SPHERICAL NEWTON DISTANCE FOR OSCILLATORY INTEGRALS WITH HOMOGENEOUS PHASE FUNCTIONS

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Abstract. In this paper we study oscillatory integrals with analytic homogeneous phase functions for smooth radial functions. We give their sharp asymptotic behavior in terms of spherical Newton distance.

1. Introduction

In this paper we consider oscillatory integrals $I(\lambda, \varphi)$ defined by

$$I(\lambda, \varphi) = \int_{\mathbb{R}^k} e^{i\lambda f(x)} \varphi(x) \, dx,$$

where $\lambda$ is a large positive real parameter. Here $f$ is a homogeneous polynomial and $\varphi$ is a smooth cut-off function. In this chapter we consider the cases $k = 2, 3$. Main purpose of this chapter is to reduce the dimension by one to treat the higher dimensional problems in lower dimension. For the case $k = 3$ we introduce “spherical Newton distance” in “superadapted coordinates”. A superadapted coordinate system was introduced by Greenblatt in [8] to describe the oscillatory index of smooth phase function in $\mathbb{R}^2$ and it turned out to be a stronger notion of adapted coordinate system in [9].

In view of the work of Varchenko [9], it is not surprising that the asymptotic behavior of $I(\lambda, \varphi)$ bears a close connection to the blow-up properties of integrals of the form

$$\int_{U} \frac{1}{|f(x)|^3} \, dx$$
where $U$ is a small neighborhood of the origin. It has been known that if the support of $\varphi$ is contained in a sufficiently small neighborhood of the origin, then the oscillatory integral $I(\lambda, \varphi)$ has the following asymptotic expansion as $\lambda \to \infty$:

\begin{equation}
I(\lambda, \varphi) \approx e^{i\lambda f(0)} \sum_{p} \sum_{l=0}^{k-1} a_{p,l} \lambda^p (\ln \lambda)^l
\end{equation}

where $p$ runs through a decreasing arithmetic progression of negative rational numbers depending only on the phase function $f$.

The oscillatory index $\beta(f)$ of $f$ at the origin is defined by the minimum number of $-p$ which has the following property: For any neighborhood $U$ of the origin of $\mathbb{R}^k$ there exists a smooth cut-off function $\varphi$ supported in $U$ such that in the asymptotic expansion of $I(\lambda, \varphi)$ in (1.1) there exists $l$ satisfying $a_{p,l}(\varphi) \neq 0$.

If we denote by $\tilde{\beta}(f)$

$$
sup \left\{ \delta : \int_U |f(x)|^{-\delta} dx < \infty \right\}
$$

where $U$ is a small neighborhood of the origin, then the arguments in [9] lead us to the fact that when $\tilde{\beta}(f) < 1$

\begin{equation}
\beta(f) = \tilde{\beta}(f).
\end{equation}

He also found a close connection between the oscillatory index and the Newton distance. To be more precise we consider the power series of $f$

$$
f(x) = \sum_{\alpha \in \mathbb{N}^k} a_\alpha x^\alpha
$$

where $\mathbb{N}$ is the set of all nonnegative integers. We set

$$
\text{supp} f = \{\alpha \in \mathbb{N}^k | a_\alpha \neq 0\}.
$$

We define Newton’s polyhedron $\Gamma_+(f)$ of $f$ as

$$
\Gamma_+(f) = \text{the convex hull in } \mathbb{R}_+^k \text{ of the set } \bigcup_{\alpha \in \text{supp} f} (\alpha + \mathbb{R}_+^k)
$$

where $\mathbb{R}_+$ is the set of all nonnegative real numbers. Newton’s diagram $\Gamma(f)$ of $f$ is defined as the boundary of $\Gamma_+(f)$. For any compact face $\gamma \subset \Gamma(f)$ we denote the polynomial $\sum_{\alpha \in \gamma} a_\alpha x^\alpha$ by $f_\gamma$ and we define the principal part of $f$ by the polynomial $\sum_{\gamma} \sum_{\alpha \in \gamma} a_\alpha x^\alpha$ where $\gamma$ runs through all compact faces of $\Gamma(f)$. The principal part of $f$ is said to be nonsingular if for any compact face $\gamma \subset \Gamma(f)$, $\nabla f_\gamma = (\frac{\partial f_\gamma}{\partial x_1}, \ldots, \frac{\partial f_\gamma}{\partial x_k})$
Spherical Newton distance with homogeneous phase functions

does not vanish in $\left(\mathbb{R} \setminus \{0\}\right)^k$. The Newton distance $d(f)$ of $f$ is defined as

$$d(f) = \min\{t \mid (t, \cdots, t) \in \Gamma_+(f)\}.$$

In [9], Varchenko established following asymptotic behavior of $I(\lambda, \varphi)$ when the principal part of $f$ is nonsingular:

\begin{align}
\beta(f) &\geq \frac{1}{d(f)}; \\
\text{If } d(f) > 1, \text{ then } \beta(f) &= \frac{1}{d(f)}. \tag{1.4}
\end{align}

We note that $-\frac{1}{d(f)}$ is called the remoteness of $f$ at the origin.

In the same paper he also considered the case $k = 2$. In this case he showed that $\beta(f) \leq \frac{1}{d(f)}$ and that there is necessarily a coordinate system where $\beta(f) = \frac{1}{d(f)}$. This coordinate system is called “adapted coordinates” and the coordinate change to such coordinates can be always be made of the form $(x, y) \to (x, y - r(x))$ or $(x, y) \to (x - r(y), y)$ for real analytic function $r$.

In [8] Greenblatt introduced a stronger notion of adapted coordinate system, which was called “superadapted coordinates”. Precisely a real analytic function $f$ in $\mathbb{R}^2$ is said to be in superadapted coordinates if whenever $\gamma$ is a compact face of $\Gamma_+(f)$ intersecting the bisectrix, both of the function $f\gamma(1, y)$ and $f\gamma(-1, y)$ have no real zero of order $d(f)$ or greater other than possibly $y = 0$.

He showed that any phase function can be put in superadapted coordinates and superadaptedness is stronger notion of adaptedness in the sense that a superadapted coordinate system is also an adapted coordinate system. It was also proved that in a superadapted coordinate system, a critical point of $f(x, y)$ at the origin is nondegenerate if and only if $d(f) = 1$.

Further results for asymptotic behaviors of oscillatory integrals with large $\lambda$ were considered in [1]-[7].

Now we introduce polar Newton distance and spherical Newton distance for homogeneous polynomials in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

Let $f$ be a homogeneous polynomial of degree $d$ in $\mathbb{R}^2$. Let $\theta_i \ (i = 1, 2, \cdots, n)$ be real roots of the equation $F(\theta) = f(\cos \theta, \sin \theta) = 0$ and let $d_i$ be the multiplicity of $\theta_i$. We set $\alpha = \max\{d_i \mid i = 1, 2, \cdots, n\}$ and define the polar Newton distance $\delta_p(f)$ of $f$ by

$$\delta_p = \min\{2/d, 1/\alpha\}.$$
To define the spherical Newton distance of homogeneous polynomials in \( \mathbb{R}^3 \) we define Newton distance of analytic function in \( \mathbb{R}^2 \) in super-adapted coordinates. Let \( g \) be an analytic function in \( \mathbb{R}^2 \) and let \((x_0, y_0)\) be a solution of the equation \( g(x, y) = 0 \). We define the Newton distance \( d_{(x_0, y_0)}(g) \) in superadapted coordinates at \((x_0, y_0)\) as the Newton distance of the Taylor series expansion of \( g \) at \((x_0, y_0)\) with a coordinate change into superadapted coordinates.

Let \( f \) be a homogeneous polynomial of degree \( d \) in \( \mathbb{R}^3 \). We set \( A = \{ (\theta, \phi) : f(\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) = 0 \} \) and
\[
\tilde{\alpha} = \sup \{ d_{(\theta_0, \phi_0)}(f(\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)) : (\theta_0, \phi_0) \in A \}.
\]
We define the spherical Newton distance \( \delta_s \) of \( f \) in superadapted coordinates by
\[
\delta_s = \min \{ 3/d, 1/\tilde{\alpha} \}.
\]
In view of (1.2) we state our theorems in terms of \( \tilde{\beta}(f) \).

**Theorem 1.** Let \( f \) be a homogeneous polynomial in \( \mathbb{R}^2 \) and let \( U \) be a small neighborhood of the origin. Then we have
\[
(1.5) \quad \tilde{\beta}(f) = \frac{1}{\delta_p(f)}.
\]

**Theorem 2.** Let \( f \) be a homogeneous polynomial of degree \( d \) in \( \mathbb{R}^3 \). Suppose that \( f(0, 0, z) \) is not identically zero. Then we have
\[
(1.6) \quad \tilde{\beta}(f) = \frac{1}{\delta_s(f)}.
\]

In Theorems 1 and 2, we abuse the notation of function \( f \) to stand for phase functions of \( I(\lambda, \varphi) \) in dimension 2 and 3, respectively.

### 2. Preliminaries

In this section we state preliminary propositions: the one of which is one-dimensional analogue of our main theorems and the other of which is elementary lemma in topology. The one reason of reviewing elementary mathematical theorems is pedagogical purpose and the other is to make the paper self-contained.

**Proposition 1.** Let \( f \) be an analytic function on \( \mathbb{R} \) and let \( \mathbb{I} \) be a sufficiently small open interval containing 0. Then
\[
\frac{1}{m} = \sup \left\{ \varepsilon : \int_{\mathbb{I}} |f(x)|^{-\varepsilon} \, dx < +\infty \right\},
\]
Spherical Newton distance with homogeneous phase functions

where $m$ is the lowest degree of the Taylor series of $f$ at 0.

**Proof.** We write

$$f(x) = a_m x^m + a_{m+1} x^{m+1} + \cdots,$$

where $a_m \neq 0$. We obtain

$$\int_{\mathbb{R}} \frac{1}{|f(x)|^\varepsilon} \psi(x) = \int_{-r}^{r} \frac{1}{|a_m x^m + a_{m+1} x^{m+1} + \cdots|^\varepsilon} \psi(x) \, dx$$

$$= \int_{-r}^{r} \frac{1}{|x|^{m\varepsilon} |a_m + a_{m+1} x + \cdots|^\varepsilon} \psi(x) \, dx. \quad (2.1)$$

We set

$$g(x) = a_m + a_{m+1} x + \cdots.$$ 

Then for sufficiently small $r$, we have

$$\int_{-r}^{r} \frac{1}{|g(x)|^\varepsilon} < +\infty.$$ 

Thus we see that the integral in $(2.1)$ converges if $m\varepsilon < 1$ and diverges if $m\varepsilon \geq 1$. We therefore obtain the desired results and this completes the proof of Proposition 1.

**Proposition 2.** Suppose $V_i (i = 1, 2, \cdots, n)$ are open subsets of a locally compact Hausdorff space $X$, $K$ is compact and $K \subset \bigcup_{i=1}^{n} V_i$. Then there exist functions $\eta_i \prec V_i (i = 1, 2, \cdots, n)$ such that

$$\sum_{i=1}^{n} \eta_i(x) = 1, \quad \text{for all } x \in K.$$ 

The collection $\eta_1, \cdots, \eta_n$ is called partition of unity on $K$, subordinate to the cover $V_1, \cdots, V_n$.

The proof of Proposition 2 can be found in [10]. We shall use Proposition 1 and 2 in the remaining sections.

**3. Polar Newton Distance**

In this section we prove Theorem 1. The idea is to make the change of variables in polar coordinates and write the homogeneous function $f$ as a form of tensor product.

**Proof of Theorem 1.** Without loss of generality we may take a small open neighborhood $U$ of the origin as a small open disc centered at the
origin. We make use of the change of variables \((x_1, x_2) = (\rho \cos \theta, \rho \sin \theta)\) with \(\theta \in [0, 2\pi]\) to write
\[
\int_{U} |f(x)|^{-\epsilon} \, dx = \int_{0}^{2\pi} \int_{0}^{r} |f(\rho \cos \theta, \rho \sin \theta)|^{-\epsilon} \rho \, d\rho \, d\theta,
\]
where \(r\) is a small positive real number. Since \(f\) is assumed to be homogeneous of degree \(d\) the right-hand side integral in (3.1) can be rewritten as
\[
\int_{0}^{r} \rho^{1-d\epsilon} \, d\rho \int_{0}^{2\pi} |f(\cos \theta, \sin \theta)|^{-\epsilon} \, d\theta.
\]
It is obvious that
\[
\int_{0}^{r} \rho^{1-d\epsilon} \, d\rho < +\infty \quad \text{if and only if} \quad \epsilon < \frac{2}{d}.
\]
If we set \(F(\theta) = f(\cos \theta, \sin \theta)\), then \(F(\theta)\) is an analytic function so the equation \(F(\theta) = 0\) has the finite number of real roots. Let \(\theta_1, \theta_2, \cdots, \theta_n\) be the distinct real roots of the equation in \([0, 2\pi]\). Let \(\zeta > 0\) be so small that intervals \([\theta_i - \zeta, \theta_i + \zeta]\) are pairwise disjoint. With such \(\zeta\) we write
\[
\int_{0}^{2\pi} |F(\theta)|^{-\epsilon} \, d\theta = \sum_{i=1}^{n} \int_{\theta_i - \zeta}^{\theta_i + \zeta} |F(\theta)|^{-\epsilon} \, d\theta + \int_{[0,2\pi] \setminus \bigcup_{i=1}^{n} (\theta_i - \zeta, \theta_i + \zeta)} |F(\theta)|^{-\epsilon} \, d\theta.
\]
Since \([0, 2\pi] \setminus \bigcup_{i=1}^{n} (\theta_i - \zeta, \theta_i + \zeta)\) does not contain any real roots, it is clear that
\[
\int_{[0,2\pi] \setminus \bigcup_{i=1}^{n} (\theta_i - \zeta, \theta_i + \zeta)} |F(\theta)|^{-\epsilon} \, d\theta < +\infty.
\]
By Proposition 1, we immediately obtain that
\[
\int_{\theta_i - \zeta}^{\theta_i + \zeta} |F(\theta)|^{-\epsilon} \, d\theta < +\infty, \quad \text{if and only if} \quad \epsilon < \frac{1}{d_i},
\]
where \(d_i\) is the multiplicity of the real root \(\theta_i\). Then we clearly have
\[
\sum_{i=1}^{n} \int_{\theta_i - \zeta}^{\theta_i + \zeta} |F(\theta)|^{-\epsilon} \, d\theta < +\infty \quad \text{if and only if} \quad \epsilon < \frac{1}{\alpha},
\]
where
\[
\alpha = \max \{d_i \mid i = 1, 2, \cdots, n\}.
\]
Therefore
\[
\int_{U} |F(\theta)|^{-\epsilon} \, dx < +\infty \quad \text{if and only if} \quad \epsilon < \min\{2/d, 1/\alpha\}.
\]
This completes the proof of Theorem 1.

\[\square\]

4. Spherical Newton Distance

In this section we prove Theorem 2.

Proof of Theorem 2. Without loss of generality we may take a small open neighborhood \(U\) of the origin as a small open ball centered at the origin. We use the change of variables \((x_1, x_2, x_3) = (\rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi)\) with \(0 \leq \theta \leq 2\pi\) and \(0 \leq \phi \leq \pi\), and homogeneity of \(f\) to write

\[
\int_{U} |f(x)|^{-\varepsilon} \, dx = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{r \rho^2 - d \varepsilon} \frac{\rho^2 - d \varepsilon \sin \phi}{|f(\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)| \varepsilon} \, d\rho \, d\phi \, d\theta.
\]

(4.1)

By setting \(F(\theta, \phi) = f(\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)\) we rewrite the integral in (4.1) as

\[
\int_{0}^{r} \rho^2 - d \varepsilon \, d\rho \int_{0}^{2\pi} \int_{0}^{\pi} \sin \phi |F(\theta, \phi)|^{-\varepsilon} \, d\phi \, d\theta.
\]

(4.2)

It is clear that

\[
\int_{0}^{r} \rho^2 - d \varepsilon \, d\rho < +\infty, \quad \text{if and only if} \quad \varepsilon < 3/d.
\]

We take into account of the second factor of (4.2). It is easy to see that for any solution \((\theta_0, \phi_0)\) of the equation \(F(\theta, \phi) = 0\) there exists \(r_0 > 0\) such that for any \((\theta, \phi) \in D((\theta_0, \phi_0), r_0)\)

\[
d_{(\theta_0, \phi_0)}(F) \geq d_{(\theta, \phi)}(F)
\]

where \(D(x, r)\) is a disc centered at \(x\) with radius \(r\). By using the compactness of the zero set in \([0, 2\pi] \times [0, \pi]\) we can find finite disc \(D((\theta_i, \phi_i), r_i), i = 1, 2, \ldots, n\) such that \(\{D((\theta_i, \phi_i), r_i)\}\) covers the zero set, \(F(\theta_i, \phi_i) = 0\), and

\[
d_{(\theta_i, \phi_i)}(F) \geq d_{(\theta, \phi)}(F) \quad \text{if} \quad (\theta, \phi) \in D((\theta_i, \phi_i), r_i).
\]
By Proposition 2, there exists a partition of unity \( \eta_i, i = 1, 2, \ldots, n \) such that \( \eta_i \prec D((\theta_i, \phi_i), r_i) \) and \( \sum_{i=1}^{n} \eta_i = 1 \). Then we obtain
\[
\int_{0}^{\pi} \int_{0}^{2\pi} \frac{\sin \phi}{|F(\theta, \phi)|^\varepsilon} d\theta d\phi = \int_{0}^{\pi} \int_{0}^{2\pi} \frac{\sin \phi}{|F(\theta, \phi)|^\varepsilon} \sum_{i=1}^{n} \eta_i d\theta d\phi
\]
\[
= \sum_{i=1}^{n} \int_{0}^{\pi} \int_{0}^{2\pi} \frac{\sin \phi}{|F(\theta, \phi)|^\varepsilon} \eta_i d\theta d\phi
\]
\[
= \sum_{i=1}^{n} \int_{D((\theta_i, \phi_i), r_i)} \frac{\sin \phi}{|F(\theta, \phi)|^\varepsilon} \eta_i d\theta d\phi
\]
\[
+ \int_{(0,2\pi] \times [0,\pi]} \bigcup_{i=1}^{n} D((\theta_i, \phi_i), r_i) \frac{\sin \phi}{|F(\theta, \phi)|^\varepsilon} \eta_i d\theta d\phi.
\]
Since \( F(\theta, \phi) \neq 0 \) in \((0,2\pi] \times [0,\pi] \setminus \bigcup_{i=1}^{n} D((\theta_i, \phi_i), r_i)\), we obtain
\[
\int_{(0,2\pi] \times [0,\pi]} \bigcup_{i=1}^{n} D((\theta_i, \phi_i), r_i) \frac{\sin \phi}{|F(\theta, \phi)|^\varepsilon} d\theta d\phi < +\infty.
\]
If we set \( \bar{\alpha} = \max\{d(\theta_i, \phi_i) : i = 1, \ldots, n\} \), then
\[
\bar{\alpha} = \sup\{d(\theta, \phi) : (\theta, \phi) \in [0, 2\pi] \times [0, \pi]\}
\]
and clearly
\[
\sum_{i=1}^{n} \int_{D((\theta_i, \phi_i), r_i)} \frac{\sin \phi}{|F(\theta, \phi)|^\varepsilon} d\theta d\phi < +\infty \quad \text{if} \quad \varepsilon < 1/\bar{\alpha},
\]
and
\[
\sum_{i=1}^{n} \int_{D((\theta_i, \phi_i), r_i)} \frac{\sin \phi}{|F(\theta, \phi)|^\varepsilon} d\theta d\phi = +\infty \quad \text{if} \quad \varepsilon > 1/\bar{\alpha}.
\]
This completes the proof of Theorem 2.

\[\square\]

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