SURFACES WITH PLANAR LINES OF CURVATURE

DONG-SOO KIM$^1$ AND YOUNG HO KIM$^2$

Abstract. We study surfaces in the 3-dimensional Euclidean space with two families of planar lines of curvature. As a result, we establish some characterization theorems for such surfaces.

1. Introduction

Consider a smooth surface $M$ in the Euclidean space $\mathbb{E}^3$ with a unit normal vector field $U$. Then on each tangent plane $T_p M$ the shape operator $S$ is defined as follows:

$$S(v) = -\nabla_v U,$$

where $\nabla_v U$ denotes the covariant derivative of $U$ in the $v$ direction.

For a unit vector $u$ tangent to $M$ at a point $p$, the number $k(u) = \langle S(u), u \rangle$ is called the normal curvature of $M$ in the $u$ direction. The maximum and minimum values of the normal curvature $k(u)$ of $M$ at $p$ are called the principal curvatures of $M$ at $p$, and are denoted by $k_1$ and $k_2$. The directions in which these extreme values occur are called principal directions of $M$ at $p$.

A regular curve $X$ in $M$ is called a line of curvature provided that the velocity $X'$ of $X$ always points in a principal direction. Through each non-umbilic point of $M$, there are exactly two lines of curvature, which necessarily cut orthogonally across each other.

---

Received November 1, 2010. Accepted December 09, 2010.

2000 Mathematics Subject Classification. 53A04, 53A05, 53A07.

Keywords and phrases. lines of curvature, surface of revolution, slant cylinder, generalized slant cylinder, Weingarten surface.

$^1$ was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0022926).

$^2$ was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0007184).
The next theorem is useful to find lines of curvature on some classes of surfaces:

**Theorem of Joachimstahl.** Suppose that $M_1$ and $M_2$ intersect along a regular curve $X$ and make an angle $\theta(p)$, $p \in X$. Assume that $X$ is a line of curvature of $M_1$. Then $X$ is a line of curvature of $M_2$ if and only if $\theta(p)$ is constant.

**Proof.** See the proof of Theorem 9 in ([10], p. 296).

Note that every regular curve on a plane is a line of curvature. Using above theorem, it is easy to show the following: The meridians and parallels on a surface of revolution are its lines of curvature.

For a plane curve $X$ in a plane $P$, the cylinder $M$ over $X$ is a ruled surface generated by a one-parameter family of straight lines through each point $X(s)$ which are orthogonal to the plane $P$. Theorem of Joachimstahl also shows that the straight lines, and the intersection of $M$ and each plane parallel to the plane $P$ are lines of curvature of $M$.

Hence we see that cylinders and surfaces of revolution satisfy the following condition:

(C) Around each point $p \in M$, there exists a local orthonormal frame $\{E_1, E_2\}$ whose integral curves are planar lines of curvature.

In this paper, we study smooth surfaces $M$ in the Euclidean space $\mathbb{E}^3$ which satisfy the condition (C). As a result, we establish some characterization theorems for such surfaces. Furthermore, we give a condition for such a surface to be a surface of revolution.

### 2. Slant cylinders and generalized slant cylinders

For a fixed unit speed plane curve $X(s) = (x(s), y(s), 0)$, let $T(s) = X'(s)$ and $N(s) = (-y'(s), x'(s), 0)$ denote the unit tangent and principal normal vector, respectively. The curvature $\kappa(s)$ of $X(s)$ is defined by $T'(s) = \kappa(s)N(s)$ and we have $T(s) \times N(s) = V$, where $V$ denotes the unit vector $(0, 0, 1)$. For a constant $\theta$, we let $Y(s) = \cos \theta N(s) + \sin \theta V$.

Then the ruled surface $M$ defined by

$$F(s, t) = X(s) + tY(s)$$

is regular at $(s, t)$ where $1 - \cos \theta \kappa(s)t$ does not vanish. This ruled surface $M$ is called a *slant cylinder* over $X(s)$. For the unit normal
vector $U = -\sin \theta N(s) + \cos \theta V$, $M$ satisfies
\[
\langle F_s, F_t \rangle = 0, \langle F_{st}, U \rangle = 0.
\]
This shows that the coordinates lines of $F$ are lines of curvature of $M$ with corresponding principal curvatures
\[
k_1(s, t) = \frac{-\kappa(s) \sin \theta}{1 - \kappa(s) \cos \theta}, k_2(s, t) = 0,
\]
respectively. Hence $F(s, t)$ is a principal curvature coordinate system of the flat slant cylinder $M$ ([6], p. 53). Since the coordinate lines of $F$ are planar, it follows that the slant cylinder $M$ satisfies the condition (C). The slant cylinder with $\sin \theta = 0$ or $\cos \theta = 0$ is nothing but a parametrization of either a plane or a usual cylinder.

In general, we consider another unit speed plane curve $W(t) = (z(t), w(t))$. If we let $Y_s(t) = z(t)N(s) + w(t)V$, then the parametrized surface defined by
\[
H(s, t) = X(s) + Y_s(t)
\]
is regular at $(s, t)$ where $1 - \kappa(s)z(t)$ does not vanish. This parametrized surface $M$ is called a generalized slant cylinder over $X(s)$. For the unit normal vector $U(s, t) = -w'(t)N(s) + z'(t)V$, $M$ satisfies
\[
\langle H_s, H_t \rangle = 0, \langle H_{st}, U \rangle = 0.
\]
This shows that $H(s, t)$ is a principal curvature coordinate system of $M$ with corresponding principal curvatures
\[
k_1(s, t) = \frac{-\kappa(s) w'(t)}{1 - \kappa(s) z(t)}, k_2(s, t) = \kappa(t),
\]
respectively, where $\kappa(t) = z'(t)w''(t) - z''(t)w'(t)$ denotes the curvature of $W(t)$. It is obvious that the coordinate lines of $H$ are planar. Hence we see that the generalized slant cylinder also satisfies the condition (C).

If $W(t)$ is a straight line, then the generalized slant cylinder $H(s, t)$ is nothing but a slant cylinder. Furthermore, we prove the following.

**Proposition 1.** If a plane curve $X(s)$ is a circle, then the generalized slant cylinder $M$ over $X(s)$ is a surface of revolution.

**Proof.** Suppose that $X(s)$ is a circle of radius $r$. Then it is straightforward to show that for each fixed $t$, $s$ curve of the generalized slant cylinder $H$ defined in (2.4) is a circle of radius $r - z(t)$ with principal normal vector $N(s)$. Hence the $s$ curve through $H(0, t)$ is a circle centered
at
\[ C(t) = H(0,t) + \{r - z(t)\}N(0) = X(0) + rN(0) + w(t)V, \]
which parametrizes a fixed straight line \( l \) in the direction of \( V \). Thus \( M \) is a surface of revolution with axis \( l \).

Therefore the class of generalized slant cylinders contains both the class of slant cylinders and the class of surfaces of revolution.

3. Some characterizations

Suppose that a smooth surface \( M \) in the Euclidean space \( \mathbb{E}^3 \) satisfies the condition (C). If we let \( E_3 = E_1 \times E_2 \), then \( \{ E_1, E_2, E_3 \} \) is a principal frame field on \( M([9], \text{p. 261}) \). For the dual 1-forms \( \theta_1, \theta_2 \) of \( E_1, E_2 \) the connection forms are given by
\[ \omega_{12} = g_1 \theta_1 + g_2 \theta_2, \omega_{13} = k_1 \theta_1, \omega_{23} = k_2 \theta_2, \]
where \( g_1, g_2 \) are some functions and \( k_1, k_2 \) denote the principal curvatures in the direction of \( E_1, E_2 \), respectively. Hence the covariant derivatives of \( E_i(i = 1, 2, 3) \) with respect to \( E_j(j = 1, 2) \) are given by
\[ \nabla_{E_1} E_1 = g_1 E_2 + k_1 E_3, \nabla_{E_1} E_2 = -g_1 E_1, \nabla_{E_1} E_3 = -k_1 E_1, \]
\[ \nabla_{E_2} E_1 = g_2 E_2, \nabla_{E_2} E_2 = -g_2 E_1 + k_2 E_3, \nabla_{E_2} E_3 = -k_2 E_2, \]
respectively.

From the Codazzi equations we have([9], p. 262)
\[ E_1(k_2) = (k_1 - k_2)g_2, \]
\[ E_2(k_1) = (k_1 - k_2)g_1. \]
For the Gaussian curvature \( K \) of \( M \) the second structural equation gives([9], p. 263)
\[ K = k_1 k_2 = E_2(g_1) - E_1(g_2) - g_1^2 - g_2^2. \]
It follows from (3.2) that the integral curves of \( E_1 \) are planar if and only if
\[ g_1 E_1(k_1) - k_1 E_1(g_1) = 0. \]
Similarly, we see that the integral curves of $E_2$ are planar if and only if
\begin{equation}
(3.8) \quad g_2E_2(k_2) - k_2E_2(g_2) = 0.
\end{equation}
Furthermore, for each $i = 1, 2$, the integral curves of $E_i$ lie on a plane $V_i$ normal to $V_i$, which is given by
\begin{equation}
(3.9) \quad V_1 = \frac{-k_1E_2 + g_1E_3}{\sqrt{k_1^2 + g_1^2}}, V_2 = \frac{k_2E_1 + g_2E_3}{\sqrt{k_2^2 + g_2^2}},
\end{equation}
unless the denominators vanish. It is obvious from the condition (C) that
\begin{equation}
(3.10) \quad \nabla_{E_i} V_i = 0, i = 1, 2.
\end{equation}

First of all we prove the following:

**Theorem 2.** A flat surface $M$ in the Euclidean space $E^3$ satisfies the condition (C) if and only if it is locally a slant cylinder over a plane curve.

**Proof.** Suppose that a flat surface $M$ satisfies the condition (C). We denote by $P$ the set of planar points and by $W = M - P$ the set of parabolic points. Then $P$ is closed and $W$ is open in $M$. On a connected component $W_1$ of $W$, we may assume that $k_1$ does not vanish. Hence $k_2$ vanishes identically on $W_1$. By reversing the direction of $E_1$ if necessary, we may assume that $k_1 > 0$. Hence (3.4) shows that $g_2 = 0$. Thus it follows from (3.3) that the $E_2$ curve through a point $p \in W_1$ is an open segment of a straight line, which parametrizes a unique asymptotic line segment through $p$. Using (3.7), we see that $g_1 = h_1k_1$ for a function $h_1$ satisfying $E_1(h_1) = 0$. Therefore we get from (3.5) and (3.6) that
\begin{equation}
(3.11) \quad g_1^2 = E_2(g_1) = g_1^2 + E_2(h_1)k_1,
\end{equation}
which shows that $h_1$ is a constant $c$, that is, $g_1 = ck_1$. Thus we obtain from (3.9) that
\begin{equation}
(3.11) \quad V_1 = \frac{-E_2 + cE_3}{\sqrt{1 + c^2}}.
\end{equation}
Since $g_2 = k_2 = 0$, (3.3) and (3.10) show that $V_1$ is a constant vector. Hence every $E_1$ curve lies in a plane $V_1^\perp$.

We now prove Theorem 2 in the following procedures.

**Step 1.** Let $\ell(p)$ be the maximal asymptotic line segment through a point $p \in W$. Then we have $\ell(p) \subset W$. 

\begin{flushright}
\textbf{Surfaces with planar lines of curvature} 781
\end{flushright}
We parametrize not vanish on $\psi$ and $q$ line segment before, we see that the unique trajectory $k_1$ simultaneously. Since $p$ from (3.5) that $\frac{dk_1}{dt} = ck_1^2$. Hence we have $k_1(t) = \frac{1}{c-dt}$, which cannot vanish along $\ell(p)$. This completes the proof.

For a point $p$ in the boundary $bd(W)$ of the set $W$, we prove the following.

**Step 2.** Let $p \in bd(W) \subset M$. Then through $p$ there passes a unique open segment of straight line $\ell(p) \subset M$. Furthermore, $\ell(p) \subset bd(W)$, that is, $bd(W)$ consists of open segments of asymptotic lines.

Proof. Let $p \in bd(W)$. On a neighborhood $O$ around $p$, let $\{E_1, E_2\}$ be a principal orthonormal frame on $O$ with principal curvatures $k_1, k_2$, respectively, which appears in the condition (C). On $O \cap W$ the Gaussian curvature $k_1k_2$ vanishes everywhere, but $k_1$ and $k_2$ does not vanish simultaneously. Since $p$ is a limit point of $W$, it is possible to choose a sequence $\{p_n\}$ in $O \cap W$ which converges to $p$ as $n \to \infty$.

Without loss of generality, we may assume that there exists such a sequence $\{p_n\}$ as above with $k_1(p_n) \neq 0, n = 1, 2, \cdots$. Then in a neighborhood of $p_n$, $k_2$ vanishes identically. Put $\phi : (-\delta_1, \delta_1) \times U \to O$ be the unique trajectory of $E_2$ with $\phi(0, q) = q$ in a neighborhood $U$ of $p$. Then $\phi(t, p_n)$ is nothing but a parametrization of the asymptotic line segment $\ell(p_n)$ through $p_n$. This shows that $\nabla_{E_2}(E_2(\phi(t, p_n)) = 0$ for each $n = 1, 2, \cdots$ and $|t| < \delta_1$. By letting $n \to \infty$, we see that $\nabla_{E_2}(E_2(\phi(t, p)) = 0$ for all $t$ with $|t| < \delta_1$. Thus $\phi(t, p)$ is an asymptotic line segment through $p$ in the direction of $E_2$.

Suppose that there exists another sequence $\{q_n\}$ in $O \cap W$ with $k_2(q_n) \neq 0, n = 1, 2, \cdots$, which converges to $p$ as $n \to \infty$. Then, as before, we see that the unique trajectory $\psi(t, q_n)$ of $E_1$, $|t| < \delta_2$, converges to a line segment $\psi(t, p)$ through $p$. For sufficiently large $n$, the line segment $\phi(t, p_n)$ through $p_n$ should meet the line segment $\psi(t, p)$ at a point $q$ in $O$. This is a contradiction, because Step 1 shows that $\phi(t, p_n)$ and $\psi(t, p)$ belong to the sets $W$ and $P$, respectively. This contradiction shows that for a sufficiently small neighborhood $O$ of $p$, $k_1$ does not vanish on $O \cap W$ and the integral curve $\phi(t, p)$ of $E_2$ is the unique asymptotic line segment through $p$, which we will denote by $\ell(p)$.

Next, we assert that every point of $\ell(p)$ on $M$ is a boundary point of $W$. In fact, if $q \in \ell(p)$, there exists a sequence $q_n = \phi(t, p_n)$ in $W$ with $p_n \to p$, and hence $q_n \to q$ as $n \to \infty$. Thus $q$ belongs to the closure of $W$. Assume that $q$ does not belong to $bd(W)$. Then $q \in W$. Since $\ell(p)$ is the unique asymptotic line segment through $q \in W$, we get $p \in W$, which is a contradiction.
Note that each connected component of int(P) is an open part of a plane.

Now we give a proof of Theorem 2. It suffices to show that the theorem holds in a neighborhood of a point \( p \in \text{bd}(W) \). Let \( p \) be a point in the boundary of \( W \), and \( \{E_1, E_2\} \) an orthonormal frame in a neighborhood of \( p \) as in the proof of Step 2. Without loss of generality, we may assume that the line segment \( \ell(p) \) is in the direction of \( E_2 \). Then the proof of Step 2 shows that there exists a neighborhood \( O \) of \( p \) such that \( \nabla_{E_2} E_2 = 0 \) and \( k_1 \) does not vanish on \( O \cap W \). It follows from the condition (C) that for the constant vector \( V_1 \) in (3.11), every \( E_1 \) curve on \( O \cap \text{int}(P) \) parametrizes an open segment of the straight line \( V_1 \cap \text{int}(P) \) which is orthogonal to \( \ell(p) \). Every \( E_2 \) curve on \( O \cap \text{int}(P) \) is also an open segment of a straight line which is parallel to \( \ell(p) \).

Let \( X(s) \) denote an \( E_1 \) curve through \( p \) which lies in the plane \( V_1 \) and \( N(s) = V_1 \times E_1(s) \) the principal normal. It follows from Theorem of Joachimsthal that \( \langle E_3, V_1 \rangle \) is constant along \( X(s) \), hence we have for a constant \( \theta \),
\[
E_2(s) = \cos \theta N(s) + \sin \theta V_1.
\]
Hence \( O \) is an open part of the following slant cylinder:
\[
F(s, t) = X(s) + tE_2(s).
\]
This completes the proof of Theorem 2.

Example 1 in ([4], p.409) describes a flat surface which satisfies the condition (C). It is locally (but not globally) an open part of a slant cylinder.

Now, suppose that a non-flat surface \( M \) satisfies the condition (C). Then by reversing the unit vector \( E_1 \) (hence \( E_3 = E_1 \times E_2 \) is also reversed) if necessary, we may assume that \( k_1 > 0, k_2 \neq 0 \). It follows from (3.7) and (3.8) that
\[
(3.12) \quad g_i = h_i k_i, \quad E_i(h_i) = 0, \quad i = 1, 2.
\]
We prove the following:

**Theorem 3** Suppose that a non-flat surface \( M \) satisfies the condition (C). Then every \( E_2 \) curve is a geodesic (that is, \( g_2 = 0 \)) if and only if it is a generalized slant cylinder over an \( E_1 \) curve. In either case, we have
\[
(3.13) \quad E_2(h_1) = (1 + h_1^2)k_2.
\]

**Proof.** Suppose that \( g_2 \) vanishes identically on \( M \). Then from (3.3) we get
\[
(3.14) \quad \nabla_{E_2} E_1 = 0, \quad \nabla_{E_2} E_2 = k_2 E_3, \quad \nabla_{E_2} E_3 = -k_2 E_2,
\]
Furthermore, (3.13) follows from (3.5), (3.6) and (3.12). Since $M$ is non-flat, it follows from (3.9) that

$$V_1 = \frac{-E_2 + h_1 E_3}{\sqrt{1 + h_1^2}}, V_2 = E_1,$$

which shows that $V_1, V_2$ are orthogonal to each other. By differentiating $V_1$ in (3.15) with respect to $E_2$, (3.14) shows that

$$\frac{(1 + h_1)^{3/2}}{E_1} V_1 = g_2 (1 + h_1^2) E_1 + h_1 \{ E_2 (h_1) - (1 + h_1^2) k_2 \} E_2 + \{ E_2 (h_1) - (1 + h_1^2) k_2 \} E_3.$$

Together with (3.10), (3.13) and (3.16) show that $V_1$ is a constant vector.

We denote by $X(s)$ an $E_1$ curve. Then $X(s)$ lies on a plane $V_1^\perp$ perpendicular to $V_1$ and $N(s) = V_1 \times E_1(s)$ is the principal normal to $X(s)$. Note that for each $s$, the $E_2$ curve through $X(s)$ lies in the plane $V_2^\perp$. Since $V_2^\perp$ is orthogonal to $V_2(s) = E_1(s)$, it is spanned by \{N(s), V_1\}. Thus we see that

$$H(s, t) = X(s) + z(s, t) N(s) + w(s, t) V_1$$

is a parametrization of the surface $M$, where $z(s, t) and w(s, t)$ are some functions which satisfy

$$z(s, 0) = w(s, 0) = 0, z_t^2 + w_t^2 = 1.$$

Now we show that $z(s, t), w(s, t)$ can be chosen so that they depend only on $t$. For this purpose, first of all we assert that for any $(s_0, t_0)$, $w_t(s_0, t_0) \neq 0$. Otherwise, differentiating the last equation in (3.18) with respect to $t$, we have $z_t(s_0, t_0) = 0$. Hence we get at $(s_0, t_0)$

$$k_2 E_3 = \nabla_{E_2} E_2 = H_{tt} = w_{tt} V_1,$$

where the first equality follows from (3.14). Since $M$ is non-flat, $k_2(s_0, t_0) \neq 0$. Thus (3.19) shows that

$$V_1 = \pm E_3(s_0, t_0),$$

which contradicts to (3.15). This contradiction implies that $w_t(s_0, t_0) \neq 0$.

Note that the $E_1$ curve through $H(s_0, t_0)$ is contained in the plane $V_1^\perp$ through $H(s_0, t_0)$. Hence it follows from (3.17) that the $E_1$ curve is contained in the set \{ $H(s, t) | w(s, t) = w(s_0, t_0)$ \}. Since $w_t(s_0, t_0) \neq 0$, we see that

$$X_{t_0}(s) = H(s, f(s)),$$
is a reparametrization of the $E_1$ curve through $H(s_0, t_0)$, where $f(s)$ satisfies

\begin{equation}
(3.21) \quad f(s_0) = t_0, w(s, f(s)) = w(s_0, t_0).
\end{equation}

By differentiating (3.20) with respect to $s$, (3.17) and (3.21) show that

\begin{equation}
(3.22) \quad X'_{t_0}(s) = \{1 - \kappa(s)z(s, f(s))\}E_1(s) + \left\{\frac{d}{ds}z(s, f(s))\right\}N(s).
\end{equation}

On the other hand, it follows from (3.20) that $X'_{t_0}(s)$ is proportional to $E_1(s, f(s))$. Furthermore, the first equation in (3.14) shows that $E_1$ is parallel along $t$-curve of $H$ so that we have $E_1(s, f(s)) = E_1(s, 0) = E_1(s)$. Hence it follows from (3.21) and (3.22) that

\begin{equation}
(3.23) \quad z(s, f(s)) = z(s_0, t_0).
\end{equation}

Thus we have

\begin{equation}
(3.24) \quad X_{t_0}(s) = X(s) + z(s, f(s))N(s) + w(s, f(s))V_1
= X(s) + z(s_0, t_0)N(s) + w(s_0, t_0)V_1,
\end{equation}

where the second equality follows from (3.21) and (3.23). Since $t_0$ is arbitrary, if we let $z(t) = z(s_0, t)$, $w(t) = w(s_0, t)$, then (3.24) implies that

\[ H(s, t) = X(s) + z(t)N(s) + w(t)V_1 \]

is a reparametrization of $M$. This shows that $M$ is a generalized slant cylinder over an $E_1$ curve $X(s)$.

Finally, suppose that $M$ is a generalized slant cylinder over an $E_1$ curve $X(s)$ of which parametrization $H(s, t)$ is given in (2.3). Then every $E_2$ curve is a $t$-curve of $H$. Since $H_{tt}$ is orthogonal to $H_t$ and $H_s$, every $t$ curve of $H$ is a geodesic of $M$, that is, $g_2$ vanishes identically. Together with (3.16), constancy of $V = V_1$ shows that (3.13) holds. This completes the proof.

There exist surfaces in the Euclidean space $\mathbb{E}^3$ which satisfy the condition (C), but not an open part of a generalized slant cylinder. For example, the Enneper’s minimal surface and the family of associated Bonnet surfaces are cases of these kinds([1], [3], [8]).
4. Linear Weingarten surfaces with planar lines of curvature

Suppose that a non-flat and non-minimal linear Weingarten surface \( M \) in the Euclidean space \( \mathbb{E}^3 \) satisfies the condition (C). Hence we have\[ k_2 = ak_1 + b, \quad k_1 \neq 0, \quad k_2 \neq 0, \]where \( a, b \) are constant with\[ (a+1)^2 + b^2 \neq 0 \]and\[ a^2 + b^2 \neq 0. \]Furthermore we assume that \( M \) has no umbilic points, that is, \( k_1 \neq k_2 \). By reversing the unit vector \( E_1 \) (hence \( E_3 = E_1 \times E_2 \) is also reversed) if necessary, we assume that \( k_1 > 0 \).

From (3.4), (3.5) and (3.6) we obtain
\[
\begin{align*}
(4.1) & \quad aE_1(k_1) = \{(1-a)k_1 - b\}g_2, \\
(4.2) & \quad E_2(k_1) = \{(1-a)k_1 - b\}g_1, \\
(4.3) & \quad k_1(ak_1 + b) = E_2(g_1) - E_1(g_2) - g_1^2 - g_2^2.
\end{align*}
\]

By differentiating (4.1) and (4.2) with respect to \( E_2, E_1 \), respectively, we obtain
\[
\begin{align*}
(4.4) & \quad aE_2E_1(k_1) = (1-a)\{(1-a)k_1 - b\}g_1g_2 + \{(1-a)k_1 - b\}E_2(g_2), \\
(4.5) & \quad aE_1E_2(k_1) = (1-a)\{(1-a)k_1 - b\}g_1g_2 + a\{(1-a)k_1 - b\}E_1(g_1).
\end{align*}
\]

On the other hand, from (3.2), (3.3), (4.1) and (4.2) we have
\[
a\{E_2E_1(k_1) - E_1E_2(k_1)\} = a\{\nabla E_2E_1(k_1) - \nabla E_1E_2(k_1)\}
\quad = a\{g_2E_2(k_1) + g_1E_1(k_1)\}
\quad = (a + 1)\{(1-a)k_1 - b\}g_1g_2.
\]

Hence (4.4) and (4.5) show that
\[
(4.6) \quad E_2(g_2) - aE_1(g_1) = (a + 1)g_1g_2.
\]

1) First, we consider the case \( a \neq 0 \). It follows from (3.12) that \( g_1 = h_1k_1, \quad g_2 = h_2(ak_1 + b) \) for some functions satisfying \( E_1(h_1) = E_2(h_2) = 0 \). Substituting these into (4.6), we get
\[
(4.7) \quad h_1h_2\{a(a + 1)k_1^2 + 2bk_1 - b^2\} = 0.
\]

Suppose that \( h_1h_2 \neq 0 \) on an open set \( W \). Then (4.7) shows that \( k_1 \) is a root of a nontrivial polynomial of degree 1 or 2. Hence \( k_1 \) (and hence \( k_2 \)) is constant. This shows that \( W \) is an open part of either a circular cylinder (flat) or a sphere (umbilic) ([C]), which contradicts to the hypotheses. Thus \( h_1h_2 \) (hence \( g_1g_2 \)) vanishes identically on \( M \).
Since $M$ is non-flat, (4.3) shows that $g_1, g_2$ cannot vanish simultaneously on an open set. Hence we may assume that $W_1 =\{ p \in M | g_1(p) \neq 0 \}$ is nonempty. Since $g_2$ vanishes identically on $W_1$, Theorem 3 shows that $W_1$ is a generalized slant cylinder over an $E_1$ curve $X(s)$. It follows from (4.1) and (4.6) that

\[(4.8) \quad E_1(k_1) = E_1(g_1) = 0.\]

Since $X''(s) = \nabla E_1 E_1 = g_1 E_2 + k_1 E_3$, (4.8) shows that the plane curve $X(s)$ has nonzero constant curvature $\sqrt{k_1^2 + g_1^2}$. Hence $X(s)$ is a circle. It follows from Proposition 1 that $W_1$ is a surface of revolution and each parallel(that is, $E_1$ curve) on $W_1$ lies on a plane $V_1^\perp$, where $V_1$ is given by

\[(4.9) \quad V_1 = \frac{-E_2 + h_1 E_3}{\sqrt{1 + h_1^2}}.\]

It follows from (4.8) that $g_1$ is constant on each parallel. Hence the closure $\overline{W}_1$ of $W_1 \subset M$ has boundary $bd(\overline{W}_1)$ (if any) consisting of open segments of parallels which lie on some planes $V_1^\perp$.

Now suppose that $W_2 = \{ p \in M | g_2(p) \neq 0 \}$ is nonempty. Then, as before, it follows from Proposition 1 and Theorem 3 that $W_2$ is a surface of revolution and each parallel(that is, $E_2$ curve) on $W_2$ lies on a plane $V_2^\perp$, where $V_2$ is given by

\[(4.10) \quad V_2 = \frac{-E_1 + h_2 E_3}{\sqrt{1 + h_2^2}}.\]

For a point $p \in bd(\overline{W}_2)$, the parallel $C(p)$ through $p$ on $\overline{W}_2$ is also a parallel on $\overline{W}_1$. This implies that $C(p)$ lies on both $V_1^\perp$ and $V_2^\perp$, which shows that $V_1$ is parallel to $V_2$. But from (4.9) and (4.10) we see that $V_1$ cannot be parallel to $V_2$. This contradiction shows that $W_2$ is empty, and hence $M$ is a surface of revolution.

2) Finally, we consider the case $a = 0$. Then we have $k_2 = b(\neq 0)$. Hence, (3.4) shows that $g_2$ vanishes identically. It follows from (3.3) that every $E_2$ curve $Y(t)$ is a circle of radius $1/|b|$. Thus Theorem 3 shows that $M$ is a tube along an $E_1$ curve $X(s)$. 
5. Weingarten surfaces with planar lines of curvature

Suppose that a non-flat surface $M$ satisfying the condition (C) also satisfies the Weingarten condition:
\[(W)\]
$$k_2 = f(k_1),$$
for some polynomial function $f(x)$ of degree $n(\geq 2)$ in $x$. Furthermore we assume that $M$ has no umbilic points. As in Section 4, we may assume that $k = k_1 > 0$. From (3.4), (3.5) and (3.6) we obtain
\[(5.1)\]
$$f'(k)E_1(k) = \{k - f(k)\}g_2,$$
\[(5.2)\]
$$E_2(k) = \{k - f(k)\}g_1,$$
\[(5.3)\]
$$kf(k) = E_2(g_1) - E_1(g_2) - g_1^2 - g_2^2.$$

By differentiating (5.1) and (5.2) with respect to $E_2, E_1$, respectively, we obtain
\[(5.4)\]
$$f'(k)\{E_2E_1(k) - E_1E_2(k)\} = \{k - f(k)\}\{-f''(k)E_1(k)g_1 + E_2(g_2) - f'(k)E_1(g_1)\}.$$ 

On the other hand, from (3.2), (3.3), (5.1) and (5.2) we have
\[(5.5)\]
$$f'(k)\{E_2E_1(k) - E_1E_2(k)\} = f'(k)\{\nabla E_2E_1(k) - \nabla E_1E_2(k)\}$$
$$= f'(k)\{g_2E_2(k) + g_1E_1(k)\}$$
$$= \{f'(k) + 1\}\{k - f(k)\}g_1g_2.$$ 

Hence (5.4) and (5.5) show that
\[(5.6)\]
$$E_2(g_2) - f'(k)E_1(g_1) = f''(k)E_1(k)g_1 + \{f'(k) + 1\}g_1g_2.$$

It follows from (3.12) that $g_1 = h_1k, g_2 = h_2f(k)$ for some functions $h_1$ and $h_2$ satisfying $E_1(h_1) = E_2(h_2) = 0$. Substituting these into (5.6), we get
\[(5.7)\]
$$h_1h_2\{k - f(k)\}\{f''(k)kf(k) - f'(k)^2k + f(k)\} + \{f'(k) + 1\}kf(k) = 0.$$

Suppose that $h_1h_2 \neq 0$ on an open set $W$. Then (5.7) shows that $k = k_1$ is a root of some nontrivial polynomial of degree $3n - 1$. Hence $k = k_1$ (and hence $k_2$) is constant there. Thus $W$ is an open part of either a circular cylinder (flat) or a sphere (umbilic) ([2]). This contradiction shows that $h_1h_2$ (and hence $g_1g_2$) vanishes identically on $M$. Hence we can proceed as in Section 4 to conclude that $M$ is a surface of revolution.

Summarizing the results in Section 4 and 5, we establish the following.
Theorem 4 Let $M$ be a non-flat and non-minimal surface without umbilic points which satisfies the condition (C). Suppose that $M$ is a Weingarten surface with

\[(W) \quad k_2 = f(k_1),\]

where $f$ is a polynomial of degree $n(\geq 1)$. Then $M$ is a surface of revolution.

It is well-known that every surface of revolution is a Weingarten surface ([7], pp. 91-92). According to H. Hopf([5]), surfaces of revolution satisfying $k_2 = ak_1 (a \in R)$ are classified in ([7], pp. 92-93).

References
Department of Mathematics,
Kyungpook National University,
Taegu 702-701, Korea
E-mail: yhkim@knu.ac.kr