FUZZY TRANSLATIONS AND FUZZY MULTIPLICATIONS OF HYPER BCK-ALGEBRAS

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Abstract. Fuzzy $\alpha$-translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications of fuzzy hyper $BCK$-subalgebras in hyper $BCK$-algebras are discussed. Relations among fuzzy $\alpha$-translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications are investigated.

1. Introduction

The hyper structure theory (called also multialgebras) was introduced in 1934 by Marty [8] at the 8th congress of Scandinavian Mathematicians. Around the 40’s, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia, and Japan. Over the following decades, many important results appeared, but above all since the 70’s onwards the most luxuriant flourishing of hyper structures has been seen. Hyper structures have many applications to several sectors of both pure and applied sciences. In [7], Jun et al. applied the hyper structures to $BCK$-algebras, and introduced the concept of a hyper $BCK$-algebra which is a generalization of a $BCK$-algebra. They also introduced the notion of a (weak, $s$-weak, strong) hyper $BCK$-ideal, and gave relations among them. Harizavi [2] studied prime weak hyper $BCK$-ideals of lower hyper $BCK$-semilattices. Jun et al. discussed the notion of hyperatoms and scalar elements of hyper $BCK$-algebras (see [4]). Jun et al. also discussed the fuzzy structures of (implicative) hyper $BCK$-ideals in hyper $BCK$-algebras (see [3, 5]).

Fuzzy set theory is established in the paper [9]. In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0, 1]$. The membership degree expresses the degree of belongingness
of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 indicates that an element does not belong to the fuzzy set. The membership degrees on the interval (0, 1) indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set (see [1, 10]). In the viewpoint of satisfaction degree, the membership degree 0 is assigned to elements which do not satisfy some property. The elements with membership degree 0 are usually regarded as having the same characteristics in the fuzzy set representation. By the way, among such elements, some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements.

In this paper, we discuss fuzzy $\alpha$-translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications of fuzzy hyper $BCK$-subalgebras in hyper $BCK$-algebras. We investigate relations among fuzzy $\alpha$-translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications.

2. Preliminaries

We include some elementary aspects of hyper $BCK$-algebras that are necessary for this paper, and for more details we refer to [5], [6], and [7].

Let $H$ be a nonempty set endowed with a hyperoperation “$\circ$”. For two subsets $A$ and $B$ of $H$, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

By a hyper $BCK$-algebra we mean a nonempty set $H$ endowed with a hyperoperation “$\circ$” and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$. In such case, we call “$\ll$” the hyperorder in $H$. 
Note that the condition (HK3) is equivalent to the condition:

\[(\forall x, y \in H)(x \circ y \ll x).\]

In any hyper BCK-algebra \(H\), the following hold:

\[(A \circ B) \circ C = (A \circ C) \circ B, \ A \circ B \ll A, \ 0 \circ A \ll \{0\}.\]

\[0 \circ 0 = \{0\} \text{.}\]

\[A \ll A.\]

\[A \subseteq B \Rightarrow A \ll B.\]

\[0 \circ A \ll \{0\}.\]

\[A \ll \{0\} \Rightarrow A = \{0\}.\]

\[x \in x \circ 0.\]

\[x \circ 0 \ll \{y\} \Rightarrow x \ll y.\]

\[y \ll z \Rightarrow x \circ z \ll x \circ y.\]

\[x \circ y = \{0\} \Rightarrow (x \circ z) \circ (y \circ z) = \{0\}, \ x \circ z \ll y \circ z.\]

for all \(x, y, z \in H\) and for all nonempty subsets \(A, B\) and \(C\) of \(H\).

We now review some fuzzy logic concepts. A fuzzy set in a set \(H\) is a function \(\mu: X \rightarrow [0, 1]\). For any \(t \in [0, 1]\) and a fuzzy set \(\mu\) in a nonempty set \(H\), the set

\[U(\mu; t) = \{x \in H \mid \mu(x) \geq t\}\] (resp. \(L(\mu; t) = \{x \in H \mid \mu(x) \leq t\}\))

is called an upper (resp. lower) level set of \(\mu\).

A fuzzy set \(\mu\) in a hyper BCK-algebra \(H\) is called a fuzzy hyper BCK-ideal of \(H\) if it satisfies

\[(\forall x, y \in H)(x \ll y \Rightarrow \mu(y) \leq \mu(x)).\]

\[(\forall x, y \in H)(\mu(x) \geq \min \{x \circ a, y \circ b \mid (x \circ a, y \circ b) \in \mu\}).\]

3. Fuzzy translations and fuzzy multiplications of fuzzy hyper BCK-subalgebras

In what follows let \(H = (H, \circ, 0)\) denote a hyper BCK-algebra, and for any fuzzy set \(\mu\) of \(H\), we denote \(\top := 1 - \sup \{\mu(x) \mid x \in H\}\) unless otherwise specified.

**Definition 3.1.** [7] Let \(S\) be a subset of \(H\). If \(S\) is a hyper BCK-algebra with respect to the hyper operation “\(\circ\)” on \(H\), we say that \(S\) is a hyper subalgebra of \(H\).
Theorem 3.2. [7] Let $S$ be a non-empty subset of a $H$. Then $S$ is a hyper subalgebra of $H$ if and only if $x \circ y \subseteq S$ for all $x, y \in S$.

Definition 3.3. Let $\mu$ be a fuzzy subset of $H$ and let $\alpha \in [0, \top]$. A mapping $\mu^T_\alpha : H \rightarrow [0, 1]$ is called a fuzzy $\alpha$-translation of $\mu$ if it satisfies:

$$(\forall x \in H)(\mu^T_\alpha(x) = \mu(x) + \alpha).$$

Definition 3.4. A fuzzy set $\mu$ in $H$ is called a fuzzy hyper $BCK$-subalgebra of $H$ if it satisfies:

$$(3.1) \quad (\forall x, y \in H) \left( \inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\} \right).$$

Theorem 3.5. Let $\mu$ be a fuzzy hyper $BCK$-subalgebra of $H$ and $\alpha \in [0, \top]$. Then the fuzzy $\alpha$-translation $\mu^T_\alpha$ of $\mu$ is a fuzzy hyper $BCK$-subalgebra of $H$.

Proof. Let $x, y \in H$. Then

$$\inf_{z \in x \circ y} \mu^T_\alpha(z) = \inf_{z \in x \circ y} (\mu(z) + \alpha) = \alpha + \inf_{z \in x \circ y} \mu(z) \geq \alpha + \min\{\mu(x), \mu(y)\} = \min\{\mu(x) + \alpha, \mu(y) + \alpha\} = \min\{\mu^T_\alpha(x), \mu^T_\alpha(y)\}.$$ 

Hence $\mu^T_\alpha$ is a fuzzy hyper $BCK$-subalgebra of $H$. \hfill \Box

Theorem 3.6. Let $\mu$ be a fuzzy subset of $H$ such that the fuzzy $\alpha$-translation $\mu^T_\alpha$ of $\mu$ is a fuzzy hyper $BCK$-subalgebra of $H$ for some $\alpha \in [0, \top]$. Then $\mu$ is a fuzzy hyper $BCK$-subalgebra of $H$.

Proof. Assume that $\mu^T_\alpha$ is a fuzzy hyper $BCK$-subalgebra of $H$ for some $\alpha \in [0, \top]$. Let $x, y \in H$, we have

$$\alpha + \inf_{z \in x \circ y} \mu(z) = \inf_{z \in x \circ y} (\mu(z) + \alpha) = \inf_{z \in x \circ y} \mu^T_\alpha(z) \geq \min\{\mu^T_\alpha(x), \mu^T_\alpha(y)\} = \min\{\mu(x) + \alpha, \mu(y) + \alpha\} = \min\{\mu(x), \mu(y)\} + \alpha$$

which implies that $\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in H$. Hence $\mu$ is a fuzzy hyper $BCK$-subalgebra of $H$. \hfill \Box

Definition 3.7. Let $\mu_1$ and $\mu_2$ be fuzzy subsets of $H$. If $\mu_1(x) \leq \mu_2(x)$ for all $x \in H$, then we say that $\mu_2$ is a fuzzy extension of $\mu_1$.

Definition 3.8. Let $\mu_1$ and $\mu_2$ be fuzzy subsets of $H$. Then $\mu_2$ is called a fuzzy $S$-extension of $\mu_1$ if the following assertions are valid:

(i) $\mu_2$ is a fuzzy extension of $\mu_1$. 

(ii) If $\mu_1$ is a fuzzy hyper $BCK$-subalgebra of $H$, then $\mu_2$ is a fuzzy hyper $BCK$-subalgebra of $H$.

**Example 3.9.** Consider a hyper $BCK$-algebra $H = \{0, 1, 2\}$ with the following Cayley table:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
<td>${0}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${0, 1}$</td>
<td>${0, 1}$</td>
</tr>
<tr>
<td>2</td>
<td>${2}$</td>
<td>${2}$</td>
<td>${0, 1}$</td>
</tr>
</tbody>
</table>

Define a fuzzy subsets $\mu_1$ and $\mu_2$ of $H$ by

<table>
<thead>
<tr>
<th>$H$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>0.7</td>
<td>0.5</td>
<td>0.3</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>0.7</td>
<td>0.5</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Then $\mu_1$ is a fuzzy hyper $BCK$-subalgebra of $H$ and $\mu_2$ is fuzzy extension of $\mu_1$. Since

$$\inf_{z \in 2 \circ 2} \mu_2(z) = \mu_2(1) = 0.5 \not\geq 0.6 = \min\{\mu_2(2), \mu_2(2)\},$$

$\mu_2$ is not a fuzzy hyper $BCK$-subalgebra of $H$. Hence $\mu_2$ is not a fuzzy $S$-extension of $\mu_1$.

**Example 3.10.** Consider a hyper $BCK$-algebra $H = [0, \infty)$, define a hyper operation “$\circ$” in $H$ by

$$x \circ y = \begin{cases} [0, x] & \text{if } x \leq y, \\ (0, y] & \text{if } x > y \neq 0, \\ \{x\} & \text{if } y = 0, \end{cases}$$

for all $x, y \in H$. Define a fuzzy subset $\mu_i$ of $H$ by

$$\mu_i(x) = 1/(i + x),$$

for all $i \in \mathbb{N}$, for all $x \in H$. Then $\mu_{i_1}$ and $\mu_{i_2}$ are fuzzy hyper $BCK$-subalgebras of $H$ for all $i_1, i_2 \in \mathbb{N}$. If $i_1 < i_2$, then

$$\mu_{i_1}(x) = 1/(i_1 + x) > 1/(i_2 + x) = \mu_{i_2}$$

for all $x \in H$. Hence $\mu_{i_1}$ is a fuzzy $S$-extension of $\mu_{i_2}$.

By means of the definition of fuzzy $\alpha$-translation, we know that $\mu^\alpha_{\alpha}(x) \geq \mu(x)$ for all $x \in H$. Hence we have the following theorem.

**Theorem 3.11.** Let $\mu$ be a fuzzy hyper $BCK$-subalgebra of $H$ and $\alpha \in [0, \top]$. Then the fuzzy $\alpha$-translation $\mu^\alpha_{\alpha}$ of $\mu$ is a fuzzy $S$-extension of $\mu$. 
Proof. \( \mu^T_\alpha(x) = \mu(x) + \alpha \geq \mu(x) \) for all \( x \in H \). Thus \( \mu^T_\alpha \) is a fuzzy extension of \( \mu \). Assume that \( \mu \) is a fuzzy hyper \( BCK \)-subalgebra of \( H \) and \( \alpha \in [0, \top] \). By Theorem 3.5, \( \mu^T_\alpha \) is a fuzzy hyper \( BCK \)-subalgebra of \( H \). Hence \( \mu^T_\alpha \) is a fuzzy \( S \)-extension of \( \mu \). \( \square \)

The converse of Theorem 3.11 is not true in general as seen in the following example.

**Example 3.12.** Consider \( \mu_{i_1} \) and \( \mu_{i_2} \) in Example 3.10. \( \mu_{i_1} \) is a fuzzy \( S \)-extension of \( \mu_{i_2} \). But \( \mu_{i_1} \) is not a fuzzy \( \alpha \)-translation \( (\mu_{i_2})^T_\alpha \) of \( \mu_{i_2} \) for all \( \alpha \in [0, \top] \), since \( \mu_{i_1}(5) - \mu_{i_2}(5) \neq \mu_{i_1}(6) - \mu_{i_2}(6) \).

**Theorem 3.13.** Let \( \mu \) be a fuzzy hyper \( BCK \)-subalgebra of \( H \). If \( \mu_1 \) and \( \mu_2 \) are fuzzy \( S \)-extensions of \( \mu \), then \( \nu := \mu_1 \cap \mu_2 \) is a fuzzy \( S \)-extension of \( \mu \).

**Proof.** Assume that \( \mu_1 \) and \( \mu_2 \) are fuzzy \( S \)-extensions of \( \mu \). Then

\[
\nu(x) = (\mu_1 \cap \mu_2)(x) = \min\{\mu_1(x), \mu_2(x)\} \geq \mu(x)
\]

for all \( x \in H \). Hence \( \nu \) is a fuzzy extension of \( \mu \). Let \( \mu \) be a fuzzy hyper \( BCK \)-subalgebra of \( H \). Then \( \mu_1 \) and \( \mu_2 \) are fuzzy hyper \( BCK \)-subalgebras of \( H \). Then

\[
\inf_{z \in x \cup y} \nu(z) = \inf_{z \in x \cup y} (\mu_1 \cap \mu_2)(z) = \inf_{z \in x \cup y} (\min\{\mu_1(z), \mu_2(z)\})
\]

\[
= \min\{\inf_{z \in x \cup y} \mu_1(z), \inf_{z \in x \cup y} \mu_2(z)\}
\]

\[
\geq \min\{\min\{\mu_1(x), \mu_1(y)\}, \min\{\mu_2(x), \mu_2(y)\}\}
\]

\[
= \min\{\min\{\mu_1(x), \mu_2(x)\}, \min\{\mu_1(y), \mu_2(y)\}\}
\]

\[
= \min\{\mu_1(x) \cap \mu_2(x), (\mu_1 \cap \mu_2)(y)\} = \min\{\nu(x), \nu(y)\}
\]

for all \( x, y \in H \). Thus \( \inf_{z \in x \cup y} \nu(z) \geq \min\{\nu(x), \nu(y)\} \) for all \( x, y \in H \). Hence \( \nu \) is a fuzzy hyper \( BCK \)-subalgebra of \( H \). Therefore \( \nu \) is a fuzzy \( S \)-extension of \( \nu \). \( \square \)

**Example 3.14.** Consider a hyper \( BCK \)-algebra \( H = \{0, 1, 2, 3\} \) with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0,1}</td>
<td>{0,1}</td>
<td>{0,1}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{2}</td>
<td>{0,2}</td>
<td>{0,1,2}</td>
</tr>
<tr>
<td>3</td>
<td>{3}</td>
<td>{3}</td>
<td>{3}</td>
<td>{0,3}</td>
</tr>
</tbody>
</table>
Define a fuzzy subsets $\mu$, $\mu_1$ and $\mu_2$ of $H$ by

<table>
<thead>
<tr>
<th>$H$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.9</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>0.95</td>
<td>0.3</td>
<td>0.6</td>
<td></td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>0.95</td>
<td>0.3</td>
<td>0.7</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Then $\mu$ is a fuzzy hyper BCK-subalgebra of $H$, $\mu_1$ and $\mu_2$ are fuzzy $S$-extensions of $\mu$. But

$$\nu := \mu_1 \cup \mu_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.95 & 0.3 & 0.7 & 0.6 \end{pmatrix}$$

is not a fuzzy $S$-extension of $\mu$, since $\inf_{z \in 2 \circ y} \nu(1) = 0.3 \nleq 0.6 = \min\{\nu(2), \nu(3)\}$.

For a fuzzy subset $\mu$ of $H$, $\alpha \in [0, \top]$ and $t \in [0, 1]$ with $t \geq \alpha$, let

$$U_\alpha(\mu; t) := \{x \in H | \mu(x) \geq t - \alpha\}.$$  

**Theorem 3.15.** If $\mu$ is a fuzzy hyper BCK-subalgebra of $H$, then $U_\alpha(\mu; t)$ is a hyper subalgebra of $H$ for all $t \in \text{Im}(\mu)$ with $t \geq \alpha$, $\alpha \in [0, \top]$.

**Proof.** Let $x, y \in U_\alpha(\mu; t)$. Then $\mu(x) \geq t - \alpha$ and $\mu(y) \geq t - \alpha$. Then $\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\} \geq t - \alpha$. Then $\mu(z) \geq t - \alpha$ for all $z \in x \circ y$. i.e., $z \in U_\alpha(\mu; t)$ for all $z \in x \circ y$. Then $x \circ y \subseteq U_\alpha(\mu; t)$. Hence $U_\alpha(\mu; t)$ is a hyper subalgebra of $H$. \hfill $\square$

In Theorem 3.15, if we do not give a condition that $\mu$ is a fuzzy hyper BCK-subalgebra of $H$, then $U_\alpha(\mu; t)$ is not a hyper subalgebra of $H$ as seen in the following example.

**Example 3.16.** Consider a hyper $BCK$-algebra $H = \{0, 1, 2\}$ with the following Cayley table:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
<td>${0}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${0, 1}$</td>
<td>${0, 1}$</td>
</tr>
<tr>
<td>2</td>
<td>${2}$</td>
<td>${1, 2}$</td>
<td>${0, 1, 2}$</td>
</tr>
</tbody>
</table>

Define a fuzzy subset $\mu$ of $H$ by

$$\mu : H \rightarrow [0, 1], \quad x \mapsto \begin{cases} 0.7 & \text{if } x = 0, \\ 0.3 & \text{if } x = 1, \\ 0.5 & \text{if } x = 2. \end{cases}$$
Then $\mu$ is not a fuzzy hyper $BCK$-subalgebra of $H$ since

$$\inf_{z \in \mathbb{R}^2} \mu(z) = \mu(1) = 0.3 \not\geq 0.5 = \mu(2) = \min\{\mu(2), \mu(2)\}.$$  

For $\alpha = 0.2$ and $t = 0.7$, we obtain $U_\alpha(\mu; t) = \{0, 2\}$ which is not a hyper subalgebra of $H$ since $2 \odot 2 = \{0, 1, 2\} \not\subseteq \{0, 2\} = U_\alpha(\mu; t)$.

**Theorem 3.17.** If $U_\alpha(\mu; t)$ is a hyper subalgebra of $H$ for all $t \in \text{Im}(\mu)$ with $t \geq \alpha$, $\alpha \in [0, \top]$, then $\mu$ is a fuzzy hyper $BCK$-subalgebra of $H$.

**Proof.** Assume that $U_\alpha(\mu; t)$ is a hyper subalgebra of $H$, for all $t \in \text{Im}(\mu)$ with $t \geq \alpha$, $\alpha \in [0, \top]$. Then $U_0(\mu; t)$ is a hyper subalgebra of $H$, for all $t \in \text{Im}(\mu)$ with $t \geq 0$. Assume that there exists $a, b \in H$ such that

$$\inf_{c \in a \circ b} \mu(c) < \min\{\mu(a), \mu(b)\}.$$  

Let $\beta := \min\{\mu(a), \mu(b)\}$. Then $\mu(a) \geq \beta$ and $\mu(b) \geq \beta$. Then $\mu(a) \geq \beta - 0$ and $\mu(b) \geq \beta - 0$, so $a, b \in U_0(\mu; \beta)$. $\mu(w) < \beta$ for some $w \in a \circ b$, since $\inf_{c \in a \circ b} \mu(c) < \beta$. Then $\mu(w) < \beta - 0$ for some $w \in a \circ b$. i.e., $w \notin U_0(\mu; \beta)$ for some $w \in a \circ b$. Then $a \circ b \not\subseteq U_0(\mu; \beta)$, which is contradiction. So

$$\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\}$$  

for all $x, y \in H$. Hence $\mu$ is a fuzzy hyper $BCK$-subalgebra of $H$. \hfill $\square$

**Theorem 3.18.** Let $\mu$ be a fuzzy subset of $H$ and $\alpha \in [0, \top]$. Then the fuzzy $\alpha$-translation $\mu_\alpha^T$ of $\mu$ is a fuzzy hyper $BCK$-subalgebra of $H$ if and only if $U_\alpha(\mu; t)$ is a hyper subalgebra of $H$ for all $t \in \text{Im}(\mu)$ with $t \geq \alpha$.

**Proof.** Let $x, y \in U_\alpha(\mu; t)$. Then $\mu(x) \geq t - \alpha$ and $\mu(y) \geq t - \alpha$. Then

$$\alpha + \inf_{z \in x \circ y} \mu(z) = \inf_{z \in x \circ y} (\mu(z) + \alpha) = \inf_{z \in x \circ y} \mu_\alpha^T(z) \geq \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\} = \min\{\mu(x) + \alpha, \mu(y) + \alpha\} = \alpha + \min\{\mu(x), \mu(y)\} \geq \alpha + t - \alpha = t.$$  

Thus $\inf_{z \in x \circ y} \mu(z) \geq t - \alpha$. Then $\mu(z) \geq t - \alpha$ for all $z \in x \circ y$, i.e., $z \in U_\alpha(\mu; t)$ for all $z \in x \circ y$. Then $x \circ y \subseteq U_\alpha(\mu; t)$. Hence $U_\alpha(\mu; t)$ is a hyper subalgebra of $H$.

Conversely assume that there exists $a, b \in H$ such that

$$\inf_{c \in a \circ b} \mu_\alpha^T(c) < \beta \leq \min\{\mu_\alpha^T(a), \mu_\alpha^T(b)\}.$$
Then $\mu_{T}^{\alpha}(a) \geq \beta$ and $\mu_{T}^{\alpha}(b) \geq \beta$. Then $\mu(a) \geq \beta - \alpha$ and $\mu(b) \geq \beta - \alpha$, so $a, b \in U_{\alpha}(\mu; \beta)$, $\mu_{T}^{\alpha}(w) < \beta$ for some $w \in a \circ b$, since $\inf_{c \in a \circ b} \mu_{T}^{\alpha}(c) < \beta$. Then $\mu(w) < \beta - \alpha$ for some $w \in a \circ b$, i.e., $w \notin U_{\alpha}(\mu; \beta)$ for some $w \in a \circ b$. Then $a \circ b \notin U_{\alpha}(\mu; \beta)$, which is contradiction. So

$$\inf_{z \in x \circ y} \mu_{T}^{\alpha}(z) \geq \min\{\mu_{T}^{\alpha}(x), \mu_{T}^{\alpha}(y)\}$$

for all $x, y \in H$. Hence $\mu_{T}^{\alpha}$ is a fuzzy hyper $BCK$-subalgebra of $H$. □

**Theorem 3.19.** Let $\mu$ be a fuzzy hyper $BCK$-subalgebra of $H$ and let $\alpha, \beta \in [0, \top]$. If $\alpha \geq \beta$, then the fuzzy $\alpha$-translation $\mu_{T}^{\alpha}$ of $\mu$ is a fuzzy $S$-extension of the fuzzy $\beta$-translation $\mu_{T}^{\beta}$ of $\mu$.

**Proof.** Straightforward. □

For every fuzzy hyper $BCK$-subalgebra $\mu$ of $H$ and $\beta \in [0, \top]$, the fuzzy $\beta$-translation $\mu_{T}^{\beta}$ of $\mu$ is a fuzzy hyper $BCK$-subalgebra of $H$. If $\nu$ is a fuzzy $S$-extension of $\mu_{T}^{\beta}$, then there exists $\alpha \in [0, \top]$ such that $\alpha \geq \beta$ and $\nu(x) \geq \mu_{T}^{\alpha}(x)$ for all $x \in H$. Thus we have the following theorem.

**Theorem 3.20.** Let $\mu$ be a fuzzy hyper $BCK$-subalgebra of $H$ and $\beta \in [0, \top]$. For every fuzzy $S$-extension $\nu$ of the fuzzy $\beta$-translation $\mu_{T}^{\beta}$ of $\mu$, there exists $\alpha \in [0, \top]$ such that $\alpha \geq \beta$ and $\nu$ is a fuzzy $S$-extension of the fuzzy $\alpha$-translation $\mu_{T}^{\alpha}$ of $\mu$.

The following example illustrates Theorem 3.20.

**Example 3.21.** Consider a hyper $BCK$-algebra $H = \{0, 1, 2\}$ with the following Cayley table:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0, 1}</td>
<td>{0, 1}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{2}</td>
<td>{0, 1}</td>
</tr>
</tbody>
</table>

Define a fuzzy subset $\mu$ of $H$ by

<table>
<thead>
<tr>
<th>$H$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.7</td>
<td>0.5</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Then $\mu$ is a fuzzy hyper $BCK$-subalgebra of $H$, and $\top = 0.3$. If we take $\beta = 0.1$, then the fuzzy $\beta$-translation $\mu_{T}^{\beta}$ of $\mu$ is given by

<table>
<thead>
<tr>
<th>$H$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_{T}^{\beta}$</td>
<td>0.8</td>
<td>0.6</td>
<td>0.4</td>
</tr>
</tbody>
</table>
Let $\nu$ be a fuzzy subset of $H$ defined by

\[
\begin{array}{c|ccc}
H & 0 & 1 & 2 \\
\hline
\nu & 0.93 & 0.77 & 0.56
\end{array}
\]

Then $\nu$ is clearly a fuzzy hyper $BCK$-subalgebra of $H$ which is fuzzy extension of $\mu^T_\beta$, and hence $\nu$ is a fuzzy $S$-extension of the fuzzy $\beta$-translation $\mu^T_\beta$ of $\mu$. But $\nu$ is not a fuzzy $\alpha$-translation of $\mu$ for all $\alpha \in [0, 1]$. Take $\alpha = 0.2$. Then $\alpha = 0.2 > 0.1 = \beta$, and the fuzzy $\alpha$-translation $\mu^T_\alpha$ of $\mu$ is given as follows:

\[
\begin{array}{c|ccc}
H & 0 & 1 & 2 \\
\hline
\mu^T_\alpha & 0.9 & 0.7 & 0.5
\end{array}
\]

Note that $\nu(x) \geq \mu^T_\alpha(x)$ for all $x \in H$, and hence $\nu$ is a fuzzy $S$-extension of the fuzzy $\alpha$-translation $\mu^T_\alpha$ of $\mu$.

A fuzzy $S$-extension $\nu$ of a fuzzy hyper $BCK$-subalgebra $\mu$ of $H$ is said to be normalized if there exists $x_0 \in H$ such that $\nu(x_0) = 1$. Let $\mu$ be a fuzzy hyper $BCK$-subalgebra of $H$. A fuzzy subset $\nu$ of $H$ is called a maximal fuzzy $S$-extension of $\mu$ if it satisfies:

(i) $\nu$ is a fuzzy $S$-extension of $\mu$,
(ii) there does not exist any fuzzy hyper $BCK$-subalgebra of $H$ which is a fuzzy extension of $\nu$.

**Example 3.22.** Consider a hyper $BCK$-algebra $H = [0, \infty)$, define a hyper operation “$\circ$” in $H$ by

\[
x \circ y = \begin{cases} 
[0, x] & \text{if } x \leq y, \\
(0, y) & \text{if } x > y \neq 0, \\
\{x\} & \text{if } y = 0,
\end{cases}
\]

for all $x, y \in H$. Then $H$ is a hyper $BCK$-algebra. Let $\mu$ and $\nu$ be fuzzy subsets of $H$ which are defined by $\mu(x) = \frac{2}{x}$ and $\nu(x) = 1$ for all $x \in H$. Clearly $\mu$ and $\nu$ are fuzzy hyper $BCK$-subalgebras of $H$. It is easy to verify that $\nu$ is a maximal fuzzy $S$-extension of $\mu$.

**Proposition 3.23.** If a fuzzy subset $\nu$ of $H$ is a normalized fuzzy $S$-extension of a fuzzy hyper $BCK$-subalgebra $\mu$ of $H$, then $\nu(0) = 1$.

**Proof.** Assume that $\mu$ is a fuzzy hyper $BCK$-subalgebra of $H$. Since $\nu$ is a fuzzy $S$-extension of $\mu$, $\nu$ is a fuzzy hyper $BCK$-subalgebra of $H$. 
Then
\[ \nu(0) \geq \inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\} = \mu(x) \]
for all \( x \in H \). Since \( \nu \) is a normalized fuzzy \( S \)-extension of \( \mu \), \( \nu(0) = 1 \).

**Theorem 3.24.** Let \( \mu \) be a fuzzy hyper \( BCK \)-subalgebra of \( H \). Then every maximal fuzzy \( S \)-extension of \( \mu \) is normalized.

**Proof.** This follows from the definitions of the maximal and normalized fuzzy \( S \)-extensions.

**Definition 3.25.** Let \( \mu \) be a fuzzy subset of \( H \) and \( \gamma \in [0, 1] \). A fuzzy \( \gamma \)-multiplication of \( \mu \), denoted by \( \mu_m^\gamma \), is defined to be a mapping
\[ \mu_m^\gamma : H \to [0, 1], x \mapsto \mu(x) \cdot \gamma. \]

For any fuzzy subset \( \mu \) of \( H \), a fuzzy 0-multiplication \( \mu_m^0 \) of \( \mu \) is clearly a fuzzy hyper \( BCK \)-subalgebra of \( H \).

**Theorem 3.26.** If \( \mu \) is a fuzzy hyper \( BCK \)-subalgebra of \( H \), then the fuzzy \( \gamma \)-multiplication of \( \mu \) is a fuzzy hyper \( BCK \)-subalgebra of \( H \) for all \( \gamma \in [0, 1] \).

**Proof.**
\[ \inf_{z \in x \circ y} \mu_m^\gamma(z) = \inf_{z \in x \circ y} (\mu(z) \cdot \gamma) = \gamma \cdot \inf_{z \in x \circ y} \mu(z) \]
\[ \geq \gamma \cdot \min\{\mu(x), \mu(y)\} = \min\{\mu(x) \cdot \gamma, \mu(y) \cdot \gamma\} \]
\[ = \min\{\mu_m^\gamma(x), \mu_m^\gamma(y)\} \]
for all \( \gamma \in [0, 1] \) for all \( x, y \in H \). Hence \( \mu_m^\gamma \) is a fuzzy hyper \( BCK \)-subalgebra of \( H \).

**Theorem 3.27.** For any fuzzy subset \( \mu \) of \( H \), the following are equivalent:

(i) \( \mu \) is a fuzzy hyper \( BCK \)-subalgebra of \( H \).

(ii) \( (\forall \gamma \in (0, 1]) (\mu_m^\gamma \) is a fuzzy hyper \( BCK \)-subalgebra of \( H \)).

**Proof.** Necessity follows from Theorem 3.26. Let \( \gamma \in (0, 1] \) be such that \( \mu_m^\gamma \) is a fuzzy hyper \( BCK \)-subalgebra of \( H \). Then
\[ \gamma \cdot \inf_{z \in x \circ y} \mu(z) = \inf_{z \in x \circ y} (\mu(z) \cdot \gamma) = \inf_{z \in x \circ y} \mu_m^\gamma(z) \]
\[ \geq \min\{\mu_m^\gamma(x), \mu_m^\gamma(y)\} = \min\{\mu(x) \cdot \gamma, \mu(y) \cdot \gamma\} \]
\[ = \gamma \cdot \min\{\mu(x), \mu(y)\} \]
for all \( x, y \in H \), and so \( \inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\} \) for all \( x, y \in H \) since \( \gamma \in (0, 1] \). Hence \( \mu \) is a fuzzy hyper \( BCK \)-subalgebra of \( H \).
Theorem 3.28. Let \( \mu \) be a fuzzy subset of \( H \), \( \alpha \in [0, \top] \) and \( \gamma \in (0, 1] \). Then every fuzzy \( \alpha \)-translation \( \mu^T_\alpha \) of \( \mu \) is a fuzzy \( S \)-extension of the fuzzy \( \gamma \)-multiplication \( \mu^m_\gamma \) of \( \mu \).

Proof. For every \( x \in H \), we have
\[
\mu^T_\alpha(x) = \mu(x) + \alpha \geq \mu(x) \geq \mu(x) \cdot \gamma = \mu^m_\gamma(x),
\]
and so \( \mu^T_\alpha \) is a fuzzy extension of \( \mu^m_\gamma \). Assume that \( \mu^m_\gamma \) is a fuzzy hyper \( \text{BCK} \)-subalgebra of \( H \). Then \( \mu \) is a fuzzy hyper \( \text{BCK} \)-subalgebra of \( H \) by Theorem 3.27. It follows from Theorem 3.5 that \( \mu^T_\alpha \) is a fuzzy hyper \( \text{BCK} \)-subalgebra of \( H \) for all \( \alpha \in [0, \top] \). Hence every fuzzy \( \alpha \)-translation \( \mu^T_\alpha \) is a fuzzy \( S \)-extension of the fuzzy \( \gamma \)-multiplication \( \mu^m_\gamma \).

The following example shows that Theorem 3.28 is not valid for \( \gamma = 0 \).

Example 3.29. Consider a hyper \( \text{BCK} \)-algebra \( H := \mathbb{N} \cup \{0\} \). Define a fuzzy set \( \mu : H \to [0, 1] \) by
\[
\mu(x) := \begin{cases} 
0 & \text{if } x = 0, \\
\frac{1}{2} \cdot (1 - \frac{1}{2}) & \text{if } x \neq 0.
\end{cases}
\]
Taking \( \gamma = 0 \), we see that
\[
\inf_{z \in x \cdot y} \mu^m_0(z) = 0 = \min\{\mu^m_0(x), \mu^m_0(y)\}
\]
for all \( x, y \in H \), that is, \( \mu^m_0 \) is a fuzzy hyper \( \text{BCK} \)-subalgebra of \( H \). But if we take \( x = 2 \) and \( y = 4 \), then
\[
\inf_{z \in 2 \cdot 4} \mu^T_\alpha(z) = \mu^T_\alpha(0) = \mu(0) + \alpha = \alpha < \frac{1}{4} + \alpha = \min\{\mu(2) + \alpha, \mu(4) + \alpha\}
\]
for all \( \alpha \in [0, \frac{1}{2}] \). Hence \( \mu^T_\alpha \) is not a fuzzy \( S \)-extension of \( \mu^m_0 \) for all \( \alpha \in [0, \frac{1}{2}] \).

The following example illustrates Theorem 3.28.

Example 3.30. Consider a hyper \( \text{BCK} \)-algebra \( H := \mathbb{N} \cup \{0\} \). Define a fuzzy set \( \mu : H \to [0, 1] \) by
\[
\mu(x) := \begin{cases} 
\frac{1}{2} & \text{if } x = 0, \\
\frac{1}{2x} & \text{if } x \neq 0.
\end{cases}
\]
Clearly \( \mu \) is a fuzzy hyper \( \text{BCK} \)-subalgebra of \( H \). If we take \( \gamma = \frac{1}{2} \), then the fuzzy \( \gamma \)-multiplication \( \mu^m_{\frac{1}{2}} \) of \( \mu \) is given by
\[
\mu^m_{\frac{1}{2}}(x) := \begin{cases} 
\frac{1}{4} & \text{if } x = 0, \\
\frac{1}{4x} & \text{if } x \neq 0.
\end{cases}
\]
Clearly $\mu_{1/2}^m$ is a fuzzy hyper $BCK$-subalgebra of $H$. Also, for any $\alpha \in [0, 0.5]$, the fuzzy $\alpha$-translation $\mu^T_\alpha$ of $\mu$ is given by

$$
\mu^T_\alpha(x) := \begin{cases} 
\frac{1}{2} + \alpha & \text{if } x = 0, \\
\frac{1}{2x} + \alpha & \text{if } x \neq 0.
\end{cases}
$$

Then $\mu^T_\alpha$ is a fuzzy extension of $\mu_{1/2}^m$ and $\mu^T_\alpha$ is always a fuzzy hyper $BCK$-subalgebra of $H$ for all $\alpha \in [0, 0.5]$. Hence $\mu^T_\alpha$ is a fuzzy $S$-extension of $\mu_{1/2}^m$ for all $\alpha \in [0, 0.5]$.

References


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