HOMOGENEOUS IDEAL $I(+)IM$ OF $R(+)M$

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Abstract. In this short paper, we show that properties of an ideal $I$ of a ring $R$ are related to those of the homogeneous ideal $I(+)IM$ of a ring $R(+)M$.

1. Introduction

Throughout this paper, all rings are commutative rings with unity and all modules are unital. $R$-module $M$ is called multiplication module if every submodule $N$ of $M$ has the form $IM$ for some ideal $I$ of $R$. Equivalently, $N = (N : M)M$. $R$-module $M$ is said to be divisible if $M = rM$ whenever $r$ is an element of $R$ which is not a zero divisor.

$R$-module $M$ is called cancellation if whenever $AM = BM$ for ideals $A$ and $B$ of $R$, then $A = B$.

Let $M$ be an $R$-module. Consider $R(M) = \{(r, m) | r \in R, m \in M\}$ and let $(r, m)$ and $(s, n)$ be two elements of $R(M)$. Define:

1. $(r, m) = (s, n)$ if $r = s$ and $m = n$
2. $(r, m) + (s, n) = (r + s, m + n)$
3. $(r, m)(s, n) = (rs, rn + sm)$

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Under these definition, \( R(M) \) becomes a commutative ring with unity and \( R(M) \) is called the \textit{idealization} of a ring \( R \) and an \( R \)-module \( M \). Sometimes \( R(M) \) is also denoted by \( R(+)M \). We can find some basic results about \( R(M) \) ([8]). An ideal \( H \) of \( R(+)M \) is called \textit{homogeneous} if \( H = I(+)N \) where \( I \) is an ideal of \( R \) and \( N \) a submodule of \( M \). In this case, \( I(+)N = (R(+)M)(I(+)N) = I(+)(IM + N) \) gives \( IM \subseteq N \). If \( IM \subseteq N \), then \( M/N \) is an \( R/I \)-module and \( g : R(+)M \to R/I(+)M/N \) defined by \( g(r, m) = (r + I, m + N) \) is a ring homomorphism and \( \ker(g) = I(+)N \) and hence \( I(+)N \) is an ideal of \( R(+)M \). So we know that \( I(+)N \) is an ideal of \( R(+)M \) if and only if \( IM \subseteq N \). Every ideal of \( R(+)M \) of the form \( I(+)N \) is homogeneous. However, ideals of \( R(+)M \) need not have the form \( I(+)N \), that is, need not be homogeneous. \( \mathbb{Z}(+)2\mathbb{Z}(2,2) \) is not a homogeneous ideal of \( \mathbb{Z}(+)2\mathbb{Z}([2]) \). In this paper, we show that properties of an ideal \( I \) of a ring \( R \) are related to those of the homogeneous ideal \( I(+)IM \) of a ring \( R(+)M \).

2. Ideals \( I \) and \( I(+)IM \).

Compare the following Theorem with Proposition 11 in [3]

**Theorem 2.1.** Let \( R \) be a ring and \( M \) an \( R \)-module and \( I \) an ideal of \( R \). If \( I(+)IM \) is a cancellation ideal of \( R(+)M \) then \( I \) is a cancellation ideal of \( R \). The converse is true if \( M \) is a divisible multiplication module over a domain \( R \) with \( \text{ann}(M \otimes R/A) = A \) for every ideal \( A \) of \( R \).

**Proof.** Suppose that \( I(+)IM \) is a cancellation ideal of \( R(+)M \) and let \( AI = BI \) for ideals \( A, B \) of \( R \). (\( A(+)M)(I(+)IM) = AI(+) (AIM + IM) = AI(+I)IM = BIM + IM) = (B(+)M)(I+)IM. Since \( I(+)IM \) is a cancellation ideal of \( R(+)M \), \( A(+)M = B(+)M \) and so \( A = B \). Now we prove the converse.
If $M$ is a divisible module over a domain $R$, then every ideal of $R(+M)$ is homogeneous ([4]-Theorem 3.3). So, for ideals $H, H'$ of $R(+M)$ such that $H(I(+IM)) = H'(I(+IM))$ we have $H = \mathcal{A}(+N)$ and $H' = \mathcal{B}(+K)$, where $\mathcal{A}$ and $\mathcal{B}$ are ideals of $R$ and $N, K$ are submodules of $M$

\[ H(I(+IM)) = (\mathcal{A}(+N))(I(+IM)) = \mathcal{A}(+)(\mathcal{A}M + IN) \]

\[ = \mathcal{A}(+)IN \text{ since } \mathcal{A}M \subseteq N. \]

Similarly $H'(I(+IM)) = (\mathcal{B}(+K))(I(+IM)) = \mathcal{B}(+)IK$. Hence $\mathcal{A}I = \mathcal{B}I$ and $IN = IK$. Since $I$ is cancellation, $\mathcal{A} = \mathcal{B}$ and from $N = (N : M)M$ and $K = (K : M)M$ we have $I(N : M)M = I(K : M)M$. On the other hand, $\text{ann}(M \otimes R/A) = \mathcal{A}$ implies that $\text{ann}(M/\mathcal{A}M) = \mathcal{A}$ and hence $(\mathcal{A}M : M) = \mathcal{A}$ for any ideal $\mathcal{A}$ of $R$. Now, let $\mathcal{I}M = \mathcal{J}M$ for ideals $\mathcal{I}, \mathcal{J}$ of $R$. Then $\mathcal{I} = (\mathcal{I}M : M) = (\mathcal{J}M : M) = \mathcal{J}$. So $M$ is cancellation and $I(N : M) = I(K : M)$. Again, since $I$ is cancellation $(N : M) = (K : M)$. Therefore $N = (N : M)M = (K : M)M = K$ and $\mathcal{A}(+).N = \mathcal{B}(+).K$. i.e, $H = H'$

**Corollary 2.2.** Let $R$ be a ring and $M$ an $R$-module. If every ideal of $R(+M)$ is cancellation then every ideal of $R$ is cancellation.

**Proof.** Suppose that every ideal of $R(+M)$ is cancellation and let $\mathcal{A}$ be any ideal of $R$. $\mathcal{A}(+).\mathcal{A}M$ is an ideal of $R(+M)$ and so cancellation. By Theorem 2.1 $\mathcal{A}$ is cancellation.

**Corollary 2.3.** Let $R$ be a ring and $M$ a faithful multiplication $R$-module. If every faithful ideal of $R(+M)$ is cancellation then every faithful ideal of $R$ is cancellation.

**Proof.** Let $\mathcal{A}$ be any faithful ideal of $R$. Since $M$ is a faithful multiplication $R$-module, we know that $\text{ann}(\mathcal{A}(+).\mathcal{A}M) = \text{ann}.\mathcal{A}M$
$(+)\text{ann}(\mathcal{A})M$ ([5]-Remark 1) and $\text{ann}(\mathcal{A}M) \subseteq \text{ann}(\mathcal{A}) = 0$. Therefore $\text{ann}(\mathcal{A}(+)\mathcal{A}M) = 0(+0)$. So $(\mathcal{A}(+)\mathcal{A}M)$ is a faithful ideal of $R(+)^{}M$ and by our assumption $(\mathcal{A}(+)\mathcal{A}M)$ is cancellation. Therefore $\mathcal{A}$ is cancellation by the above Theorem 2.1

**Proposition 2.4.** Let $I$ be an ideal of $R$ and $M$ an $R-$ module. Then $I$ is idempotent in $R$ if and only if $I(+)^{}IM$ is idempotent in $R(+)^{}M$.

**Proof.** Let $I^2 = I$. Then $(I(+)^{}IM)^2 = (I(+)^{}IM)(I(+)^{}IM) = I^2(+)^{}I^2M + I^2M = (I^2(+)^{}IM) = (I(+)^{}IM)$. So $(I(+)^{}IM)$ is idempotent. Conversely, if $(I(+)^{}IM)$ is idempotent then $I^2(+)^{}IM = I(+)^{}IM$ and so $I^2 = I$. □

Ali([1]) defined idempotent submodule as follows. A submodule $N$ of an $R$-module $M$ is called *idempotent* if $N = (N : M)^{}N$. If we put $N = I$ for an ideal $I$ of $R$ and $M = R$ then we know that this is a generalized concept of idempotent ideal.

**Proposition 2.5.** Let $I$ be an ideal of $R$ and $M$ an $R-$ module. If $I(+)^{}IM$ is an idempotent ideal of $R(+)^{}M$ then $IM$ is an idempotent submodule of $M$.

**Proof.** Since $I(+)^{}IM$ is idempotent, $(I(+)^{}IM)^2 = I^2(+)^{}I^2M = I(+)^{}IM$. Then $I^2M = IM$ and $I^2M = I(IM) \subseteq (IM : M)^{}IM \subseteq IM$. Hence $IM = (IM : M)^{}IM$ and $IM$ is idempotent. □

**Proposition 2.6.** Let $I$ be an ideal of $R$ and $M$ a finitely generated faithful multiplication $R-$ module. Then $I$ is an idempotent ideal of $R$ if and only if $IM$ is an idempotent submodule of $M$.

**Proof.** If $I$ is an idempotent ideal of $R$ then $I(+)^{}IM$ is an idempotent ideal of $R(+)^{}M$. Hence $IM$ is an idempotent submodule
of $M$ (Proposition 2.5). Conversely, if $IM$ is an idempotent submodule of $M$ then $IM = (IM : M)IM$ and by our assumption $IM = I(IM : M)M = I^2M$. Since $M$ is cancellation module([9]-Theorem 6.1) $I = I^2$.

Ali([1]) introduced the concept of nilpotent submodule which is a generalized concept of nilpotent ideal. A submodule $N$ of $M$ is called a nilpotent submodule if $(N : M)^k N = 0$

**Proposition 2.7.** Let $I$ be an ideal of $R$ and $M$ an $R-$module. Then $I$ is nilpotent in $R$ if and only if $I(+)IM$ is nilpotent in $R(+)M$.

*Proof.* Let $I^n = 0$ for some positive integer $n$. Then $(I(+)IM)^n = (I^n(+)I^nM) = 0(+)0$. Hence $I(+)IM$ is nilpotent. Conversely, if $I(+)IM$ is nilpotent then there exists a positive integer $k$ such that $(I(+)IM)^k = 0(+)0$ and hence $I^k = 0$. □

**Proposition 2.8.** Let $I$ be an ideal of $R$ and $M$ a finitely generated faithful multiplication $R-$module. If $I(+)IM$ is a nilpotent ideal of $R(+)M$ then $IM$ is a nilpotent submodule of $M$.

*Proof.* By our assumption there exists a positive integer $k$ such that $(I(+)IM)^k = 0(+)0$. So $I^kM = 0$. Since $M$ is cancellation module([9]-Theorem 6.1) and faithful, $I = (IM : M)$([9]-Proposition 1.4) and hence $(IM : M)^k IM = I^kIM = I(I^kM) = 0$. Therefore $IM$ is nilpotent. □

**Proposition 2.9.** Let $I$ be an ideal of $R$ and $M$ a faithful multiplication $R-$module. Then $I$ is a nilpotent ideal of $R$ if and only if $IM$ is a nilpotent submodule of $M$.

*Proof.* Suppose that $IM$ is nilpotent. Then $(IM : M)^k IM = 0$ for some positive integer $k$. So $(IM : M)^{k-1}(IM : M)IM = 0$. Since $M$ is a multiplication module, $(IM : M)^{k-1}I^2M = 0$. 

\[(IM : M)^{k-2}(IM : M)I^2M = 0\] and hence \[(IM : M)^{k-2}I^3M = 0.\] Continue this way. Then we arrive at \((IM : M)I^kM = I^k(IM : M)M = I^{k+1}M = 0.\) Since \(M\) is faithful, \(I^{k+1} = 0.\) \(\square\)

An ideal \(A\) of a ring \(R\) is said to be regular if it contains a nonzero divisor element.

**Theorem 2.10.** Let \(I\) be an ideal of a ring \(R\) and \(M\) an \(R\)-module. If \(I(+)IM\) is a regular ideal of \(R(+)M\) then \(I\) is a regular ideal of \(R\). The converse is true if \(M\) is torsion free.

**Proof.** Suppose that \(I(+)IM\) is regular and let \((i, m) \in I(+)IM\) be a regular element in \(R(+)M\). To show that \(i \in I\) is a regular element in \(R\), let \(ij = 0\) for \(j \in R\). Then \((0, jm)(i, m) = (0, ijm) = (0, 0).\) Since \((i, m)\) is regular, \(jm = 0\) and \((j, 0)(i, m) = (0, 0).\) Thus \(j = 0.\) i.e., \(i \in I\) is regular and \(I\) is regular. Conversely, suppose that \(M\) is torsion free and \(i \in I\) is regular. Consider an element \((i, m') \in I(+)IM\) and let \((j, n)(i, m') = (0, 0)\) for an element \((j, n) \in R(+)M.\) Then \(ji = 0, jm' + in = 0.\) Since \(i\) is regular, \(j = 0\) and hence \(0 = jm' + in = in.\) Thus \(n = 0\) because \(M\) is torsion free. So \((i, m')\) is regular in \(R(+)M\) and \(I(+)IM\) is regular. \(\square\)

**Theorem 2.11.** Let \(I\) be a nonzero ideal of a ring \(R\) and \(M\) an \(R\)-module. If \(I(+)IM\) is an invertible ideal of \(R(+)M\) then \(I\) is an invertible ideal of \(R.\) The converse is true if \(M\) is faithful and multiplication.

**Proof.** In a ring the concepts of invertible ideal and regular multiplication ideal coincide ([7]-Theorem 7.2). Therefore, if \(I(+)IM\) is invertible, \(I(+)IM\) is both regular and multiplication ideal. So, \(I\) is a multiplication ideal([2]-Theorem 7). Further \(I\) is regular by Theorem 2.10. Therefore \(I\) is invertible. Conversely, If \(I\) is invertible then \(I\) is regular and multiplication and so, \(I(+)IM\)
is a multiplication ideal([2]-Theorem 7). Since $M$ is faithful multiplication, $M$ is torsion free ([6]-Lemma 4.1) and since $I$ is regular, $I(+IM)$ is regular by Theorem 2.10. Therefore $I(+IM)$ invertible.

□

A ring $R$ is presimplifiable if for $x, y \in R$, $xy = x$ implies $x = 0$ or $y$ is a unit. $R-$ module $M$ is $R$-presimplifiable if for $r \in R$ and $m \in M$, $rm = m$ implies $r$ is a unit or $m = 0$. This generalizes the previous definition of $R$ being presimplifiable.

**Theorem 2.12.** Let $I$ be an ideal of a ring $R$ and $M$ an $R-$module. Then, $I$ and $IM$ are $R$-simplifiable if and only if $I(+IM)$ is $R(+)M$-presimplifiable.

**Proof.** Suppose that $I$ and $IM$ are $R$-simplifiable. Let $(r, m)$ $(i, m') = (i, m')$, where $i \in I, m' \in IM, r \in R$ and $m \in M$. Then $ri = i$ and $rm' + im = m'$. Since $I$ is $R$-presimplifiable $r \in U(R)$, the set of all unit elements in $R$ or $i = 0$. However $U(R(+)M) = U(R)(+M)$([4]-Theorem 3.7) and hence if $r \in U(R)$ then $(r, m) \in U(R(+)M)$ and if $i = 0$ then $rm' = m'$. Since $IM$ is $R-$presimplifiable $m' = 0$ and $(i, m') = (0, 0)$ or $(r, m) \in U(R)(+M) = U(R(+)M)$. In any case we have $(r, m) \in U(R(+)M)$ or $(i, m') = (0, 0)$. Conversely, Assume that $I(+IM)$ is $R(+)M$-presimplifiable. Let $ri = i$ for $r \in R$ and $i \in I$. Then $(r, 0)(i, 0) = (i, 0)$ and $(r, 0) \in U(R(+)M)$ or $(i, 0) = (0, 0)$. Therefore $r \in U(R)$ or $i = 0$.

□

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