ON n-FOLD STRONG IDEALS OF BH-ALGEBRAS

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Abstract. The notion of n-fold strong ideal in BH-algebra is introduced and some related properties of it are investigated. The role of initial segments in BH-algebras is described.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([2,3]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. BCK-algebras have some connections with other areas: D. Mundici [5] proved MV-algebras are categorically equivalent to bounded commutative algebra, and J. Meng [6] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. Y. B. Jun, E. H. Roh, and H. S. Kim [4] introduced the notion of a BH-algebra, which is a generalization of BCK/BCI-algebras. They defined the notions of ideal, maximal ideal and translation ideal and investigated some properties. E. H. Roh and S. Y. Kim [8] estimated the number of BH∗-subalgebras of order i in a transitive BH∗-algebras by using Hao’s method. S. S. Ahn and J. H. Lee ([1]) introduced the notion of strong ideals in BH-algebra and investigated some properties of it.

In this paper, we introduce the notion of n-fold strong ideal in BH-algebra and investigated some related properties of it. We also describe the role of initial segments in BH-algebras.

2. Preliminaries

By a BH-algebra ([4]), we mean an algebra (X; *, 0) of type (2,0) satisfying the following conditions:

(I) \( x * x = 0 \),
(II) \( x * 0 = x \),

Received May 4, 2011. Accepted June 2, 2011.
2000 Mathematics Subject Classification. 06F35, 03G25.
Key words and phrases. Initial segments, ideals, (n-fold) strong ideals.
(III) $x \ast y = 0$ and $y \ast x = 0$ imply $x = y$, for all $x, y \in X$.

For brevity, we also call $X$ a BH-algebra. In $X$ we can define a binary operation “$\leq$” by $x \leq y$ if and only if $x \ast y = 0$. A non-empty subset $S$ of a BH-algebra $X$ is called a subalgebra of $X$ if, for any $x, y \in S$, $x \ast y \in S$, i.e., $S$ is a closed under binary operation.

**Definition 2.1.** ([4]) A non-empty subset $A$ of a BH-algebra $X$ is called an ideal of $X$ if it satisfies:

(I1) $0 \in A$,

(I2) $x \ast y \in A$ and $y \in A$ imply $x \in A$, $\forall x, y \in X$.

An ideal $A$ of a BH-algebra $X$ is said to be a translation ideal of $X$ if it satisfies:

(I3) $x \ast y \in A$ and $y \ast x \in A$ imply $(x \ast z) \ast (y \ast z) \in A$ and $(z \ast x) \ast (z \ast y) \in A$, $\forall x, y, z \in X$.

Obviously, $\{0\}$ and $X$ are ideals of $X$. A mapping $f : X \rightarrow Y$ of BH-algebras is called a homomorphism if $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in X$. For a homomorphism $f : X \rightarrow Y$ of BH-algebras, the kernel of $f$, denoted by $\text{Ker}(f)$, defined to be the set

$$\text{Ker}(f) = \{x \in X | f(x) = 0\}.$$

**Definition 2.2.** ([8]) A BH-algebra $X$ is called a BH*-algebra if it satisfies the identity $(x \ast y) \ast x = 0$ for all $x, y \in X$.

**Example 2.3.** ([4]) Let $X := \{0, 1, 2, 3\}$ be a BH-algebra which is not a BCK-algebra with the following Cayley table:

\[
\begin{array}{c|cccc}
* & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 3 \\
3 & 3 & 3 & 3 & 0 \\
\end{array}
\]

Then $A := \{0, 1\}$ is a translation ideal of $X$.

**Lemma 2.4.** Let $X$ be a BH*-algebra. Then the following identity holds:

$$0 \ast x = 0, \forall x \in X.$$ 

**Definition 2.5.** A BH-algebra $(X; \ast, 0)$ is said to be transitive if $x \ast y = 0$ and $y \ast z = 0$ imply $x \ast z = 0$. 
Definition 2.6. ([1]) A non-empty subset $A$ of a $BH$-algebra $X$ is called a strong ideal of $X$ if it satisfies (I1) and
(I4) $(x * y) * z, y \in A$ imply $x * z \in A$.

Lemma 2.7. ([1]) In a $BH$-algebra, any strong ideal is an ideal.

Lemma 2.8. ([1]) In a $BH^*$-algebra $X$, any ideal is a subalgebra.

Corollary 2.9. ([1]) Any strong ideal of $BH^*$-algebra is a subalgebra.

Definition 2.10. Let $X$ be a $BH$-algebra. $X$ is said to be positive implicative if it satisfies the following identity:

$$(x * y) * z = (x * z) * (y * z), \forall x, y, z \in X.$$ 

Lemma 2.11. Let $X$ be a positive implicative $BH^*$-algebra. If $x \leq y$, then $x * z \leq y * z$ for any $x, y, z \in X$.

Proposition 2.12. If $X$ is a positive implicative $BH^*$-algebra, then $X$ is a transitive $BH^*$-algebra.

Definition 2.13. ([1]) A non-zero element $a \in X$ is called an atom of a $BH$-algebra $X$ if $x \leq a$ implies $x = 0$ or $x = a$. Let $n$ be a positive integer. A non-zero element $a$ of a $BH$-algebra $X$ is called an $n$-atom of $X$ if $x * a^n = 0$ implies $x = 0$ or $x = a$.

3. $n$-fold strong ideals

For any elements $x$ and $y$ of a $BH$-algebra $X$, $x * y^n$ denotes $(\cdots ((x * y) * y) * \cdots) * y$ in which $y$ occurs $n$ times.

Definition 3.1. A non-empty subset $A$ of a $BH$-algebra $X$ is called an $n$-fold strong ideal of $X$ if it satisfies (I1) and
(I5) for every $x, y, z \in X$ there exists a natural number $n$ such that $x * z^n \in A$ whenever $(x * y) * z^n$ and $y \in A$.

For a $BH$-algebra $X$, obviously $\{0\}$ and $X$ itself are $n$-fold strong ideals of $X$ for every positively integer $n$.

Example 3.2. Let $X := \{0, 1, 2, 3\}$ be a set with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<tr>
<td>0</td>
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<td>1</td>
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<td>2</td>
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<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
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</table>
Then $(X; *, 0)$ is a BH-algebra. It is easy to check that $A := \{0, 1, 2\}$ is an $n$-fold strong ideal of $X$ for every positive integer $n$.

The 1-fold strong ideal is precisely a strong ideal. Taking $z := 0$ in (I5) and using (II), then $x * y = (x * y) * 0^n \in A$ and $y \in A$ which imply that $x = x * 0^n \in A$. Hence we have the following theorem.

**Theorem 3.3.** In a BH-algebra, every $n$-fold strong ideal is an ideal.

Combining Lemma 2.8 and Theorem 3.3, we have the following corollary.

**Corollary 3.4.** In a BH*-algebra, every $n$-fold strong ideal is a subalgebra.

The converse of Corollary 3.4 may not be true as seen in the following example.

**Example 3.5.** Let $X = \{0, 1, 2, 3\}$ be a BH-algebra as in Example 3.2. Then $(X; *, 0)$ is a BH*-algebra. The subset $C = \{0, 2\}$ of $X$ is a subalgebra, but not an $n$-fold strong ideal for every positive integer $n$, since $(3 * 2) * 0^n = 2 * 0^n = 2 \in C$ and $3 * 0^n = 3 \notin C$.

Now we give a condition for a BH-algebra to be an $n$-fold strong ideal.

**Theorem 3.6.** Let $A$ be a subalgebra a BH*-algebra $X$. Then $A$ is an $n$-fold strong ideal of $X$ if and only if $(y * x) * z^n \notin A$ whenever $y * z^n \notin A$ and $x \in A$.

**Proof.** Let $A$ be an $n$-fold strong ideal of $X$ and let $x, y, z \in X$ be such that $y * z^n \notin A$ and $x \in A$. If $(y * x) * z^n \in A$, then $y * z^n \in A$ by (I5). This is a contradiction. Hence $(y * x) * z^n \notin A$.

Conversely, let $A$ be a subalgebra $X$ in which $y * z^n \notin A$ and $x \in A$ imply $(y * x) * z^n \notin A$. Obviously, $0 \in A$. Let $x, y, z \in X$ be such that $(x * y) * z^n \in A$ and $y \in A$. If $x * z^n \notin A$, then $(x * y) * z^n \notin A$ by assumption. This is impossible. Hence $A$ is an $n$-fold strong ideal of $X$. \qed

**Theorem 3.7.** Let $f : X \to Y$ be a homomorphism of a BH-algebra $X$. If $B$ is an $n$-fold strong ideal of $Y$, then $f^{-1}(B)$ is an $n$-fold strong ideal of $X$ for every positive integer $n$.

**Proof.** Since $f(0) = 0 \in B$, we have $0 \in f^{-1}(B)$. Let $x, y, z \in X$ be such that $(x * y) * z^n \in f^{-1}(B)$ and $y \in f^{-1}(B)$. Then $(f(x) * f(y)) * f(z)^n = f((x * y) * z^n) \in B$ and $f(y) \in B$. Since $B$ is an $n$-fold strong
ideal of $Y$, it follows from (I5) that $f(x * z^n) = f(x) * f(z)^n \in B$ so that $x * z^n \in f^{-1}(B)$. Hence $f^{-1}(B)$ is an $n$-fold strong ideal of $X$.

Corollary 3.8. Let $f : X \to Y$ be a homomorphism of BH-algebras. Then $\text{Ker} f := \{ x \in X | f(x) = 0 \}$ is an $n$-fold strong ideal of $X$ for every positive integer $n$.

Theorem 3.9. Let $f : X \to Y$ be an isomorphism of BH-algebras. If a non-zero element $a$ is an $n$-atom of $X$, then $f(a)$ is an $n$-atom of $Y$ where $n$ is a positive integer.

Proof. Let $y$ be a non-zero element of $Y$ such that $y * f(a)^n = 0$. Then there exists a non-zero element $x \in X$ such that $f(x) = y$. Thus
\[
 f(0) = 0 = y * f(a)^n = f(x) * f(a)^n = f(x * a^n).
\]
Since $f$ is 1-1, it follows that $x * a^n = 0$ so that $x = a$ because $a$ is an $n$-atom if $X$. Hence $y = f(x) = f(a)$, and $f(a)$ is an $n$-atom of $Y$.

Let $A(\leq)$ be a partially ordered set with the least element 0. If we define a binary operation $*$ on $A$ as follows:
\[
x * y := \begin{cases} 
0 & \text{if } x \leq y \\
x & \text{otherwise}
\end{cases}
\]
then the algebraic structure $(A; *, 0)$ is a BH-algebra. Hence any partially ordered set is regarded as a BH-algebra. We say that a BH-algebra with such defined a multiplication has the trivial structure.

For any fixed element $a \leq b$ of a BH-algebra $X$, the set
\[
[a, b] = \{ x \in X | a \leq x \leq b \} = \{ x \in X | a * x = x * b = 0 \}
\]
is called the segment of $X$. Note that the segment
\[
[0, b] = \{ x \in X | x \leq b \} = \{ x \in X | x * b = 0 \}
\]
is called initial, is the left annihilator of $b$. Since $[0, b]$ has two elements only in the case when $b \in X$ is an atom of $X$, a BH-algebra in which all initial segments have at most two elements has the trivial structure.

Proposition 3.10. Every initial segment of a positive implicative BH*-algebra is a subalgebra of $X$.

Proof. Obviously, $0 \in [0, c]$. If $x, y \in [0, c]$, then $x \leq c$ and $y \leq c$. By Definition 2.2 and Lemma 2.11, we have $x * y \leq c * y \leq c$. Thus $x * y \in [0, c]$, which proves that $[0, c]$ is a subalgebra of $X$. 

\[
\square
\]
Proposition 3.11. The set-theoretic union of any two initial segments of a given positive implicative BH*-algebra is a subalgebra.

Proof. Straightforward.

In general, initial segments of a BH*-algebra $X$ may not be an $n$-fold strong ideal of $X$, where $n$ is a positive integer, as seen in the following example.

Example 3.12. Let $X := \{0, a, b, c\}$ be a set with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tbody>
<tr>
<td>0</td>
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<td>0</td>
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<tr>
<td>a</td>
<td>a</td>
<td>0</td>
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<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

It is easy to check that $(X; *, 0)$ is a BH*-algebra. Then $[0, b] = \{0, a, b\}$ is not an ideal of $X$, since $c * b = a \in [0, b]$ but $c \notin [0, b]$. Also, it is not both a strong ideal and an $n$-fold strong ideal of $X$, because $(c * b) * 0^n = a * 0^n = a \in [0, b]$, but $c * 0^n = c \notin [0, b]$.

Proposition 3.13. Let $c$ be a fixed element of a BH-algebra $X$ and let $n$ be a positive integer. If the initial segment $[0, c]$ is an $n$-fold strong ideal of $X$, then for all $x, z \in X$,

$$(x * c) * z^n \leq c \Rightarrow x * z^n \leq c.$$  

Proof. Assume that for all $x, z \in X$, $(x * c) * z^n \leq c$. Hence $(x * c) * z^n \in [0, c]$. Since $c \in [0, c]$ and $[0, c]$ is an $n$-fold strong ideal of $X$, we have $x * z^n \in [0, c]$. Hence $x * z^n \leq c$.

Corollary 3.14. Let $c$ be a fixed element of a BH-algebra $X$. If the initial segment $[0, c]$ is a strong ideal of $X$, then for all $x \in X$,

$$x * c \leq c \Rightarrow x \leq c.$$  

Corollary 3.15. If a non-trivial segment $[0, c]$ is an ideal or a strong ideal of a BH-algebra, then $x * c \neq c$ for all non-zero $x \in X$.

Proof. Let $[0, c]$, where $c \neq 0$, be an ideal. If $x * c = c$ for some $x \in X$, then $x * c \in [0, c]$. Hence $x * c \leq c$. By Corollary 3.14, we have $x \leq c$, which is a contradiction since in this case we obtain $0 = x * c = c$. 


References


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