mainly on operators and relations on intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. Çoker\[8\] introduced the idea of the topology of intuitionistic fuzzy sets, and Hur et.al\[11,12\] investigated some properties of intuitionistic fuzzy topological groups and intuitionistic fuzzy topological spaces. Samanta and Mondal\[16,17\] introduced the definitions of the topology of interval-valued fuzzy sets and the topology of interval-valued intuitionistic fuzzy sets, respectively. In particular, recently, Mondal and Samanta\[14,15\] introduced the notion of intuitionistic gradation of openness.

In this paper, we list two kinds of gradation of openness and we the sense of the followings:

(i) We give the definition of IVGO of fuzzy sets and obtain some basic results.

(ii) We give the definition of interval-valued gradation of clopeness and obtain some properties.

(iii) We give the definition of a subspace of an interval-valued smooth topological space and obtain some properties.

(iv) We investigate some properties of gradation preserving (In short, IVGP) mappings.

2. Preliminaries

Throughout this paper, \( X \) will denote a nonempty set; \( I = [0, 1] \), the closed unit interval of the real line; \( I_0 = (0, 1] \); \( I_1 = [0, 1) \); \( I^X \) = the set of all fuzzy sets in \( X \). In particular, \( \emptyset \) and \( X \) denote the empty fuzzy set and the whole fuzzy set in \( X \) defined by \( \emptyset(x) = 0 \) and \( X(x) = 1, \forall x \in X \), respectively. All other notations are standard notations of fuzzy set theory. A complex mapping \( A = (\mu_A, \nu_A) : X \to I \times I \) satisfying
the condition $\mu_A(x) + \nu_A(x) \leq 1$, $\forall x \in X$, is called an intuitionistic fuzzy set in $X$, and $0_\sim$ and $1_\sim$ denote the empty intuitionistic fuzzy set and the whole intuitionistic fuzzy set in $X$ defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$, $\forall x \in X$, respectively. We will denote the set of all intuitionistic fuzzy sets in $X$ as IFS($X$). Also all the notations are standard notations of intuitionistic fuzzy set theory.

Let $D(I)$ be the set of all closed subintervals of the unit interval $I$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \ldots$, and note that $M = [M^L, M^U]$, where $M^L$ and $M^U$ are the lower and upper end points respectively. Especially, we denote $a = [a, a]$ for every $a \in (0, 1)$. We also note that

(i) $(\forall M, N \in D(I))(M = N \iff M^L = N^L, M^U = N^U)$.

(ii) $(\forall M, N \in D(I))(M \leq N \iff M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the complement of $M$, denoted by $M^c$, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$.

**Definition 2.1**[20]. Let $X$ be a given nonempty set. A mapping $A = [A^L, A^U] : X \to D(I)$ is called an interval valued fuzzy set (brieﬂy, IVFS) in $X$, where $A^L$ and $A^U$ are fuzzy sets in $X$ satisfying $A^L(x) \leq A^U(x)$ and $A(x) = [A^L(x), A^U(x)]$ for each $x \in X$, and $A^L(x)$ and $A^U(x)$ are called the lower and upper end points of $A(x)$, respectively

It is clear that every fuzzy set $A$ in $X$ is an IVFS of the form $A = [A, A]$. For any $[a, b] \in D(I)$, the IVFS whose value is the interval $[a, b]$ for all $x \in X$ is denoted by $[a, b]$, i.e., $[a, b](x) = [a, b]$ for each $x \in X$. For any $a \in I$, the IVFS whose value is $a$ for all $x \in X$ is denoted by simply $\tilde{a}$, i.e., $\tilde{a}(x) = a$ for each $x \in X$. $\tilde{0}$ and $\tilde{1}$ denote the empty interval-valued fuzzy set and the whole interval-valued fuzzy set in $X$, respectively. For a point $p \in X$ and for $[a, b] \in D(I)$ with $b > 0$, the IVFS which takes the value $[a, b] \in D(I)$ at $p$ and $0$ elsewhere in $X$ is called an interval-valued fuzzy point (briefly, an IVFP) and is denoted by $p_{[a, b]}$. In particular, if $b = a$, it is also denoted by $p_a$. We will denote by $D(I)^X$ and IVF$_F$($X$) the set of all IVFS$_F$ and the set of all IVF points in $X$ by $D(I)^X$ and IVF$_F$($X$), respectively.

**Notation.** Let $X = \{x_1, x_2, \ldots, x_n\}$. Then $A = ([a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n])$ denotes an IVFS in $X$ such that $A^L(x_i) = a_i$ and $A^U(x_i) = b_i$, for all $i = 1, 2, \ldots, n$. 


Definition 2.2[16]. Let $A, B \in D(I)^X$. Then:

(a) $A \subset B$ iff $A_L(x) \leq B_L(x)$ and $A_U(x) \leq B_U(x)$ for all $x \in X$.

(b) $A = B$ iff $A \subset B$ and $B \subset A$.

(c) The complement $A^c$ of $A$ is defined by $A^c = [1 - A_U(x), 1 - A_L(x)]$ for all $x \in X$.

(d) If $\{A_\alpha : \alpha \in \Gamma\}$ is an arbitrary subset of $D(I)^X$, then

\[ \bigcap_{\alpha \in \Gamma} A_\alpha(x) = \left[ \bigwedge_{\alpha \in \Gamma} A^L_\alpha(x), \bigwedge_{\alpha \in \Gamma} A^U_\alpha(x) \right], \]

\[ \bigcup_{\alpha \in \Gamma} A_\alpha(x) = \left[ \bigvee_{\alpha \in \Gamma} A^L_\alpha(x), \bigvee_{\alpha \in \Gamma} A^U_\alpha(x) \right]. \]

Definition 2.3[16]. Let $T \subset D(I)^X$. Then $T$ is called an interval-valued fuzzy topology (in short, IVFT) on $X$ if it satisfies the following conditions:

(i) $\tilde{0}, \tilde{1} \in T$,

(ii) $A, B \in T \Rightarrow A \cap B \in T$,

(iii) $\{A_\alpha\}_{\alpha \in \Gamma} \subset T \Rightarrow \bigcup_{\alpha \in \Gamma} A_\alpha \in T$.

In this case, each member of $T$ is called an IVF open set and the pair $(X,T)$ is called an interval-valued fuzzy topological space (in short, IVFTS). $A \in D(I)^X$ is called closed in $(X,T)$ if $A^c \in T$.

As in ordinary topologies, the indiscrete topology of IVF sets contains only $\tilde{1}$ and $\tilde{0}$, while the discrete topology of IVF sets contains all IVF sets. These two topologies are denoted by $T^0$ and $T^1$, respectively.

3. Interval-valued gradation of openness

Definition 3.1[7,18]. A mapping $\tau : I^X \rightarrow I$ is called a gradation of openness (in short, GO) or a smooth topology on $X$ if it satisfies the following conditions:

\[ (\text{GO1}) \quad \tau(\emptyset) = \tau(X) = 1, \]

\[ (\text{GO2}) \quad \tau(A) \geq r \text{ and } \tau(B) \geq r \Rightarrow \tau(A \cap B) \geq r, \text{ for any } A, B \in I^X, \]

\[ (\text{GO3}) \quad \tau(A_\alpha) \geq r, \forall \alpha \in \Gamma \Rightarrow \tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq r, \text{ for any } \{A_\alpha\}_{\alpha \in \Gamma} \subset I^X, \]

where $r \in I_0$; or equivalently:

\[ (\text{GO1})' \quad \tau(\emptyset) = \tau(X) = 1, \]

\[ (\text{GO2})' \quad \tau(A \cap B) \geq \tau(A) \land \tau(B), \text{ for any } A, B \in I^X, \]
The pair \((X, \tau)\) is called a smooth topological space (in short, \(STS\)).

**Definition 3.2**\(^{[14]}\). A complex mapping \(\tau = (\mu_\tau, \nu_\tau) : I^X \to I \times I\) is called an intuitionistic gradation of openness (in short, \(IGO\)) an intuitionistic smooth topology on \(X\) if it satisfies the following conditions:

\(IGO1\) \(\mu_\tau(A) + \nu_\tau(A) \leq 1\), for each \(A \in I^X\),

\(IGO2\) \(\tau(\emptyset) = \tau(X) = (1, 0)\),

\(IGO3\) \(\mu_\tau(A \cap B) \geq \mu_\tau(A) \land \mu_\tau(B)\) and \(\nu_\tau(A \cap B) \leq \nu_\tau(A) \lor \nu_\tau(B)\), for any \(A, B \in I^X\),

\(IGO4\) \(\mu_\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \mu_\tau(A_\alpha)\) and \(\nu_\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \leq \bigvee_{\alpha \in \Gamma} \nu_\tau(A_\alpha)\), for any \(\{A_\alpha\}_{\alpha \in \Gamma} \subseteq I^X\).

The triple \((X, \mu_\tau, \nu_\tau)\) is called an intuitionistic smooth topological space (in short, \(ISTS\)), and \(\mu_\tau\) and \(\nu_\tau\) may be interpreted as gradation of openness and nonopenness, respectively.

**Definition 3.3.** A mapping \(\tau = [\tau^L, \tau^U] : I^X \to D(I)\) is called an interval-valued gradation of openness (in short, \(IVGO\)) or an interval-valued smooth topology on \(X\) if it satisfies the following conditions:

\(IVGO1\) \(\tau^L(A) \leq \tau^U(A)\), for each \(A \in I^X\),

\(IVGO2\) \(\tau(\emptyset) = \tau(X) = 1\),

\(IVGO3\) \(\tau^L(A \cap B) \geq \tau^L(A) \land \tau^L(B)\) and \(\tau^U(A \cap B) \geq \tau^U(A) \land \tau^U(B)\), for any \(A, B \in I^X\),

\(IVGO4\) \(\tau^L(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \tau^L(A_\alpha)\) and \(\tau^U(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \tau^U(A_\alpha)\), for any \(\{A_\alpha\}_{\alpha \in \Gamma} \subseteq I^X\).

The pair \((X, \tau)\) is called an interval-valued smooth topological space (in short, \(IVSTS\)).

We will denote the set of all GOs [resp. IGOs and IVGOs] on \(X\) as GO\((X)\) [resp. IGO\((X)\) and IVGO\((X)\)].

**Example 3.3.** (a) Let \(T\) be the topology on \(\mathbb{R}\) generated by \(\mathcal{B} = \{(a, b] : a, b \in \mathbb{R} \text{ and } a < b\}\) as a subbase, and let \(T_\alpha\) be the family of all open sets in \(\mathbb{R}\) with respect to (in short, w.r.t.) the usual topology on \(\mathbb{R}\), where \(\mathbb{R}\) denotes the set of all real numbers. We define the mapping \(\tau = [\tau^L, \tau^U] : I^\mathbb{R} \to D(I)\) as follows: For each \(A \in I^\mathbb{R}\),
\[ \tau(A) = \begin{cases} 
1 & \text{if } A \in T_o, \\
[0.5, 0.7] & \text{if } A \in T \setminus T_o, \\
0 & \text{otherwise.} 
\end{cases} \]

Then it can easily be seen that \( \tau \in \text{IVGO}(X) \).

(b) Let \( a < b \) in \( \mathbb{R} \) and let \( \lambda \in I_o \). We define the mapping \( A : \mathbb{R} \to I \) as follows: For each \( x \in \mathbb{R} \),

\[ A(x) = \begin{cases} 
1 & \text{if } x \in (a, b), \\
\lambda & \text{if } x = b, \\
0 & \text{otherwise.} 
\end{cases} \]

Then clearly \( A \in I^X \) and we write \( A = (a, b)_\lambda \). Let \( \mathcal{B} = \{(a, b)_\lambda : a, b \in \mathbb{R}, a < b \text{ and } \lambda \in I_o\} \), let \( T \) be the chang’s fuzzy topology generated by \( \mathcal{B} \) as a subbase and let \( T_o = \{\chi_o : O \text{ is an open set in } \mathbb{R}\} \). Any \( A \in T \setminus T_o \) can be expressed as

\[ A = \bigcup_{\alpha \in \Gamma} A_\alpha \text{ (3.1)} \]

where \( A_\alpha = (a_\alpha, b_\alpha)_\lambda \) and \( \Gamma \) is countable. We define the mapping \( \tau = [\tau^L, \tau^U] : I^\mathbb{R} \to D(I) \) as follows: For each \( A \in I^X \),

\[ \tau(A) = \begin{cases} 
1 \text{ if } A \in T_o, \\
[1-0.5\lambda, 0.7\lambda] & \text{if } A = (a, b)_\lambda, \\
[\bigwedge_{\alpha \in \Gamma} \tau^L(A_\alpha), \bigwedge_{\alpha \in \Gamma} \tau^U(A_\alpha)] & \text{if } A \text{ is expressed in the form (3.1),} \\
0 & \text{otherwise.} 
\end{cases} \]

Then we can easily see that \( \tau \in \text{IVGO}(X) \).

The following is the immediate result of Definitions 3.1, 3.2 and 3.3.

**Proposition 3.4.** (a) If \( \tau \in \text{GO}(X) \), then \( (\tau, \tau^c) \in \text{IGO}(X) \) and \( \tau = [\tau, \tau^c] \in \text{IVGO}(X) \), where \( \tau^c(A) = 1 - \tau(A), \forall A \in I^X \).

(b) If \( \tau \in \text{IGO}(X) \) [resp. \( \text{IVGO}(X) \)], then \( \mu_\tau, \nu_\tau^c \in \text{GO}(X) \) [resp. \( \tau^L, \tau^U \in \text{GO}(X) \)].

**Proposition 3.5.** We define two mappings \( f : \text{IVGO}(X) \to \text{IGO}(X) \) and \( g : \text{IGO}(X) \to \text{IVGO}(X) \) as follows, respectively:

\[ f(\tau) = f([\tau^L, \tau^U]) = (\tau^L, (\tau^U)^c), \forall \tau \in \text{IVGO}(X) \]

and

\[ g(\tau) = g(\mu_\tau, \nu_\tau) = [\mu_\tau, \nu_\tau^c], \forall \tau \in \text{IGO}(X). \]
Then \( g \circ f = 1_{\text{IVGO}(X)} \) and \( f \circ g = 1_{\text{IGO}(X)} \).

**Proof.** It can be easily seen that \( f \) and \( g \) are functions. Let \( \tau \in \text{IVGO}(X) \). Then
\[
\begin{align*}
ge \circ f(\tau) &= g( (\tau^L, (\tau^U)^c) ) \\
&= [\tau^L, ((\tau^U)^c)^c] \\
&= [\tau^L, \tau^U] = \tau = 1_{\text{IVGO}(X)}
\end{align*}
\]
Now let \( \tau \in \text{IGO}(X) \). Then
\[
\begin{align*}
f \circ g(\tau) &= f( (\mu_\tau, (\nu_\tau)^c)^c) \\
&= (\mu_\tau, (\nu_\tau)^c)^c \\
&= (\mu_\tau, \nu_\tau) = \tau = 1_{\text{IGO}(X)}
\end{align*}
\]
This completes the proof.

**Remark 3.5.** Proposition 3.5 shows the concepts of IVGO and IGO to be equipollent generalizations of one of GO.

**Definition 3.6[7].** A mapping \( F : I^X \to I \) is called a **gradation of closedness** (in short, GC) or a **smooth cotopology** on \( X \) if it satisfies the following conditions:

(GC1) \( F(\emptyset) = F(X) = 1 \),

(GC2) \( F(A) \geq r \) and \( F(B) \geq r \Rightarrow F(A \cup B) \geq r \), for any \( A, B \in I^X \),

(GC3) \( F(A_\alpha) \geq r, \forall \alpha \in \Gamma \Rightarrow F(\bigcap_{\alpha \in \Gamma} A_\alpha) \geq r \), for any \( \{A_\alpha\} \subset I^X \),

where \( r \in I_0 \); or equivalently:

(GC1)' \( F(\emptyset) = F(X) = 1 \),

(GC2)' \( F(A \cup B) \geq F(A) \cap F(B) \), for any \( A, B \in I^X \),

(GC3)' \( F(\bigcap_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} F(A_\alpha) \), for any \( \{A_\alpha\} \subset I^X \).

**Definition 3.7[14].** A complex mapping \( F = (\mu_F, \nu_F) : I^X \to I \times I \) is called an **intuitionistic gradation of closedness** (in short, IGC) an **intuitionistic smooth cotopology** on \( X \) if it satisfies the following conditions:

(IGC1) \( \mu_F(A) + \nu_F(A) \leq 1 \), for each \( A \in I^X \),

(IGC2) \( F(\emptyset) = F(X) = (1, 0) \),

(IGC3) \( \mu_F(A \cup B) \geq \mu_F(A) \land \mu_F(B) \) and \( \nu_F(A \cup B) \leq \nu_F(A) \lor \nu_F(B) \), for any \( A, B \in I^X \).

(IGC4) \( \mu_F(\bigcap_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \mu_F(A_\alpha) \) and \( \nu_F(\bigcap_{\alpha \in \Gamma} A_\alpha) \leq \bigvee_{\alpha \in \Gamma} \nu_F(A_\alpha) \), for any \( \{A_\alpha\} \subset I^X \).
Definition 3.8. A mapping \( F = [F_L, F_U] : I^X \to D(I) \) is called an *interval-valued gradation of closedness* (in short, IVGC) an interval-valued smooth cotopology on \( X \) if it satisfies the following conditions:

\( (\text{IVGC1}) \) \( F_L(A) \leq F_U(A) \), for each \( A \in I^X \),

\( (\text{IVGC2}) \) \( F(\emptyset) = F(X) = 1 \),

\( (\text{IVGC3}) \) \( F_L(A \cup B) \geq F_L(A) \wedge F_L(B) \) and \( F_U(A \cup B) \geq F_U(A) \wedge F_U(B) \), for any \( A, B \in I^X \),

\( (\text{IVGC4}) \) \( F_L(\bigcap_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} F_L(A_\alpha) \) and \( F_U(\bigcap_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} F_U(A_\alpha) \), for any \( \{A_\alpha\}_{\alpha \in \Gamma} \subset I^X \).

We will denote the set of all GCs [resp. IGCs and IVGCs] on \( X \) as \( \text{GC}(X) \)[resp. \( \text{IGC}(X) \) and \( \text{IVGC}(X) \)].

The following is the generalization of Propositions 2.3, 2.4 and Corollary 2.5 in [7], as well as the analogue to Theorem 2.6 in [14].

**Proposition 3.9.** (a) For each \( \tau \in \text{IVGO}(X) \), we define the mapping \( F_\tau : I^X \to D(I) \) as follows: For each \( A \in I^X \),

\[ F_\tau(A) = \tau(A^c). \]

Then \( F_\tau \in \text{IVGC}(X) \).

(b) For each \( F \in \text{IVGC}(X) \), we define the mapping \( \tau_F : I^X \to D(I) \) as follows: For each \( A \in I^X \),

\[ \tau_F(A) = F(A^c). \]

Then \( \tau_F \in \text{IVGO}(X) \).

(c) \( \tau_{F_\tau} = \tau \) and \( F_{\tau_F} = F \).

**Proof.** (a) It is clear that \( F_\tau \) satisfies the conditions (IVGC1) and (IVGC2). Let \( A, B \in I^X \). Then

\[ F_\tau(A \cup B) = \tau((A \cup B)^c) = \tau(A^c \cap B^c) \geq \tau(A^c) \wedge \tau(B^c) \ [\text{By the condition (IVGO3)}] \]

\[ = F_L(A) \wedge F_L(B). \ [\text{By the definition of } F_\tau] \]

Similarly, we have \( F_\tau(U)(A \cup B) \geq F_\tau(U)(A) \wedge F_\tau(U)(B) \). Thus \( F_\tau \) satisfies the condition (IVGC3). Now let \( \{A_\alpha\}_{\alpha \in \Gamma} \subset I^X \). Then

\[ F_\tau(\bigcap_{\alpha \in \Gamma} A_\alpha) = \tau((\bigcap_{\alpha \in \Gamma} A_\alpha)^c) = \tau(\bigcup_{\alpha \in \Gamma} A_\alpha^c) \]

\[ \geq \bigwedge_{\alpha \in \Gamma} \tau(A_\alpha^c) \ [\text{By the condition (IVGO4)}] \]

\[ = \bigwedge_{\alpha \in \Gamma} F_L(A_\alpha^c). \ [\text{By the definition of } F_\tau] \]
Similarly, we have \( F_L(\bigcap_{a \in \Gamma} A_a) \geq \bigwedge_{a \in \Gamma} F^L(A_a) \). So \( F_\tau \) satisfies the condition (IVGC4). Hence \( F_\tau \in \text{IVGC}(X) \).

The proof of (b) is similar to one of (a) and (c) are the immediate results of the definitions of \( F_\tau \) and \( \tau F \).

**Definition 3.10.** Let \( \{\tau_\alpha\}_{\alpha \in \Gamma} \subset \text{IVGO}(X) \). Then the *intersection* of \( \{\tau_\alpha\}_{\alpha \in \Gamma} \), denoted by \( \bigcap_{\alpha \in \Gamma} \tau_\alpha \), is defined as follows: For each \( A \in I^X \),

\[
(\bigcap_{\alpha \in \Gamma} \tau_\alpha)(A) = [\bigwedge_{\alpha \in \Gamma} \tau^L_\alpha(A), \bigwedge_{\alpha \in \Gamma} \tau^U_\alpha(A)].
\]

The following is the immediate result of Definitions 3.3 and 3.10.

**Proposition 3.11.** Let \( \{\tau_\alpha\}_{\alpha \in \Gamma} \subset \text{IVGO}(X) \). Then \( \bigcap_{\alpha \in \Gamma} \tau_\alpha \in \text{IVGO}(X) \).

**Definition 3.12.** We define a relation “\( \leq \)” on \( \text{IVGO}(X) \) as follows:

\( \tau \leq \eta \iff \tau^L \leq \eta^L \) and \( \tau^U \leq \eta^U \), for any \( \tau, \eta \in \text{IVGO}(X) \).

It can be easily seen that \( (\text{IVGO}(X), \leq) \) is a partially ordered set.

**Remark 3.13.** We define two mappings \( \tau_0, \tau_1 : I^X \to D(I) \) as follows: For each \( A \in I^X \),

\[
\tau_0(A) = \begin{cases} 
1 & \text{if } A = \emptyset \text{ or } A = X, \\
0 & \text{if } A \in I^X \setminus \{\emptyset, X\}
\end{cases}
\]

and

\[
\tau_1(A) = 1.
\]

Then we can easily see that \( \tau_0, \tau_1 \in \text{IVGO}(X) \) and \( \tau_0 \leq \tau \leq \tau_1, \ \forall \ \tau \in \text{IVGO}(X) \).

The followings is the immediate result of Proposition 3.11 and Remark 3.13.

**Proposition 3.14.** \( (\text{IVCO}(X), \leq) \) is a complete lattice with the smallest element \( \tau_0 \) and the largest element \( \tau_1 \).

**Proposition 3.15.** Let \((X, \tau)\) be an IVFTS, where \( \tau \in \text{IVGO}(X) \) and let \([\lambda, \mu] \in D(I) \). Then

\[
\tau_{[\lambda, \mu]} = \{A \in I^X : \tau(A) \geq [\lambda, \mu], \ \text{i.e.,} \ \tau^L(A) \geq \lambda \text{ and } \tau^U(A) \geq \mu\}
\]
is a Chang’s fuzzy topology on $X$. In this case, $\tau_{[\lambda,\mu]}$ [resp. $\tau_\lambda$] is called the $[\lambda,\mu]$-level [resp. $\lambda$-level] Chang’s fuzzy topology on $X$ w.r.t. $\tau$.

**Proof.** Since $\tau \in \text{IVGO}(X)$, $\tau(\emptyset) = \tau(X) = 1$. Then

$\tau^L(\emptyset) = 1 \geq \lambda$, $\tau^U(\emptyset) = 1 \geq \mu$

and

$\tau^L(X) = 1 \geq \lambda$, $\tau^U(X) = 1 \geq \mu$.

Thus $\emptyset, X \in \tau_{[\lambda,\mu]}$. Let $A, B \in \tau_{[\lambda,\mu]}$. Then

$\tau^L(A) \geq \lambda$, $\tau^U(A) \geq \mu$

and

$\tau^L(B) \geq \lambda$, $\tau^U(B) \geq \mu$.

Since $\tau \in \text{IVGO}(X)$,

$\tau^L(A \cap B) \geq \tau^L(A) \wedge \tau^L(B) \geq \lambda$

and

$\tau^U(A \cap B) \geq \tau^U(A) \wedge \tau^U(B) \geq \mu$.

Thus $A \cap B \in \tau_{[\lambda,\mu]}$. Now let $\{A_\alpha\}_{\alpha \in \Gamma} \subset \tau_{[\lambda,\mu]}$. Then

$\tau^L(A_\alpha) \geq \lambda$ and $\tau^U(A_\alpha) \geq \mu$, $\forall \alpha \in \Gamma$.

Since $\tau \in \text{IVGO}(X)$,

$\tau^L(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \tau^L(A_\alpha) \geq \lambda$

and

$\tau^U(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \tau^U(A_\alpha) \geq \mu$.

Thus $\bigcup_{\alpha \in \Gamma} A_\alpha \in \tau_{[\lambda,\mu]}$. So $\tau_{[\lambda,\mu]}$ is a Chang’s fuzzy topology on $X$. By the process of the proof of $\tau_{[\lambda,\mu]}$, it is clear that $\tau_\lambda$ is a Chang’s fuzzy topology on $X$. $\square$

**Proposition 3.16.** Let $(X, \tau)$ be an IVFTS and let $\{\tau_{[\lambda,\mu]}\}_{[\lambda,\mu] \in D(I)}$ be the family of all $[\lambda,\mu]$-level Chang’s fuzzy topologies w.r.t. $\tau$. Then $\{\tau_{[\lambda,\mu]}\}_{[\lambda,\mu] \in D(I)}$ is descending and for each $[\lambda,\mu] \in D(I_0)$, $\tau_{[\lambda,\mu]} = \bigcap_{[a,b] < [\lambda,\mu]} \tau_{[a,b]}$.

In this case, $\{\tau_{[\lambda,\mu]}\}_{[\lambda,\mu] \in D(I_0)}$ is called the family of Chang’s fuzzy topologies associated with the gradation of $\tau$.

**Proof.** Suppose $[a,b] \leq [\lambda,\mu]$. Then clearly $\tau_{[\lambda,\mu]} \subset \tau_{[a,b]}$. Thus $\{\tau_{[\lambda,\mu]}\}_{[\lambda,\mu] \in D(I)}$ is a descending family of Chang’s fuzzy topologies. So

$\tau_{[\lambda,\mu]} \subset \bigcap_{[a,b] < [\lambda,\mu]} \tau_{[a,b]}$, for each $[\lambda,\mu] \in D(I_0)$. 


Assume that $A \notin \tau_{[\lambda, \mu]}$. Then $\tau^L(A) < \lambda$ or $\tau^U(A) < \mu$. Thus $\exists [a, b] \in D(I_o)$ such that $\tau^L(A) < a < \lambda$ or $\tau^U(A) < b < \mu$. So $A \notin \bigcap_{[a,b] < [\lambda, \mu]} \tau_{[a,b]}$. Hence $\bigcap_{[a,b] < [\lambda, \mu]} \tau_{[a,b]} \subset \tau_{[\lambda, \mu]}$. Therefore $\tau_{[\lambda, \mu]} = \bigcap_{[a,b] < [\lambda, \mu]} \tau_{[a,b]}$. \hfill \qed

The following is the immediate result of Proposition 3.16.

**Corollary 3.16.** Let $(X, \tau)$ be an IVFTS and let $\{\tau_r\}_{r \in D(I)}$ be the family of all $r$-level Chang’s fuzzy topologies w.r.t. $\tau$. Then $\{\tau_r\}_{r \in D(I)}$ is descending and for each $r \in D(I_o)$, $\tau_r = \bigcap_{s < r} \tau_s$.

**Proposition 3.17.** Let $\{T_{[\lambda, \mu]}\}_{[\lambda, \mu] \in D(I_o)}$ be a nonempty descending family of Chang’s fuzzy topologies on $X$. We define the mapping $\tau = [\tau^L, \tau^U] : I^X \to D(I)$ as follows:

$$\tau(A) = \bigvee\{[\lambda, \mu] \in D(I_o) : A \in T_{[\lambda, \mu]}\}, \forall A \in I^X.$$  

Then $\tau \in$ IVGO(X). If, for each $[a, b] \in D(I_o)$,

$$T_{[\lambda, \mu]} = \bigcap_{[a, b] < [\lambda, \mu]} T_{[a, b]}, \quad (3.2)$$

then $\tau_{[\lambda, \mu]} = T_{[\lambda, \mu]}$ for each $[\lambda, \mu] \in D(I_o)$.

**Proof.** Since $T_{[\lambda, \mu]}$ is a Chang’s fuzzy topology on $X$, $\emptyset, X \in T_{[\lambda, \mu]}$. Then, by the definition of $\tau$,

$$\tau(\emptyset) = \tau(X) = 1.$$  

Furthermore, $\tau^L(A) \leq \tau^U(A)$, for each $A \in I^X$. Thus $\tau$ satisfies the conditions (IVGO1) and (IVGO2).

For any $A_i \in I^X$, let $\tau(A_i) = [a_i, b_i]$ for $i = 1, 2$. Suppose $\tau(A_i) = 0$ for some $i$. Then clearly

$$\tau(A_1 \cap A_2) \geq \tau(A_1) \cap \tau(A_2).$$

Thus, without loss of generality, suppose $[a_i, b_i] > 0$ for $i = 1, 2$. Let $[s, t] \leq \tau(A_i)$ for $i = 1, 2$ and let $\varepsilon > 0$. Then, by the definition of $\tau$,

$$\exists [\lambda_1, \mu_1], [\lambda_2, \mu_2] \in D(I_o) \text{ such that } a_i - \varepsilon < \lambda_i \leq a_i, \ b_i - \varepsilon < \mu_i \leq b_i \text{ and } A_i \in T_{[\lambda_i, \mu_i]} \text{ for } i = 1, 2.$$  

Let $[\lambda, \mu] = [\lambda_1, \mu_1] \wedge [\lambda_2, \mu_2]$ and let $[a, b] = [a_1, b_1] \wedge [a_2, b_2]$. Then clearly $A_1, A_2 \in T_{[\lambda, \mu]}$. Thus $A_1 \cap A_2 \in T_{[\lambda, \mu]}$. So

$$\tau^L(A_1 \cap A_2) \geq \lambda > a - \varepsilon > s - \varepsilon$$

and

$$\tau^U(A_1 \cap A_2) \geq \mu > b - \varepsilon > t - \varepsilon.$$
Since $\varepsilon > 0$ is arbitrary,
$$\tau(A_1 \cap A_2) \geq s$$ and $$\tau^U(A_1 \cap A_2) \geq t.$$ Hence $\tau(A_1, A_2) \geq [s, t]$, i.e., $\tau(A_1 \cap A_2) \geq \tau(A_1) \wedge \tau(A_2)$. Therefore $\tau$ satisfies the condition (IVGO3).

Now suppose $\tau(A_\alpha) = [l_\alpha, m_\alpha]$ for each $\alpha \in \Gamma$ and let $[l, m] = \bigwedge_{\alpha \in \Gamma} [l_\alpha, m_\alpha]$. Suppose $[l, m] = 0$. Then it is obvious that
$$\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \tau(A_\alpha).$$
Suppose $[l, m] > 0$ and let $[l, m] > \varepsilon > 0$. Then $0 < l - \varepsilon < l_\alpha$ and $0 < m - \varepsilon < m_\alpha$ for each $\alpha \in \Gamma$. Thus $A_\alpha \in T_{[l-\varepsilon,m-\varepsilon]}$, $\forall \alpha \in \Gamma$. Since $T_{[l-\varepsilon,m-\varepsilon]}$ is a Chang’s fuzzy topology, $\bigcup_{\alpha \in \Gamma} A_\alpha \in T_{[l-\varepsilon,m-\varepsilon]}$. So
$$\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq [l - \varepsilon, m - \varepsilon].$$
Since $\varepsilon > 0$ is arbitrary,
$$\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq [l, m] = \bigwedge_{\alpha \in \Gamma} \tau(A_\alpha).$$
Hence $\tau$ satisfies the condition (IVGO.4). Therefore $\tau \in \text{IVGO}(X)$.

Finally, suppose $\{T_{[\lambda, \mu]}\}_{[\lambda, \mu] \in D(I_o)}$ satisfies the condition (3.2) and let $A \in T_{[\lambda, \mu]}$. Then clearly $\tau(A) \geq [\lambda, \mu]$. Thus $A \in \tau_{[\lambda, \mu]}$. So $T_{[\lambda, \mu]} \subset \tau_{[\lambda, \mu]}$.

Now let $A \in \tau_{[\lambda, \mu]}$. Then $\tau(A) \geq [\lambda, \mu]$. Thus, by the definition of $\tau$,
$$\bigvee \{[a, b] \in D(I_o) : A \in T_{[a, b]}\} = [s, t] \geq [\lambda, \mu].$$
Let $\varepsilon > 0$. Then $\exists [a, b] \in D(I_o)$ such that
$$s - \varepsilon < a, \ t - \varepsilon < b$$ and $A \in T_{[a, b]}$.
Thus
$$\lambda - \varepsilon \leq s - \varepsilon < a, \ \mu - \varepsilon \leq t - \varepsilon < b$$ and $A \in T_{[a, b]}$.
So $A \in T_{[\lambda-\varepsilon, \mu-\varepsilon]}$. Since $\varepsilon > 0$ is arbitrary, by the condition (3.2), $A \in T_{[\lambda, \mu]}$. Hence $\tau_{[\lambda, \mu]} \subset T_{[\lambda, \mu]}$. Therefore $\tau_{[\lambda, \mu]} = T_{[\lambda, \mu]}$. This completes the proof.

The followings are the immediate results of Corollary 3.16 and Proposition 3.17.

**Corollary 3.17-1.** Let $\tau, \eta \in \text{IVGO}(X)$. Then $\tau = \eta$ if and only if $\tau_{[\lambda, \mu]} = \eta_{[\lambda, \mu]}$, $\forall [\lambda, \mu] \in D(I_o)$. 

Corollary 3.17-2. Let \( \{T_r\}_{r \in D(I_o)} \) be a nonempty descending family of Chang’s fuzzy topologies on \( X \) and let \( \tau : I^X \to D(I) \) be a mapping defined as follows: For each \( A \in I^X \),
\[
\tau(A) = \bigvee \{r \in D(I_o) : A \in T_r\}.
\]
Then \( \tau \in \text{IVGO}(X) \). If, for each \( r \in D(I_o) \),
\[
T_r = \bigcap_{s < r} T_s,
\]
then \( \tau_r = T_r \) for all \( r \in D(I_o) \).

Proposition 3.18. Let \( (X, T) \) be a Chang’s fuzzy topological space. For each \( [\lambda, \mu] \in D(I_o) \), we define a mapping \( T^{[\lambda, \mu]} : I^X \to D(I) \) as follows: For each \( A \in I^X \),
\[
T^{[\lambda, \mu]}(A) = \begin{cases} 
1 & \text{if } A = \emptyset \text{ or } A = X, \\
[\lambda, \mu] & \text{if } A \in T \setminus \{\emptyset, X\}, \\
0 & \text{otherwise}.
\end{cases}
\]
Then \( T^{[\lambda, \mu]} \in \text{IVGO}(X) \) such that \( (T^{[\lambda, \mu]})_{[\lambda, \mu]} = T \).

In this case, \( T^{[\lambda, \mu]} \) [resp. \( T^{\lambda} \)] is called a \([\lambda, \mu]-th \) [resp. \( \lambda \)-th] interval-valued gradation [in short, \( \text{IVG} \)] an \( X \), and \( (X, T^{[\lambda, \mu]}), (X, T^{\lambda}) \) is called a \([\lambda, \mu]-th \) [resp. \( \lambda \)-th] interval-valued graded fuzzy topological space.

Proof. By the definition of \( T^{[\lambda, \mu]} \), \( T^{[\lambda, \mu]}(A) \leq T^{[\lambda, \mu]}(A) \), \( \forall A \in I^X \).
Then \( \text{IVGO}1 \) holds. Also, it is clear that \( \text{IVGO}2 \) holds.

Let \( A_i \in I^X \), \( i = 1, 2 \). Suppose \( A_i = \emptyset \) for some \( i \). Then \( A_1 \cap A_2 = \emptyset \).

Thus
\[
T^{[\lambda, \mu]}(A_1 \cap A_2) = 1 \geq T^{[\lambda, \mu]}(A_1) \wedge T^{[\lambda, \mu]}(A_2).
\]
Suppose \( A_1 = X \), for some \( i \) (say \( A_1 \)). Then \( A_1 \cap A_2 = A_2 \). Thus
\[
T^{[\lambda, \mu]}(A_1 \cap A_2) = T^{[\lambda, \mu]}(A_2) \geq T^{[\lambda, \mu]}(A_1) \wedge T^{[\lambda, \mu]}(A_2).
\]
Suppose \( A_1, A_2 \in T \setminus \{\emptyset, X\} \). Then \( A_1 \cap A_2 \in T \). Thus
\[
T^{[\lambda, \mu]}(A_1 \cap A_2) \geq [\lambda, \mu] = T^{[\lambda, \mu]}(A_1) \wedge T^{[\lambda, \mu]}(A_2).
\]
Suppose \( A_i \in I^X - T \) for some \( i \) (say \( A_1 \)). Then \( T^{[\lambda, \mu]}(A_1) = 0 \). Thus
\[
T^{[\lambda, \mu]}(A_1 \cap A_2) \geq 0 = T^{[\lambda, \mu]}(A_1) \wedge T^{[\lambda, \mu]}(A_2).
\]
In all cases, \( T^{[\lambda, \mu]} \) satisfies the condition \( \text{IVGO}3 \).

Let \( \{A_\alpha\}_{\alpha \in \Gamma} \subset I^X \). Suppose \( A_{\alpha_0} = \emptyset \) for some \( \alpha_0 \in \Gamma \). Then
\[
\bigcup_{\alpha \in \Gamma} A_\alpha = \bigcup_{\alpha \in \Gamma (\alpha \neq \alpha_0)} A_\alpha.
\]
Thus \[
\bigwedge_{\alpha \in \Gamma} T^{[\lambda, \mu]}(A_{\alpha}) = \bigwedge_{\alpha \in \Gamma (a \neq a_0)} T^{[\lambda, \mu]}(A_{\alpha}). \]
[Since \( T^{[\lambda, \mu]}(A_{a_0}) = 1 \)]

So, without loss of generality, assume that \( A_{a_0} = \emptyset \) \( \forall a \in \Gamma \).

Suppose \( A_{a_0} = X \) for some \( a_0 = \Gamma \). Then \[
T^{[\lambda, \mu]}(\bigcup_{a \in \Gamma} A_{a}) = T^{[\lambda, \mu]}(X) = 1 \geq \bigwedge_{a \in \Gamma} T^{[\lambda, \mu]}(A_{a}).
\]

In all cases, \( T^{[\lambda, \mu]} \) satisfies the condition (IVGO4). Hence \( T^{[\lambda, \mu]} \in \text{IVGO}(X) \).

By the above result and Proposition 3.15, \( (T^{[\lambda, \mu]}_{|_{\lambda, \mu}}) = \{ A \in I^X : T^{[\lambda, \mu]}(A) \geq [\lambda, \mu] = T \}. \)

From the process of the above proof, it can be easily seen that the remainder holds. \( \square \)

4. Interval-valued gradation of clopenness

**Definition 4.1.** A mapping \( \tau : I^X \rightarrow D(I) \) is called an interval-valued gradation of clopenness (in short, IVGCO) on \( X \) if \( \tau \in \text{IVGO}(X) \cap \text{IVGC}(X) \). We will denote the set of all IVGCOs on \( X \) as \( \text{IVGCO}(X) \). It is clear that \( \tau_0, \tau_1 \in \text{IVGCO}(X) \).

**Example 4.1.** Let \( [\lambda, \mu] \in D(I) \) be fixed. We define the mapping \( \tau : I^X \rightarrow D(I) \) as follows: For each \( A \in I^X \),

\[
\tau(A) = \begin{cases} 
1 & \text{if } A = \emptyset \text{ or } A = X, \\
[\lambda, \mu] & \text{if } A \neq \emptyset \text{ and } A \neq X.
\end{cases}
\]

Then it is obvious that \( \tau \in \text{IVGCO}(X) \). In this case, \( \tau \) is called an interval-valued constant gradation and we will denote it by \( [\lambda, \mu] \). \( \square \)

The following is the characterization of IVGCO.

**Theorem 4.2.** \( \tau \in \text{IVGCO}(X) \) if and only if
(i) $\tau^L(A) \leq \tau^U(A) \forall A \in I^X$,
(ii) $\tau(\emptyset) = \tau(X) = 1$,
(iii) $\tau(\bigcap_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \tau(A_\alpha), \forall \{A_\alpha\}_{\alpha \in \Gamma} \subset I^X$,
(iv) $\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \tau(A_\alpha), \forall \{A_\alpha\}_{\alpha \in \Gamma} \subset I^X$.

**Proof.** From Definitions 3.3, 3.8 and 4.1, it is obvious. $\square$

**Definition 4.3.** In Proposition 3.9, for each $\tau \in \text{IVGO}(X)$, $\mathcal{F}_\tau$ is called an interval-valued conjugate gradation of $\tau$. By Proposition 3.9(c), $\tau$ is the interval-valued conjugate gradation of $\mathcal{F}_\tau$.

It is clear that if $\tau \in \text{IVGCO}(X)$, $\mathcal{F}_\tau = \tau$.

The following gives a nice IVGCO.

**Proposition 4.4.** We define the mapping $\sigma : I^X \rightarrow D(I)$ as follows:

$$\sigma(A) = \begin{cases} 1 & \text{if } A = \emptyset, \\ \bigwedge_{x \in \text{supp}(A)} A(x), \bigwedge_{x \in \text{supp}(A)} A(x) & \text{if } A \neq \emptyset, \end{cases}$$

for each $A \in I^X$, where $\text{supp}(A) = \{x \in X : A(x) > 0\}$. Then $\sigma \in \text{IVGCO}(X)$. In this case, $\sigma$ is called the interval-valued support gradation.

**Proof.** It is obvious that $\sigma(\emptyset) = \sigma(X) = 1$ and $\sigma^L(A) \leq \sigma^U(A)$ for each $A \in I^X$.

Let $\{A_\alpha\}_{\alpha \in \Gamma} \subset I^X$, let $\lambda = \sigma(\bigcup_{\alpha \in \Gamma} A_\alpha)$ and let $\lambda_\alpha = \sigma(A_\alpha) \forall \alpha \in \Gamma$.

Suppose $\bigwedge_{\alpha \in \Gamma} \lambda_\alpha = \mu > \lambda$ and let $x \in \text{supp}(\bigcup_{\alpha \in \Gamma} A_\alpha)$. Since $\text{supp}(\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} \text{supp}(A_\alpha)$, $\exists \alpha_0 \in \Gamma$ such that $x \in \text{supp}(A_{\alpha_0})$. Thus $A_{\alpha_0}(x) \geq \bigwedge_{\alpha \in \Gamma} \{A_{\alpha_0}(y) : y \in \text{supp}(A_{\alpha_0})\} = \lambda_{\alpha_0} \geq \mu$.

So $\bigcup_{\alpha \in \Gamma} A_\alpha(x) \geq \mu$ and hence $\sigma(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \mu$. This is a contradiction from the fact that $\sigma(\bigcup_{\alpha \in \Gamma} A_\alpha) = \lambda < \mu$. Therefore $\sigma(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \sigma(A_\alpha)$.
Now let $\lambda = \bigwedge\{ (\bigcap_{a \in \Gamma} A_a) (x) : x \in \text{supp}(\bigcap_{a \in \Gamma} A_a) \}$. Then

$$\lambda = \bigwedge_{a \in \Gamma} (\bigwedge_{a \in \Gamma} A_a (x) : x \in \text{supp}(\bigcap_{a \in \Gamma} A_a))$$

$$= \bigwedge_{a \in \Gamma} (\bigwedge_{a \in \Gamma} A_a (x) : x \in \text{supp}(\bigcap_{a \in \Gamma} A_a))$$

$$\geq \bigwedge_{a \in \Gamma} (\bigwedge_{a \in \Gamma} A_a (x) : x \in \text{supp}(A_a)).$$

Thus, by the definition of $\sigma$,

$$\sigma(\bigcap_{a \in \Gamma} A_a) = \lambda$$

$$\geq \bigwedge_{a \in \Gamma} [\bigwedge_{a \in \Gamma} A_a (x) : x \in \text{supp}(A_a)], \bigwedge_{a \in \Gamma} [A_a (x) : x \in \text{supp}(A_a)]$$

$$= \bigwedge_{a \in \Gamma} \sigma(A_a).$$

Hence $\sigma \in \text{IVGCO}(X)$. \hfill \Box

**Remark 4.4.** Let $\sigma$ be the IVGCO on $X$ given by Proposition 4.4. Then its conjugate gradation $F_\sigma$ is given by, for each $A \in I^X$,

$$F_\sigma(A) = \sigma(A^c)$$

$$= [\bigwedge_{a \in \Gamma} A^c (x) : x \in \text{supp}(A^c)], \bigwedge_{a \in \Gamma} [A^c (x) : x \in \text{supp}(A^c)]$$

$$= [\bigwedge_{a \in \Gamma} (1 - A(x) : A(x) \neq 0), \bigwedge_{a \in \Gamma} (1 - A(x) \neq 0)]$$

$$= [1 - \bigvee_{a \in \Gamma} A(x) : A(x) \neq 0], 1 - \bigvee_{a \in \Gamma} A(x) : A(x) \neq 0].$$

**Example 4.4.** Let $X$ be a set with two points at least. We define the mapping $\delta : I^X \rightarrow D(I)$ as follows: For each $A \in I^X$,

$$\sigma(A) = \begin{cases} 1 & \text{if } A = \emptyset \text{ or } A = X \text{ or } \text{supp}(A) = X, \\ 0 & \text{if } \text{supp}(A) \neq X. \end{cases}$$

Then it can be easily seen that $\delta \in \text{IVGO}(X)$. For a fixed point $p \in X$ and for $n=1,2,\cdots$, we define the mapping $G_n : X \rightarrow I$ as follows: For each $x \in X$,

$$G_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \neq p, \\ 1 & \text{if } x = p. \end{cases}$$

Then clearly $\{G_n\}_{n \in \mathbb{N}} \subset I^X$ and $\delta(G_n) = 1 \quad \forall n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. But, $\delta(\bigcap_{n \in \mathbb{N}} G_n) = 0$. Thus $\delta(\bigcap_{n \in \mathbb{N}} G_n) < \cdots$
1 = \bigwedge_{n \in \mathbb{N}} \delta(G_n). So \delta \not\in \text{IVGC}(X). Hence \delta \not\in \text{IVGCO}(X).

The following gives a sufficient condition to be an IVGCO.

**Proposition 4.5.** Let \( \delta : I^X \to D(I) \) be a mapping. Consider the following conditions:

(a) \( \sigma^L(A) \leq \sigma^U(A), \forall A \in I^X, \)

(b) \( \sigma(\emptyset) = 1, \)

(c) \( \sigma(A) = \sigma(A^c), \forall A \in I^X, \)

(d) \( \sigma(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \sigma(A_\alpha), \forall \{A_\alpha\}_{\alpha \in \Gamma} \subset I^X, \)

(e) \( \sigma(\bigcap_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \sigma(A_\alpha), \forall \{A_\alpha\}_{\alpha \in \Gamma} \subset I^X. \)

If \( \sigma \) satisfies the condition (a)\sim(d) or (a)\sim(c) and (e), then \( \sigma \in \text{IVGCO}(X). \)

**Proof.** The condition (e) is deduced from the condition (b) and (c). Also the condition (d) is deduced from the condition (b) and (e). Hence, by Theorem 4.2, \( \sigma \in \text{IVGCO}(X). \)

The following is the immediate result of Theorem 4.2 and Proposition 4.5.

**Corollary 4.5.** If \( \sigma \in \text{IVGO}(X) \) or \( \sigma \in \text{IVGC}(X) \), and \( \sigma(A) = \sigma(A^c) \) for each \( A \in I^X \), then \( \sigma \in \text{IVGCO}(X). \)

The following is the immediate result of Definition 4.3 and Proposition 3.11.

**Proposition 4.6.** Let \( \{\tau_\alpha\}_{\alpha \in \Gamma} \subset \text{IVGC}(X) \) [resp. \( \text{IVGCO}(X) \)]. Then \( \bigcap_{\alpha \in \Gamma} \tau_\alpha \in \text{IVGC}(X) \) [resp. \( \text{IVGCO}(X) \)].

**Definition 4.7.** Let \( \{\tau_\alpha\}_{\alpha \in \Gamma} \subset \text{IVGO}(X) \). Then the union of \( \{\tau_\alpha\}_{\alpha \in \Gamma} \), denoted by \( \bigcup_{\alpha \in \Gamma} \tau_\alpha \), is defined as follows: For each \( A \in I^X, \)

\[
(\bigcup_{\alpha \in \Gamma} \tau_\alpha)(A) = \bigvee_{\alpha \in \Gamma} \tau_\alpha^L(A), \bigvee_{\alpha \in \Gamma} \tau_\alpha^U(A).
\]

The following example shows that the union of two IVGCOs is not, in general, an IVGO(IVGC) even they are conjugate.
Example 4.7. Let $X$ be a set with two points at least. Let $\{M, N\}$ be a partition of $X$, let $\frac{1}{2} < \lambda < 1$ and let $\mu = 1 - \lambda$. Consider two fuzzy sets $A$ and $B$ in $X$ defined as follows: For each $x \in X$,

$$A(x) = \begin{cases} 0 & \text{if } x \in M \\ \lambda & \text{if } x \in N \end{cases}$$

and

$$B(x) = \begin{cases} \mu & \text{if } x \in M \\ 0 & \text{if } x \in N. \end{cases}$$

Then $A \cup B$ is the fuzzy set in $X$ given by, for each $x \in X$,

$$(A \cup B)(x) = \begin{cases} \mu & \text{if } x \in M \\ \lambda & \text{if } x \in N. \end{cases}$$

Let $\sigma$ be the interval-valued support gradation and let $\delta$ be its conjugate gradation. Then

$$(\sigma \cup \delta)(A \cup B) = [\mu, \mu],$$

and

$$(\sigma \cup \delta)(A) = [\lambda, \lambda], \quad (\sigma \cup \delta)(B) = [1 - \mu, 1 - \mu] = [\lambda, \lambda].$$

Since $\frac{1}{2} < \lambda < 1$ and $\mu = 1 - \lambda$, $\mu < \lambda$. Thus

$$(\sigma \cup \delta)(A \cup B) = [\mu, \mu] < [\lambda, \lambda] = (\sigma \cup \delta)(A) \wedge (\sigma \cup \delta)(B)$$

So $\sigma \cup \delta \not\in \text{IVGCO}(X)$. \qed

Definition 4.8[9]. Let $(X, T)$ be a Chang’s fuzzy topological space. Then the fuzzy space $X$ (the fuzzy topology $T$) is said to be interpreservative[resp. super 0-dimensional] if the intersection of each family of open sets is open [resp. each open set is closed or equivalently if the family of closed sets in $X$ agrees with $T$.

It is clear that if $X$ is super 0-dimensional, then $X$ is interpreservative.

Definition 4.9. Let $\sigma \in \text{IVGO}(X)$ and let $T$ be a Chang’s fuzzy topology on $X$. We define the mapping $\sigma^* : I^X \to B(I)$ as follows: For each $A \in I^X$,

$$\sigma^*(A) = \begin{cases} \sigma(A) & \text{if } A \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sigma^*$ is called the deduced gradation from $\sigma$ and $T$. 
It is clear that $\sigma^* \in \text{IVGO}(X)$ and its $[\lambda, \mu]$-level $\sigma^*_{[\lambda, \mu]}$ is $\sigma_{[\lambda, \mu]}^* = \sigma_{[\lambda, \mu]} \cap T$ for each $[\lambda, \mu] \in D(I)$.

The following is the immediate result of Definitions 4.8 and 4.9.

**Proposition 4.10.** Let $\sigma \in \text{IVGO}(X)$ and let $T$ be a Chang’s fuzzy topology. Then

(a) If $\sigma^*$ is deduced gradation from $\sigma$ and $T$, then $\sigma^* \in \text{IVGCO}(X)$.

(b) If $\delta$ is the conjugate gradation of $\sigma$ and $T$ is super 0-dimensional, then $\delta^*$ is the conjugate gradation of $\sigma^*$ and hence $\delta^* \in \text{IVGCO}(X)$.

**Example 4.10.** Let $\sigma$ be the interval-valued support gradation on $\mathbb{R}$, let $\delta$ be its conjugate and let $T$ be the laminated indiscrete topology on $\mathbb{R}[13]$, i.e., $T$ is constituted by the constant mappings on $\mathbb{R}$. Then clearly $\sigma$ and $T$ satisfies (b) of Proposition 4.10. Let $f_\alpha \in T$ be the constant mapping given by $f_\alpha(x) = \alpha$ for each $x \in X$. Then, the deduced gradation gradation $\sigma^*$ from $\sigma$ and $T$ is given by: For each $A \in I^X$,

$$
\sigma(A) = \begin{cases}
1 & \text{if } A = \emptyset \text{ or } A = X, \\
[\alpha, \alpha] & \text{if } A = f_\alpha \in T \text{ and } \alpha \neq 0, \\
0 & \text{otherwise.}
\end{cases}
$$

Then

$$
\delta^*(\emptyset) = \delta^*(X) = 1, \\
\delta^*(f_\alpha) = \delta(f_\alpha) = \sigma(1 - f_\alpha) = [1 - \alpha, 1 - \alpha] \\
= \sigma^*(1 - f_\alpha) = \sigma^*(f_{\alpha^*}), \text{ if } \alpha \neq 1.
$$

By the definition of $T$, it is clear that $A \in T$ if and only if $A^c(= 1 - A) \in T$. Thus, for $A \notin T$, $\delta^*(A) = \sigma^*(A^c) = 0$. So $\sigma^*$ and $\delta^*$ are conjugate.

**Definition 4.11.** Let $\tau, \eta \in \text{IVGO}(X)$. Then we say that $\tau$ is equivalent to $\eta$, $\tau \approx \eta$, if their families $[\lambda, \mu]$-levels agree, i.e.,

$$
\{\tau_{[\lambda, \mu]}\}_{[\lambda, \mu] \in D(I)} = \{\eta_{[a,b]}\}_{[a,b] \in D(I)}.
$$

**Proposition 4.12.** Let $\sigma \in \text{IVGO}(X)$ [resp. $\text{IVGC}(X)$] and let $\varphi : I \rightarrow I$ be an increasing continuous mapping with $\varphi(1) = 1$. Then $\varphi \circ \sigma = [\varphi \circ \sigma^L, \varphi \circ \sigma^U] \in \text{IVGO}(X)$ [resp. $\text{IVGC}(X)$]. Moreover, if $\varphi$ is strictly increasing, then $\sigma \approx \varphi \circ \sigma$. 


Proof. Suppose $\sigma \in \text{IVGO}(X)$. Then it is clear that the condition (IVGO1) holds. On the other hand,
\[
(\varphi \circ \sigma)(\emptyset) = [(\varphi \circ \sigma^L)(\emptyset), (\varphi \circ \sigma^U)(\emptyset)]
\]
\[
= [(\varphi^L(\emptyset), (\varphi^U(\emptyset))]
\]
\[
= [(\varphi(1), (\varphi(1))]
\]
\[
= [1, 1] = 1.
\]
Similarly, $(\varphi \circ \sigma)(X) = 1$. Thus the condition (IVGO2) holds.

Let $A, B \in I^X$. Then
\[
(\varphi \circ \sigma)(A \cap B) = [(\varphi \circ \sigma^L)(A \cap B), (\varphi \circ \sigma^U)(A \cap B)]
\]
\[
\geq [(\varphi(\sigma^L(A)) \wedge (\varphi(\sigma^U(B)))]
\]
\[
\geq [(\varphi(\sigma^L(A) \wedge \sigma^L(B)), (\varphi(\sigma^U(A) \wedge \sigma^U(B))). (4.1)
\]
Since $\sigma \in \text{IVGO}(X)$

Suppose $\sigma^L(A) \leq \sigma^L(B)$. Since $\varphi$ is increasing continuous,
\[
\varphi(\sigma^L(A)) \leq \varphi(\sigma^L(B)).
\]
Thus
\[
\varphi(\sigma^L(A)) \leq \varphi(\sigma^L(B)) = \varphi(\sigma^L(A) \wedge (\varphi(\sigma^L(B))
\]
\[
= (\varphi \circ \sigma^L)(A) \wedge (\varphi \circ \sigma^L)(B). (4.2)
\]
Similarly, we have
\[
\varphi(\sigma^U(A)) \wedge \sigma^U(B) = (\varphi \circ \sigma^U)(A) \wedge (\varphi \circ \sigma^U)(B). (4.3)
\]
So, by (4.1), (4.2) and (4.3),
\[
(\varphi \circ \sigma)(A \cap B) \geq (\varphi \circ \sigma)(A) \wedge (\varphi \circ \sigma)(B).
\]

Hence the condition (IVGO3) holds.

Now let $\{A_\alpha\}_{\alpha \in \Gamma} \subset I^X$ and let $[\lambda, \mu] = \bigwedge_{\alpha \in \Gamma} \sigma(A_\alpha)$.

Suppose $\exists \alpha_0 \in \Gamma$ such that $[\lambda, \mu] = \sigma(A_{\alpha_0})$. Then
\[
(\varphi \circ \sigma)(\bigcup_{\alpha \in \Gamma} A_\alpha) = [\varphi(\sigma^L(\bigcup_{\alpha \in \Gamma} A_\alpha)), \varphi(\sigma^U(\bigcup_{\alpha \in \Gamma} A_\alpha))]
\]
\[
\geq [\varphi(\bigwedge_{\alpha \in \Gamma} \sigma^L(A_\alpha)), \varphi(\bigwedge_{\alpha \in \Gamma} \sigma^U(A_\alpha)) [\text{Since } \sigma \in \text{IVGO}(X)]
\]
\[
= [\varphi(\sigma^L(A_{\alpha_0})), \varphi(\sigma^U(A_{\alpha_0})) [\text{By the hypothesis]}
\]
\[
= [(\varphi \circ \sigma)(A_{\alpha_0})]
\]
\[
\geq \bigwedge_{\alpha \in \Gamma} (\varphi \circ \sigma)(A_\alpha).
\]

Suppose $\exists \alpha_0 \in \Gamma$ such that $[\lambda, \mu] = \sigma(A_{\alpha_0})$. Then $\lambda \in a\{\sigma^L(A_\alpha) : \alpha \in \Gamma\}$ and $\mu \in a\{\sigma^U(A_\alpha) : \alpha \in \Gamma\}$. Thus $\exists$ strictly decreasing sequences $\{\sigma^L(A_n)\}_{n=1}^\infty$ and $\{\sigma^U(A_n)\}_{n=1}^\infty$ such that they converge to $\lambda$ and $\mu$, respectively. So $\{(\varphi \circ \sigma^L)(A_n)\}_{n=1}^\infty$ and $\{(\varphi \circ \sigma^U)(A_n)\}_{n=1}^\infty$
are lower bounded sequences and thus they converge to their infimums, respectively. Hence
\[
\lim_{n \to \infty} (\varphi \circ \sigma^L)(A_n) = \bigwedge_n (\varphi \circ \sigma^L)(A_n)
\]
\[
\geq \bigwedge_{\alpha \in \Gamma} (\varphi \circ \sigma^L)(A_\alpha) \tag{4.4}
\]
Similarly, we have
\[
\lim_{n \to \infty} (\varphi \circ \sigma^U)(A_n) \geq \bigwedge_{\alpha \in \Gamma} (\varphi \circ \sigma^L)(A_\alpha). \tag{4.5}
\]
On the other hand,
\[
(\varphi \circ \sigma)(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq [\varphi(\bigwedge_{\alpha \in \Gamma} \sigma^L(A_\alpha)), \varphi(\bigwedge_{\alpha \in \Gamma} \sigma^U(A_\alpha))] \quad \text{[Since } \sigma \in \text{IVGO}(X)\text{]}.
\]
\[
= [\varphi(\lambda), \varphi(\mu)] \quad \text{[Since } [\lambda, \mu] = \bigwedge_{\alpha \in \Gamma} \sigma(A_\alpha)\text{]}
\]
\[
= [\varphi(\lim_{n \to \infty} \sigma^L(A_n)), \varphi(\lim_{n \to \infty} \sigma^U(A_n))]
\]
\[
= [(\varphi \circ \sigma^L)(A_n), \lim_{n \to \infty} (\varphi \circ \sigma^U)(A_n)]. \tag{4.6}
\]
[Since \(\varphi\) is continuous]

From (4.4), (4.5) and (4.6),
\[
(\varphi \circ \sigma)(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} (\varphi \cdot \alpha)(A_\alpha).
\]
So \(\varphi \circ \sigma\) satisfies the condition (IVGO4). Hence \(\varphi \circ \sigma \in \text{IVGO}(X)\).

Suppose \(\sigma \in \text{IVGC}(X)\). By the similar way, we can prove that
\[
(\varphi \circ \sigma)(\bigcap_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} (\varphi \circ \alpha)(A_\alpha)
\]
for each \(\{A_\alpha\} \subset I^X\). Also we can easily see that the remainders hold. Hence \(\varphi \circ \sigma \in \text{IVGC}(X)\).

The following example shows that the continuity condition for the mapping \(\varphi\) in Proposition 4.12 cannot be removed. The following is the modification of Example 2.16 in [9].

**Example 4.12.** Let \(\delta^*\) be same as in Example 4.10. Let \(\varphi : I \to I\) be the mapping defined as follows: For each \(x \in I\),
\[
\varphi(x) = \begin{cases} 
\frac{1}{2}x & \text{if } x < \frac{1}{2}, \\
\frac{1}{2} & \text{if } x = \frac{1}{2}, \\
\frac{1}{2}x + \frac{1}{2} & \text{if } x > \frac{1}{2}.
\end{cases}
\]
Then \(\varphi\) is strictly increasing and \(\varphi(1) = 1\). But it is not continuous at \(x = \frac{1}{2}\). We will show that \(\varphi \circ \delta^*\) is not an IVGO:
Consider a strictly increasing sequence \( \{k_n\}_{n=1}^\infty \) such that \( \lim_{n \to \infty} k_n = \frac{1}{2} \) and \( 0 \leq k_n \leq \frac{1}{2} \) \( \forall n \in \mathbb{N} \). For each \( n \in \mathbb{N} \) we define the constant mapping \( K_n : \mathbb{R} \to I \) as follows: For each \( x \in \mathbb{R} \), \( K_n(x) = k_n \). Then \( \delta^*(K_n) = [1-k_n, 1-k_n] \) and \( \{1-k_n\}_{n=1}^\infty \) is a strictly decreasing sequence contained in \( I \) such that \( 1-k_n = \frac{1}{2} \). Thus \( (\varphi \circ \delta^*)(K_n) = \left[ \varphi\left(1-k_n\right), \varphi\left(1-k_n\right) \right] = \left[ \frac{1-k_n}{2}, \frac{1-k_n}{2} + \frac{1}{2} \right] \). So \( \varphi(1-k_n)_{n=1}^\infty \) is a strictly decreasing sequence such that \( \lim_{n \to \infty} \varphi(1-k_n) = \frac{3}{4} \). Hence \( (\varphi \circ \delta^*)(K_n) \geq \frac{3}{4} \), for \( n = 1, 2, \ldots \).

On the other hand, \( \bigcup_{n=1}^\infty K_n \) is the constant mapping \( f_{\frac{1}{2}} : \mathbb{R} \to I \) given by \( f_{\frac{1}{2}}(x) = \frac{1}{2} \) for each \( x \in \mathbb{R} \). Then

\[
(\varphi \circ \delta^*)\left(\bigcup_{n=1}^\infty K_n\right) = \left[1 - 1/2, 1 - 1/2\right] = 1/2.
\]

Thus

\[
(\varphi \circ \delta^*)\left(\bigcup_{n=1}^\infty K_n\right) = \left[\varphi(1/2), \varphi(1/2)\right] = 1/2.
\]

So

\[
(\varphi, \delta^*)\left(\bigcup_{n=1}^\infty K_n\right) < \bigwedge_{n=1}^\infty (\varphi \circ \delta^*)K_n.
\]

Hence \( \varphi \circ \delta^* \notin \text{IVGO}(X) \).

5. Interval-valued fuzzy subspace.

**Definition 5.1**[10]. Let \( Y \) be a subset of \( X \) and let \( A \in I^X \). Then the **restriction of** \( A \) on \( Y \) is denoted by \( A|_Y \). For each \( B \in I^Y \), the **extension of** \( B \), on \( X \), denoted by \( B_X \), is defined by

\[
B_X(x) = \begin{cases} 
B(x) & \text{if } x \in Y, \\
0 & \text{if } x \in X \setminus Y, \text{ for each } x \in X.
\end{cases}
\]

**Proposition 5.2.** Let \((X, \tau)\) be an IVFTS and let \( Y \subset X \). We define the mapping \( \tau_Y : I^Y \to D(I) \) as follows: For each \( A \in I^Y \),

\[
\tau_Y(A) = \bigvee \{\tau(B) : B \in I^X \text{ and } A = B|_Y\}.
\]
Then \( \gamma_Y \in \text{IVGO}(Y) \) and \( \gamma_Y(A) \geq \tau(A_X) \). In this case, the IVFTS \((Y, \tau_Y)\) is called a \textit{subspace of} \((X, \tau)\) and \( \gamma_Y \) is called the \textit{induced IVGO on} \( Y \) \textit{from} \((X, \tau)\).

**Proof.** For each \( A \in I^Y \), let \( B \in I^X \) such that \( A = B|_Y \). Then
\[
\tau^L(B) \leq \tau^U(B).
\]
Thus
\[
\bigvee \{ \tau^L(B) : A = B|_Y \} \leq \bigvee \{ \tau^U(B) : A = B|_Y \}
\]
So, by the definition of \( \gamma_Y \),
\[
\gamma_Y^L(A) \leq \gamma_Y^U(A).
\]
Hence \( \gamma_Y \) satisfies the condition (IVGO1). It is obvious that (IVGO2) holds.

Let \( A_1, A_2 \in I^Y \). Then
\[
\gamma_Y(A_1 \cap A_2) = \bigvee \{ \tau(B) : B \in I^X \text{ and } A_1 \cap A_2 = B|_Y \}.
\]
Suppose \( \gamma_Y(A_1) \wedge \gamma_Y(A_2) = 0 \). Then clearly
\[
\gamma_Y(A_1 \cap A_2) \geq 0 = \gamma_Y(A_1) \wedge \gamma_Y(A_2).
\]
Suppose \( \gamma_Y(A_1) \wedge \gamma_Y(A_2) > 0 \). Let \( 0 < [\lambda, \mu] < \gamma_Y(A_1) \wedge \gamma_Y(A_2) \).
Then \( \exists B_i \in I^X \) such that \( A_i = B_i|_Y \) and \( \tau(B_i) > [\lambda, \mu], i = 1, 2 \). Since \( \tau \in \text{IVGO}(X) \),
\[
\tau(B_1 \cap B_2) \geq \tau(B_1) \wedge \tau(B_2) > [\lambda, \mu],
\]
on the other hand,
\[
(B_1 \cap B_2)|_Y = (B_1|_Y) \cap (B_2|_Y) = A_1 \cap A_2.
\]
Thus
\[
\gamma_Y(A_1 \cap A_2) \geq \tau(B_1 \cap B_2) > [\lambda, \mu].
\]
So, by the definition of \( \gamma_Y \),
\[
\gamma_Y(A_1 \cap A_2) \geq \gamma_Y(A_1) \wedge \gamma_Y(A_2).
\]
In either cases,
\[
\gamma_Y(A_1 \cap A_2) \geq \gamma_Y(A_1) \wedge \gamma_Y(A_2).
\]
Hence the condition (IVGO3) holds.

Now let \( \{ A_\alpha \}_{\alpha \in \Gamma} \subset I^X \). Then
\[
\gamma_Y \left( \bigcup_{\alpha \in \Gamma} A_\alpha \right) = \bigvee \{ \tau(B) : B \in I^X \text{ and } \bigcup_{\alpha \in \Gamma} A_\alpha = B|_Y \}.
\]
Suppose \( \bigwedge_{\alpha \in \Gamma} \gamma_Y(A_\alpha) = 0 \). Then clearly
\[
\gamma_Y \left( \bigcup_{\alpha \in \Gamma} A_\alpha \right) \geq 0 = \bigwedge_{\alpha \in \Gamma} \gamma_Y(A_\alpha).
\]
Suppose \( \bigwedge_{\alpha \in \Gamma} \gamma_Y(A_\alpha) > 0 \) and let \( 0 < [\lambda, \mu] < \bigwedge_{\alpha \in \Gamma} \gamma_Y(A_\alpha) \). Then
\[
\gamma_Y(A_\alpha) > [\lambda, \mu], \forall \alpha \in \Gamma.
\]
Thus \( \exists B_\alpha \in I^X \) such that \( A_\alpha = B_\alpha|_Y \) and \( \tau(B_\alpha) > [\lambda, \mu], \forall \alpha \in \Gamma \).
So \( \tau(\bigcup_{\alpha \in \Gamma} B_{\alpha}) \geq [\lambda, \mu] \).

On the other hand, 
\[
(\bigcup_{\alpha \in \Gamma} A_{\alpha})|_{Y} = \bigcup_{\alpha \in \Gamma} (A_{\alpha}|_{Y}) = (\bigcup_{\alpha \in \Gamma} B_{\alpha}) .
\]

Thus, by the definition of \( \tau_{Y} \),
\[
\tau_{Y}(\bigcup_{\alpha \in \Gamma} A_{\alpha}) \geq \bigwedge_{\alpha \in \Gamma} \tau_{Y}(A_{2}).
\]

In either cases, \( \tau_{Y} \) satisfies the condition (IVGO4). Hence \( \tau \in \text{IVGO}(Y) \).

\[ \Box \]

**Proposition 5.3.** Let \((Y, \tau_{Y})\) be an interval-valued fuzzy subspace of the IVFTS \((X, \tau)\) and let \(A \in I^{Y} \). Then
(a) \( F_{\tau_{Y}}(A) = \bigvee \{ F_{\tau}(B) : B \in I^{X} \text{ and } A = B|_{Y} \} \).
(b) If \( Z \subset Y \subset X \), then \( \tau_{Z} = (\tau_{Y})_{Z} \).

**Proof.** The proofs are very similar to that of Proposition 3.3 in (7). So they are omitted. \[ \Box \]

6. **Interval-valued gradation of preserving mappings**

**Definition 6.1.** Let \((X, \tau)\) and \((Y, \eta)\) be two IVSTSs and let \(f : X \rightarrow Y\) be a mapping. Then \(f\) is called an interval-valued gradation preserving mapping (in short, an IVGP-mapping) or interval-valued smooth continuous if for each \(B \in I^{X} \),
\[
\eta(B) \leq \tau(f^{-1}(B)), \text{i.e., } [\eta^{L}(B), \eta^{U}(B)] \leq [\tau^{L}(f^{-1}(B)), \tau^{U}(f^{-1}(B))].
\]

**Definition 6.1’[7].** Let \((X, \tau)\) and \((Y, \eta)\) be two STSs and let \(f : X \rightarrow Y\) be a mapping. Then \(f\) is called a gradation preserving mapping (in short, an GP-mapping) or smooth continuous if for each \(B \in I^{X} \), \(\eta(B) \leq \tau(f^{-1}(B))\).

**Remark 6.1.** (a) If a mapping \(f : (X, \tau) \rightarrow (Y, \eta)\) is a GP-mapping, then \(f : (X, [\tau, \tau]) \rightarrow (Y, [\eta, \eta])\) is an IVGP-mapping.
(b) If a mapping \(f : (X, \tau) \rightarrow (Y, \eta)\) is an IVGP-mapping, then \(f : (X, \tau^{L}) \rightarrow (Y, \eta^{L})\) and \(f : (X, \tau^{U}) \rightarrow (Y, \eta^{U})\) are GP-mappings, respectively.
Theorem 6.2. Let \((X, \tau)\) and \((Y, \eta)\) be two IVSTSs and let \(f : X \to Y\) be a mapping. Then \(f : (X, \tau) \to (Y, \eta)\) is an IVGP-mapping if and only if \(f : (X, \tau_{[\lambda, \mu]}) \to (Y, \eta_{[\lambda, \mu]})\) is continuous w.r.t. Chang, for each \([\lambda, \mu] \in D(I_0)\).

**Proof.** \((\Rightarrow)\): Suppose \(f\) is an IVGP-mapping. Let \([\lambda, \mu] \in D(I_0)\) and let \(B \in \eta_{[\lambda, \mu]}\). Since \(\eta \in IVGO(Y)\), \(\eta(B) \geq [\lambda, \mu]\). Then, by the hypothesis, \(\eta(B) \leq \tau(f^{-1}(B))\). Thus

\[
\tau(f^{-1}(B)) \geq [\lambda, \mu].
\]

So \(f^{-1}(B) \in \tau_{[\lambda, \mu]}\). Hence \(f : (X, \tau_{[\lambda, \mu]}) \to (Y, \eta_{[\lambda, \mu]})\) is continuous w.r.t. Chang.

\((\Leftarrow)\): Suppose \(f : (X, \tau_{[\lambda, \mu]}) \to (Y, \eta_{[\lambda, \mu]})\) is continuous for each \([\lambda, \mu] \in D(I_0)\). Let \(B \in I^Y\). If \(\eta(B) = 0\), then clearly \(\eta(B) \leq \tau(f^{-1}(B))\). If \(\eta(B) = [\lambda, \mu]\), then \(B \in \eta_{[\lambda, \mu]}\). Thus, by the hypothesis, \(f^{-1}(B) \in \tau_{[\lambda, \mu]}\). So \(\tau(f^{-1}(B)) \geq [\lambda, \mu] = \eta(B)\). In either cases, \(\eta(B) \leq \tau(f^{-1}(B))\). Hence \(f\) is an IVGP-mapping. \(\square\)

Theorem 6.3. Let \((X,T)\) and \((Y,T')\) be two Chang’s fuzzy topological space and let \(f : X \to Y\) be a mapping. Then \(f : (X, T) \to (Y, T')\) is continuous if and only if \(f : (X, T^{[\lambda, \mu]}) \to (Y, (T')^{[\lambda, \mu]})\) is an IVGP-mapping, for each \([\lambda, \mu] \in D(I_0)\).

**Proof.** \((\Rightarrow)\): Suppose \(f : (X, T) \to (Y, T')\) is continuous, let \(B \in I^Y\) and let \([\lambda, \mu] \in D(I_0)\). Then we have the following cases:

(i) \(B = \phi\) or \(Y\),
(ii) \(B \in T'\),
(iii) \(B \not\in T'\).

Case (i): \(f^{-1}(\phi) = \phi\) or \(f^{-1}(Y) = X\). Thus

\[
(T')^{[\lambda, \mu]}(B) \leq T^{[\lambda, \mu]}(f^{-1}(B)).
\]

Case (ii): Clearly \((T')^{[\lambda, \mu]}(B) = [\lambda, \mu]\). Since \(f\) is continuous, \(f^{-1}(B) \in T\). Thus

\[
T^{[\lambda, \mu]}(f^{-1}(B)) = [\lambda, \mu].
\]

So

\[
(T')^{[\lambda, \mu]}(B) \leq T^{[\lambda, \mu]}(f^{-1}(B)).
\]

Case (iii): It is clear that \((T')^{[\lambda, \mu]}(B) = 0\). Thus

\[
0 = (T')^{[\lambda, \mu]}(B) \leq T^{[\lambda, \mu]}(f^{-1}(B)).
\]

So, in all cases, \(f : (X, T^{[\lambda, \mu]}) \to (Y, (T')^{[\lambda, \mu]})\) is an IVGP-mapping.

\((\Leftarrow)\): It follows from Proposition 3.18 and Theorem 6.2. \(\square\)

The following is the immediate result of Definition 6.1.
Proposition 6.4. Let \((X, \tau), (Y, \eta)\) and \((Z, \xi)\) be IVSTSs.

(a) \(1_X : (X, \tau) \to (X, \tau)\) is an IVGP-mapping.

(b) If \(f : (X, \tau) \to (Y, \eta)\) and \(g : (Y, \eta) \to (Z, \xi)\) is IVGP-mappings, then \(g \circ f : (X, \tau) \to (Z, \xi)\) is an IVGP-mapping.

We can easily see that the collection of all IVFTSs and IVGP-mapping between then forms a concrete category and we will denote it by \(IVTop\).

Theorem 6.5. Let \((X, \tau)\) be an IVFTS and let \(f : X \to Y\) be a mapping. Let \(\{T_{\lambda, \mu}\}_{\lambda, \mu} \in D(I_0)\) be a descending family of chang’s fuzzy topologies on \(Y\). Let \(\eta\) be the IVGO on \(X\) generated by this family. For each \([\lambda, \mu] \in D(I_0)\), suppose \(B_{[\lambda, \mu]}\) or \(S_{[\lambda, \mu]}\) is a base or a subbase for \(T_{[\lambda, \mu]}\), respectively. Then the followings are equivalent:

(a) \(f : (X, \tau) \to (y, \eta)\) is an IVGP-mapping.

(b) \(\tau(f^{-1}(B)) \geq [\lambda, \mu], \forall B \in T_{[\lambda, \mu]}, \forall [\lambda, \mu] \in D(I_0).\)

(c) \(\tau(f^{-1}(B)) \geq [\lambda, \mu], \forall B \in B_{[\lambda, \mu]}, \forall [\lambda, \mu] \in D(I_0).\)

(d) \(\tau(f^{-1}(B)) \geq [\lambda, \mu], \forall B \in S_{[\lambda, \mu]}, \forall [\lambda, \mu] \in D(I_0).\)

Proof. (a) \(\Rightarrow\) (b) : Suppose (a) holds. Let \([\lambda, \mu] \in D(I_0)\) and let \(B \in T_{[\lambda, \mu]}\). Then \(\tau(f^{-1}(B)) \geq [\lambda, \mu].\)

It is obvious that (b) \(\Rightarrow\) (c) \(\Rightarrow\) (d) hold.

(d) \(\Rightarrow\) (a) : Suppose (d) holds. Let \(B \in T_{[\lambda, \mu]}\) and, without loss of generality, let \(\eta(B) = [\lambda, \mu] > 0\). Then \(B \in T_{[\lambda, \mu]}\). Now, \(B\) is of the form, \(B = \bigcup_{\alpha \in \Gamma} B_{\alpha}\), where \(B_{\alpha} \in B_{[\lambda, \mu]}\). Also, for each \(\alpha \in \Gamma, B_{\alpha}\)

is of the form, \(B_{\alpha} = \bigcup_{j=1}^{n_{\alpha}} S_{\alpha, j}\), where \(S_{\alpha, j} \in S_{[\lambda, \mu]}, \forall j = 1, 2, \cdots, n_{\alpha}\).

Thus

\[
\tau(f^{-1}(B)) = \tau(f^{-1}(\bigcup_{\alpha \in \Gamma} S_{\alpha, j}))
\]

\[
= \tau(\bigcup_{\alpha \in \Gamma} f^{-1}(S_{\alpha, j}))
\]

\[
\geq \bigwedge_{\alpha \in \Gamma} (\bigwedge_{j=1}^{n_{\alpha}} \tau f^{-1}(S_{\alpha, j})) \quad [\text{Since } \tau \in \text{IVGO}(X)]
\]

\[
\geq [\lambda, \mu] \quad [\text{By the condition (d)}]
\]

So \(\tau(f^{-1}(B)) \geq \eta(B)\). Hence \(f : (X, \tau) \to (Y, \eta)\) is on IVGP-mapping. This completes the proof.
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