INTERVAL-VALUED FUZZY GENERALIZED BI-IDEALS OF A SEMIGROUP

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Abstract. We introduce the concept of an interval-valued fuzzy generalized bi-ideal of a semigroup, which is an extension of the concept of an interval-valued fuzzy bi-ideal (and of a noninterval-valued fuzzy bi-ideal and a noninterval-valued fuzzy ideal of a semigroup), and characterize regular semigroups, and both intraregular and left quasiregular semigroup in terms of interval-valued fuzzy generalized bi-ideals.

1. Introduction

In 1975, Zadeh\cite{12} introduced the concept of an interval-valued fuzzy set as the generalization of a fuzzy set introduced by himself\cite{11}. After then, Biswas\cite{1} applied it to group theory, Garzalczany\cite{6} proposed a method of inference based on interval-valued fuzzy sets, Roy and Biswas\cite{10} studied interval-valued fuzzy relations, and Mondal and Samanta applied it to topology. Recently, Choi et al.\cite{4} investigated interval-valued smooth topological spaces, Hur et al.\cite{7} studied interval-valued fuzzy relations in the sense of lattice theory, and Kang and Hur\cite{7} applied the concept of an interval-valued fuzzy set to algebra. Furthermore, Cheong and Hur\cite{3} investigated interval-valued fuzzy ideals and bi-ideals in a semigroup.

In this paper, we will introduce the concept of an interval-valued fuzzy generalized bi-ideal of a semigroup, which is an extension of the notion
of a noninterval-valued fuzzy generalized bi-ideal (and of a noninterval-valued fuzzy bi-ideal and a noninterval-valued fuzzy ideal), and characterize such semigroups by interval-valued fuzzy generalized bi-ideals.

2. Preliminaries

We will list some concepts needed in the later sections.

Throughout this paper, we will denote the unit interval \([0, 1]\) as \(I\) and for an ordinary subset \(A\) of a set \(X\), we will denote the characteristic function of \(A\) as \(\chi_A\).

**Definition 2.1.** [6, 9, 12] Let \(X\) be a given nonempty set. A mapping \(A = [A^L, A^U] : X \to D(I)\) is called an interval-valued fuzzy set (briefly, IVFS) in \(X\), where \(A^L\) and \(A^U\) are fuzzy sets in \(X\) satisfying \(A^L(x) \leq A^U(x)\) and \(A(x) = [A^L(x), A^U(x)]\) for each \(x \in X\). In particular, \(0\) and \(1\) denote the interval-valued fuzzy empty set and the interval-valued fuzzy whole set in \(X\) defined by \(0(x) = [0, 0]\) and \(1(x) = [1, 1]\) for each \(x \in X\), respectively.

We will denote the set of all IVFSs in \(X\) as \(D(I)^X\).

**Notation.** Let \(X = \{x_1, x_2, \cdots, x_n\}\). Then \(A = ([a_1, b_1], [a_2, b_2], \cdots, [a_n, b_n])\) denotes an IVFS in \(X\) such that \(A^L(x_i) = a_i\) and \(A^U(x_i) = b_i\), for all \(i = 1, 2, \cdots, n\).

**Definition 2.2.** [9] Let \(X\) be a nonempty set and let \(A, B \in D(I)^X\). Then

(a) \(A \subset B\) iff \(A^L(x) \leq B^L(x)\) and \(A^U(x) \leq B^U(x)\) for all \(x \in X\).

(b) \(A = B\) iff \(A \subset B\) and \(B \subset A\).

(c) The complement \(A^c\) of \(A\) is defined by \(A^c(x) = [1 - A^U(x), 1 - A^L(x)]\) for all \(x \in X\).

(d) If \(\{A_i : i \in J\}\) is an arbitrary subset of \(D(I)^X\), then

\[
\bigcap_{i \in J} A_i(x) = \left[ \bigwedge_{i \in J} A^L_i(x), \bigwedge_{i \in J} A^U_i(x) \right],
\]

\[
\bigcup_{i \in J} A_i(x) = \left[ \bigvee_{i \in J} A^L_i(x), \bigvee_{i \in J} A^U_i(x) \right].
\]
3. Interval-valued fuzzy ideals of a semigroup

**Definition 3.1.** [7] Let \((X, \circ)\) be a groupoid and let \(A, B \in D(I)^X\). Then the interval-valued fuzzy product of \(A\) and \(B\), \(A \circ B\) is defined as follows: For each \(x \in X\),

\[
A \circ B(x) = \begin{cases} 
\bigvee_{x = yz} (A^L(y) \land B^L(z)), & \text{if } x = yz, \\
[0, 0], & \text{otherwise.}
\end{cases}
\]

It is clear that for any \(A, B, C \in D(I)^X\), if \(B \subset C\), then \(A \circ B \subset A \circ C\) and \(B \circ A \subset C \circ A\).

Let \(S\) be a semigroup. By a subsemigroup of \(S\) we mean a non-empty subset of \(A\) such that \(A^2 \subset A\) and by a left [resp. right] ideal of \(S\) we mean a non-empty subset \(A\) of \(S\) such that \(SA \subset A\) [resp. \(AS \subset A\)]. By two-sided ideal or simply ideal we mean a subset \(A\) of \(S\) which is both a left and a right ideal of \(S\). We well denote the set of all left ideals [resp. right ideals and ideals] of \(S\) as \(LI(S)\) [resp. \(RI(S)\) and \(I(S)\)].

**Definition 3.2.** Let \(S\) be a semigroup and let \(A \in D(I)^S\). Then \(A\) is called an :

1. interval-valued fuzzy subsemigroup (in short, IVSG) of \(S\) if

\[
A^L(xy) \geq A^L(x) \land A^L(y) \quad \text{and} \quad A^U(xy) \geq A^U(x) \land A^U(y)
\]

for any \(x, y \in S\),
2. interval-valued fuzzy left ideal (in short, IVLI) of \(S\) if

\[
A^L(xy) \geq A^L(y) \quad \text{and} \quad A^U(xy) \geq A^U(y)
\]

for any \(x, y \in S\),
3. interval-valued fuzzy right ideal (in short, IVRI) of \(S\) if

\[
A^L(xy) \geq A^L(x) \quad \text{and} \quad A^U(xy) \geq A^U(x)
\]

for any \(x, y \in S\),
4. interval-valued fuzzy (two-sided) ideal (in short, IVI) of \(S\) if it is both an interval-valued fuzzy left and an interval-valued fuzzy right ideal of \(S\).

We will denote the set of all IVSGs [resp. IVLIs, IVRIs and IVIs] of \(S\) as \(SV(G(S))\) [resp. \(VLI(S), \ VRI(S)\) and \(VI(S)\)]. It is clear that \(A \in VI(S)\) if and only if \(A^L(xy) \geq A^L(x) \land A^L(y)\) and \(A^U(xy) \geq A^U(x) \land A^U(y)\) for any \(x, y \in S\), and if \(A \in IVLI(S)\) [resp. IVRI(S) and

\[
\text{...}
\]
IVI(S)], then \( A \in \text{IVSG}(S) \).

The following is the immediate result of Definition 3.1 and 3.2.

**Theorem 3.3.** Let \( S \) be a semigroup and let \( \tilde{0} \neq A \in D(I)^S \). Then \( A \in \text{IVSG}(S) \) if and only if \( \tilde{0} \circ A \subset A \).

**Result 3.A.** [3, Theorem 3.2] Let \( A \) be a nonempty subset of a semigroup \( S \). Then, \( A \) is a subsemigroup of \( S \) if and only if \( [\chi_A, \chi_A] \in \text{IVSG}(S) \).

**Result 3.B.** [7, Proposition 6.6] Let \( A \) be a nonempty subset of a ring \( R \). Then \( A \in \text{LI}(R) \) [resp. \( \text{RI}(R) \) and \( I(R) \)] if and only if \( [\chi_A, \chi_A] \in \text{IVLI}(R) \) [resp. \( \text{IVRI}(R) \) and \( \text{IVI}(R) \)].

**Lemma 3.4.** Let \( S \) be a semigroup and let \( A \in D(I)^S \). Then \( A \in \text{IVLI}(S) \) if and only if \( 1 \circ A \subset A \).

**Proof.**\((\Rightarrow)\) : Suppose \( A \in \text{IVLI}(S) \) and let \( a \in S \).

Case (i) : Suppose \( (\tilde{1} \circ A)(a) = [0,0] \). Then clearly \( \tilde{1} \circ A \subset A \).

Case (ii) : Suppose \( (\tilde{1} \circ A)(a) \neq [0,0] \). Then there exist \( x, y \in S \) with \( a = xy \). Thus \( (\tilde{1} \circ A)^L(a) = \bigvee_{a=xy} (\tilde{1}^L(x) \land A^L(y)) \)

\[ \leq \bigvee_{a=xy} (1 \land A^L(xy)) \quad \text{(Since } A \in \text{IVLI}(S)) \]

\[ = \bigvee_{a=xy} (1 \land A^L(a)) = A^L(a). \]

Similarly, we have \( (\tilde{1} \circ A)^U(a) \leq A^U(a) \). Hence, in all, \( \tilde{1} \circ A \subset A \).

\((\Leftarrow)\) : Suppose the necessary condition holds. Let \( A \in D(I)^S \) and let \( a = xy \) for any \( x, y \in S \). Then, by the hypothesis, \( \tilde{1} \circ A \subset A \). Thus \( A^L(xy) = A^L(a) \geq (\tilde{1} \circ A)^L(a) = \bigvee_{a=bc} (\tilde{1}^L(b) \land A^L(c)) \)

\[ \geq \tilde{1}^L(x) \land A^L(y) \quad \text{(Since } a = xy) \]

\[ = 1 \land A^L(y) = A^L(y). \]

Similarly, we have \( A^U(xy) \geq A^U(y) \). Hence \( A \in \text{IVLI}(S) \). This completes the proof. \(\square\)
Lemma 3.4 [The dual of Lemma 3.4] Let $S$ be a semigroup and let $A \in D(I)^S$. Then $A \in \text{IVRI}(S)$ if and only if $A \circ \tilde{1} \subset A$.

The combined effect of these two lemmas is as follows:

Theorem 3.5. Let $S$ be a semigroup and let $A \in D(I)^S$. Then $A \in \text{IVI}(S)$ if and only if $\tilde{1} \circ A \subset A$ and $A \circ \tilde{1} \subset A$.

4. Interval-valued fuzzy generalized bi-ideals

A subsemigroup $A$ of a semigroup $S$ is called a bi-ideal of $S$ if $ASA \subset A$. We will denote the set of all bi-ideals of $S$ as $\text{BI}(S)$.

Definition 4.1. [3] Let $S$ be a semigroup and let $A \in \text{IVSG}(S)$. Then $A$ is called an interval-valued fuzzy bi-ideal (in short, IVBI) of $S$ if
\[
A^L(xyz) \geq A^L(x) \land A^L(z) \quad \text{and} \quad A^U(xyz) \geq A^U(x) \land A^U(z)
\]
for any $x, y, z \in S$.

We will denote the set of all IVBI of $S$ as $\text{IVBI}(S)$.

Result 4.1A. [3, Theorem 3.7] Let $A$ be a non-empty subset of a semigroup $S$. Then $A \in \text{BI}(S)$ if and only if $[\chi_A, \chi_A] \in \text{IVBI}(S)$.

Remark 4.2. Let $S$ be a semigroup.

(a) If $A$ is a fuzzy left ideal [resp. right ideal and bi-ideal] of $S$, then $A = [A, A] \in \text{IVLI}(S)$ [resp. IVRI(S), IVI(S) and IVBI(S)].

(b) If $A \in \text{IVBI}(S)$, then $A^L$ and $A^U$ are fuzzy bi-ideals of $S$.

A nonempty subset $A$ of a semigroup $S$ is called a generalized bi-ideal [8] if $ASA \subset A$. We will denote the set of all generalized bi-ideals of $S$ as $\text{GBI}(S)$.

Definition 4.3. Let $S$ be a semigroup and let $A \in D(I)^S$. Then $A$ is called an interval-valued fuzzy generalized bi-ideal (in short, IVGBI) of $S$ if for any $x, y, z \in S$, $A^L(xyz) \geq A^L(x) \land A^L(z)$ and $A^U(xyz) \geq A^U(x) \land A^U(z)$.

We will denote the set of all IVGBI of $S$ as $\text{IVGBI}(S)$. It is clear that $\text{IVBI}(S) \subset \text{IVGBI}(S)$. But the converse inclusion does not hold in general.
Example 4.4. Let \( S = \{ a, b, c, d \} \) be the semigroup with the following multiplication table:

\[
\begin{array}{cccc}
 a & b & c & d \\
 a & a & a & a \\
 b & a & a & a \\
 c & a & a & b \\
 d & a & a & b \\
\end{array}
\]

We define a mapping \( A : S \rightarrow \mathbb{D}(I) \) as follows:

\[
A(a) = [0.4, 0.5], \quad A(b) = [0, 0], \quad A(c) = [0.2, 0.8], \quad A(d) = [0, 0].
\]

Then we can easily show that \( A \in \text{IVGBI}(S) \) but \( A \notin \text{IVBI}(S) \).

Remark 4.5. Let \( S \) be a semigroup.

(a) If \( A \) is a fuzzy generalized bi-ideal of \( S \), then \([A, A] \in \text{IVGBI}(S)\).

(b) If \( A \in \text{IVGBI}(S) \), then \( A^L \) and \( A^U \) are fuzzy generalized bi-ideals of \( S \).

The following two lemmas are easily seen.

Lemma 4.6. Let \( A \) be a nonempty subset of a semigroup \( S \). Then \( A \in \text{GBI}(S) \) if and only if \([\chi_A, \chi_A] \in \text{IVGBI}(S)\).

Lemma 4.7. Let \( S \) be a semigroup and let \( A \in \mathbb{D}(I)^S \). Then \( A \in \text{IVGBI}(S) \) if and only if \( A \circ \tilde{1} \circ A \subset A \).

5. Regular semigroups

A semigroup \( S \) is said to be regular if for each \( a \in S \), there exists an \( x \in S \) such that \( a = axa \).

Proposition 5.1. Let \( S \) be a regular semigroup. Then \( \text{IVGBI}(S) \subset \text{IVBI}(S) \).

Proof. Let \( A \in \text{IVGBI}(S) \) and let \( a, b \in S \). Since \( S \) is regular, there exists an \( x \in S \) such that \( b = bxb \). Then \( A^L(ab) = A^L(a(bxb)) = A^L(a(bx)b) \geq A^L(a) \wedge A^U(b) \). Similarly, we have \( A^U(ab) \geq A^U(a) \wedge A^U(b) \). Thus \( A \in \text{IVSG}(S) \). So \( A \in \text{IVBI}(S) \). Hence \( \text{IVGBI}(S) \subset \text{IVBI}(S) \).

Theorem 5.2. Let \( S \) be a semigroup. Then \( S \) is regular if and only if \( A = A \circ \tilde{1} \circ A \) for each \( A \in \text{IVGBI}(S) \).
Proof. \((\Rightarrow)\): Suppose \(S\) is regular. Let \(A \in \text{IVGBI}(S)\) and let \(a \in S\). Since \(S\) is regular, there exists an \(x \in S\) such that \(a = axa\). Then
\[
(A \circ \bar{1} \circ A)^L(a) = \bigvee_{a=yz} ((A \circ \bar{1})^L(y) \wedge A^L(z))
\]
\[
\geq (A \circ \bar{1})^L(ax) \wedge A^L(a) \quad \text{(Since } a = axa) \]
\[
= \left( \bigvee_{ax=pq} A^L(p) \wedge \bar{1}^L(q) \right) \wedge A^L(a)
\]
\[
\geq A^L(a) \wedge \bar{1}^L(x) \wedge A^L(a)
\]
\[
= A^L(a) \wedge 1 \wedge A^L(a) = A^L(a).
\]
Similarly, we have \((A \circ \bar{1} \circ A)^U(a) \geq A^U(a)\). Thus \(A \subset A \circ \bar{1} \circ A\). Since \(A \in \text{IVGBI}(S)\), by Lemma 4.7, \(A \circ \bar{1} \circ A \subset A\). Hence \(A = A \circ \bar{1} \circ A\).

\((\Leftarrow)\): Suppose the necessary condition holds. Let \(A \in \text{GBI}(S)\). There, by Lemma 4.6, \([\chi_A, \chi_A] \in \text{IVGBI}(S)\). Thus, by the hypothesis, \([\chi_A, \chi_A] \circ \bar{1} \circ [\chi_A, \chi_A] = [\chi_A, \chi_A]\). Let \(a \in S\). Then
\[
([\chi_A, \chi_A] \circ \bar{1} \circ [\chi_A, \chi_A])^L(a) = \bigvee_{a=yz} (([\chi_A, \chi_A] \circ \bar{1})^L(y) \wedge \chi_A(z)) = \chi_A(a) = 1.
\]
Similarly, we have \((([\chi_A, \chi_A] \circ \bar{1})^U(a) = 1\). Thus there exist \(b, c \in S\) with \(a = bc\) such that \((([\chi_A, \chi_A] \circ \bar{1})^L(b) = \chi_A(c) = 1\) and \((([\chi_A, \chi_A] \circ \bar{1})^U(b) = \chi_A(c) = 1\). So \(\bigvee_{b=pq} ([\chi_A(p) \wedge \bar{1}^L(q)) = 1\) and \(\bigvee_{b=pq} ([\chi_A(p) \wedge \bar{1}^U(q)) = 1\). Then there exist \(d, e \in S\) with \(b = de\) such that \(\chi_A(d) = \bar{1}^L(e) = 1\) and \(\chi_A(d) = \bar{1}^U(e) = 1\). Thus \(d \in A, e \in S, c \in S\) and \(a = bc = (de)c \in \text{ASA}\). So \(A \subset \text{ASA}\). Since \(A \in \text{GBI}(S)\), it is clear that \(\text{ASA} \subset A\). Hence \(A = \text{ASA}\). Therefore \(A\) is regular. This completes the proof.  

The following result is due to Lemma 4.7 and Theorem 5.2.

**Theorem 5.3.** A semigroup \(S\) is regular if and only if \(\text{IVGBI}(S)\) is a regular semigroup.

**Theorem 5.4.** A semigroup \(S\) is regular if and only if for each \(A \in \text{IVGBI}(S)\) and each \(B \in \text{IVI}(S)\), \(A \cap B = A \circ B \circ A\).

Proof. \((\Rightarrow)\): Suppose \(S\) is regular. Let \(A \in \text{IVGBI}(S)\) and let \(B \in \text{IVI}(S)\). Then, by Lemma 4.7, \(A \circ B \circ A \subset A \circ \bar{1} \circ A \subset A\). Also, \(A \circ B \circ A \subset \bar{1} \circ B \circ \bar{1} \subset \bar{1} \circ B \subset B\). So \(A \circ B \circ A \subset A \cap B\). Now let \(a \in S\)
Since $S$ is regular, there exists an $x \in S$ such that $a = axa = axaxa$.

Since $B \in \text{IVI}(S)$,
\[
B^L(xax) \geq B^L(ax) \geq B^L(a) \quad \text{and} \quad B^U(xax) \geq B^U(ax) \geq B^U(a).
\]

Then
\[
(A \circ B \circ A)^L(a) = \bigvee_{yz} (A^L(y) \land (B \circ A)^L(z))
\]
\[
\geq A^L(a) \land (B \circ A)^L(xaxa) \quad (\text{Since } a = axaxa)
\]
\[
= A^L(a) \land \left( \bigvee_{xaxa = pq} (B^L(p) \land A^L(p)) \right)
\]
\[
\geq A^L(a) \land B^L(xaxa) \land A^L(a)
\]
\[
\geq A^L(a) \land B^L(a) \land A^L(a)
\]
\[
= A^L(a) \land B^L(a) = (A \cap B)^L(a).
\]

By the similar arguments, we have that $(A \circ B \circ A)^U(a) \geq (A \cap B)^U(a)$. So $A \cap B \subset A \circ B \circ A$. Hence $A \circ B \circ A = A \cap B$.

$(\Leftarrow)$: Suppose the necessary condition holds. It is clear that $1 \in \text{IVI}(S)$. Let $A \in \text{IVGBI}(S)$. Then, by the hypothesis, $A = A \cap \tilde{1} = A \circ 1 \circ A$. Hence, by Theorem 5.2, $S$ is regular. This completes the proof.

**Result 5.5.** [9, Theorems 1 and 4] Let $S$ be a semigroup. Then the following are equivalent:

(a) $S$ is regular.
(b) $A \cap L \subset AL$ for each $A \in \text{GBI}(S)$ and each $L \in \text{LI}(S)$.
(c) $R \cap A \cap L \subset RAL$ for each $A \in \text{GBI}(S)$, each $L \in \text{LI}(S)$ and each $R \in \text{RI}(S)$.

Now we give a characterization of a regular semigroup in terms of interval-valued fuzzy generalized bi-ideals and interval-valued fuzzy bi-ideals.

**Theorem 5.5.** Let $S$ be a semigroup. Then the following are equivalent:

(a) $S$ is regular.
(b) $A \cap B \subset A \circ B$ for each $A \in \text{IVBI}(S)$ and each $B \in \text{IVLI}(S)$.
(c) $A \cap B \subset A \circ B$ for each $A \in \text{IVGBI}(S)$ and each $B \in \text{IVLI}(S)$.
(d) $C \cap A \cap B \subset C \circ A \circ B$ for each $A \in \text{IVBI}(S)$, each $B \in \text{IVLI}(S)$ and each $C \in \text{IVRI}(S)$.
(e) $C \cap A \cap B \subset C \circ A \circ B$ for each $A \in \text{IVGBI}(S)$ and each $B \in \text{IVRI}(S)$.

Proof. (a) $\Rightarrow$ (b): Suppose $S$ is regular. Let $A \in \text{IVBI}(S)$, let $B \in \text{IVLI}(S)$ and let $a \in S$. Since $S$ is regular, there exists an $x \in S$ such that $a = axa$. Then $(A \circ B)(a) \neq [0,0]$. Thus

$$
(A \circ B)^{L}(a) = \bigvee_{a=yz} (A^{L}(y) \land B^{L}(z))
$$

$$
\geq A^{L}(a) \land B^{L}(xa) \quad \text{(Since } a = axa)$$

$$
\geq A^{L}(a) \land B^{L}(a) \quad \text{(Since } B \in \text{IVLI}(S))$$

$$
= (A \cap B)^{L}(a).
$$

Similarly, we have $(A \circ B)^{U}(a) \geq (A \cap B)^{U}(a)$. Hence $A \cap B \subset A \circ B$.

(b) $\Rightarrow$ (c): It is clear.

(c) $\Rightarrow$ (a): Suppose the condition (c) holds. Let $A \in GBI(S)$, let $L \in LI(S)$ and let $a \in A \cap L$. Then $a \in A$ and $a \in L$. Since $A \in GBI(S)$, by Lemma 4.6, $[\chi_{A}, \chi_{A}] \in \text{IVGBI}(S)$. By Result 3.B, $[\chi_{L}, \chi_{L}] \in \text{IVLI}(S)$. Thus, by the hypothesis, $[\chi_{A}, \chi_{A}] \cap [\chi_{L}, \chi_{L}] \subset [\chi_{A}, \chi_{A}] \circ [\chi_{L}, \chi_{L}]$. So

$$
([\chi_{A}, \chi_{A}] \circ [\chi_{L}, \chi_{L}])^{L}(a) \geq ([\chi_{A}, \chi_{A}] \cap [\chi_{L}, \chi_{L}])^{L}(a) = \chi_{A}(a) \land \chi_{L}(a) = 1.
$$

Similarly, we have that $([\chi_{A}, \chi_{A}] \circ [\chi_{L}, \chi_{L}])^{U}(a) \geq 1$. Then

$$
([\chi_{A}, \chi_{A}] \circ [\chi_{L}, \chi_{L}]) (a) \neq [0,0].
$$

Thus

$$
\bigvee_{a=yz} (\chi_{A}(y) \land \chi_{L}(z)) = 1 \text{ and } \bigvee_{a=yz} (\chi_{A}(y) \land \chi_{L}(z)) = 1.
$$

So there exist $b, c \in S$ with $a = bc$ such that $\chi_{A}(b) = 1$ and $\chi_{L}(c) = 1$. Thus $b \in A$ and $c \in L$, i.e., $a = bc \in AL$. So $A \cap L \subset AL$. Hence, by Result 5.A, $S$ is regular.

(a) $\Rightarrow$ (d): Suppose $S$ is regular. Let $A \in \text{IVBI}(S)$, let $B \in \text{IVLI}(S)$ and let $C \in \text{IVRI}(S)$. Since $S$ is regular, there exists an $x \in S$ such that

$$
\bigvee_{a=yz} (\chi_{A}(y) \land \chi_{L}(z)) = 1 \text{ and } \bigvee_{a=yz} (\chi_{A}(y) \land \chi_{L}(z)) = 1.
$$

So there exist $b, c \in S$ with $a = bc$ such that $\chi_{A}(b) = 1$ and $\chi_{L}(c) = 1$. Thus $b \in A$ and $c \in L$, i.e., $a = bc \in AL$. So $A \cap L \subset AL$. Hence, by Result 5.A, $S$ is regular.
\[ a = axa. \] Then
\[
(C \circ A \circ B)^L(a) = \bigvee_{a=yz} (C^L(y) \land (A \circ B)^L(z))
\]
\[
\geq C^L(ax) \land (A \circ B)^L(a) \quad (\text{Since } a = axa)
\]
\[
\geq C^L(a) \land \left( \bigvee_{a=pq} (A^L(p) \land B^L(q)) \right) \quad (\text{Since } C \in \text{IVRI}(S))
\]
\[
\geq C^L(a) \land A^L(a) \land B^L(a) \quad (\text{Since } a = axa)
\]
\[
\geq C^L(a) \land A^L(a) \land B^L(a) \quad (\text{Since } C \in \text{IVRI}(S))
\]
\[
= (C \cap A \cap B)^L(a).
\]

Similarly, we have that \((C \circ A \circ B)^U(a) \geq (C \cap A \cap B)^U(a)\). Hence \(C \cap A \cap B \subseteq C \circ A \circ B\).

(d) \(\Rightarrow\) (e): It is clear.

(e) \(\Rightarrow\) (a): Suppose the condition (e) holds. Let \(A \in \text{GBI}(S)\), let \(B \in \text{LI}(S)\) and let \(R \in \text{RI}(S)\). Let \(a \in R \cap A \cap L\). Then \(a \in R, a \in A\) and \(a \in L\). Since \(A \in \text{GBI}(S)\), by Lemma 4.6, \([\chi_A, \chi_A] \in \text{IVGBI}(S)\).

By Result 3.B, \([\chi_R, \chi_R] \in \text{IVRI}(S)\) and \([\chi_L, \chi_L] \in \text{IVLI}(S)\). By the hypothesis,
\[
[\chi_R, \chi_R] \cap [\chi_A, \chi_A] \cap [\chi_L, \chi_L] \subseteq [\chi_R, \chi_R] \circ [\chi_A, \chi_A] \circ [\chi_L, \chi_L].
\]

Then
\[
([\chi_R, \chi_R] \circ [\chi_A, \chi_A] \circ [\chi_L, \chi_L])^L(a) \geq ([\chi_R, \chi_R] \cap [\chi_A, \chi_A] \cap [\chi_L, \chi_L])^L(a) = \chi_R(a) \land \chi_A(a) \land \chi_L(a) = 1.
\]

Similarly, we have that
\[
([\chi_R, \chi_R] \circ [\chi_A, \chi_A] \circ [\chi_L, \chi_L])^U(a) \geq 1.
\]

Thus \([\chi_R, \chi_R] \circ [\chi_A, \chi_A] \circ [\chi_L, \chi_L] \neq [0, 0]\). So
\[
\bigvee_{a=yz} \left( ([\chi_R, \chi_R] \circ [\chi_A, \chi_A])^L(y) \land \chi_L(z) \right) = 1.
\]

Similarly, we have that
\[
\bigvee_{a=yz} \left( ([\chi_R, \chi_R] \circ [\chi_A, \chi_A])^U(y) \land \chi_L(z) \right) = 1.
\]

Then there exist \(b, c \in S\) with \(a = bc\) such that
\[
([\chi_R, \chi_R] \circ [\chi_A, \chi_A])^L(b) = 1,
\]
\[
([\chi_R, \chi_R] \circ [\chi_A, \chi_A])^U(b) = 1
\]
and
\[(5.1)\quad \chi_L(c) = 1.\]

Thus \([\chi_R \chi_R \circ [\chi_A, \chi_A]](b) \neq [0, 0].\)
So
\[
\bigvee_{b=pq} (\chi_R(p) \wedge \chi_A(q)) = 1 \quad \text{and} \quad \bigvee_{b=pq} (\chi_R(p) \wedge \chi_A(q)) = 1.
\]

Then there exist \(d, e \in S\) with \(b = de\) such that
\[(5.2)\quad \chi_R(d) = 1 \quad \text{and} \quad \chi_A(e) = 1.

By (5.1) and (5.2), \(d \in R, e \in A\) and \(c \in L\). Thus \(a = bc = dec \in RAL\).
So \(R \cap A \cap L \subset RAL\). Hence, by Result 4.A, \(S\) is regular. This completes the proof.

6. Left quasiregular semigroups

A semigroup \(S\) is said to be left quasiregular if every left ideal of \(S\) is globally idempotent.

**Result 6.A.** [2, Proposition 1.1] A semigroup \(S\) is left quasiregular if and only if for each \(a \in S\), there exist \(x,y \in S\) such that \(a = xaya\).

The following result can be easily proved.

**Lemma 6.1.** Let \(S\) be a semigroup. If \(S\) is left quasiregular, then \(\text{IVGBI}(S) = \text{IVBI}(S)\), i.e., \(\text{IVGBI}(S) \subset \text{IVBI}(S)\).

**Lemma 6.2.** Let \(S\) be a semigroup. Then \(S\) is left quasiregular if and only if \(A \circ A = A\) for each \(A \in \text{IVLI}(S)\).

**Proof.** (\(\Rightarrow\)): Suppose \(S\) is left quasiregular, and let \(A \in \text{IVLI}(S)\). Then, by Theorem 3.3, \(A \circ A \subset A\). Let \(a \in S\). Then, by Result 6.A, there exist \(x, y \in S\) such that \(a = xaya\). Thus
\[
(A \circ A)^L(a) = \bigvee_{a=pq} (A^L(p) \wedge A^L(q))
\geq A^L(xa) \wedge A^L(ya) \quad \text{(Since } a = xaya\text{)}
\geq A^L(a) \wedge A^L(a) \quad \text{(Since } A \in \text{IVFLI}(S)\text{)}
= A^L(a).
\]

Similarly, we have that \((A \circ A)^U(a) \geq A^U(a)\). So \(A \subset A \circ A\). Hence \(A \circ A = A\).
Suppose the necessary condition holds. Let \( L \in \text{LI}(S) \) and let \( a \in L \). By Result 3.B, \([\chi_L, \chi_L] \in \text{IVLI}(S)\). Then, by the hypothesis, \([[\chi_L, \chi_L]] \circ [[\chi_L, \chi_L]] = [\chi_L, \chi_L] \). Thus

\[
[[[\chi_L, \chi_L]] \circ [[\chi_L, \chi_L]]] = \chi_L(a) = 1
\]

So \([[\chi_L, \chi_L]] \circ [[\chi_L, \chi_L]](a) \neq [0, 0] \). Then \( \bigvee_{p=q} (\chi_L(p) \wedge \chi_L(q)) = 1 \). Thus there exist \( b, c \in S \) with \( a = bc \) such that \( \chi_L(b) = 1 \), \( \chi_L(c) = 1 \). So \( b \in L \), i.e., \( a = bc \in L \). Then \( L \subseteq LL \). It is clear that \( LL \subseteq L \). Thus \( L = LL \). Hence \( S \) is left quasiregular. This completes the proof.

A semigroup \( S \) is said to be intraregular if for each \( a \in S \), there exist \( x, y \in S \) such that \( a = xa^2y \).

Result 6.B. [9, Theorem 6] Let \( S \) be a semigroup. Then \( S \) is both intraregular and left quasiregular if and only if for each \( B \in \text{GBI}(S) \), each \( L \in \text{LI}(S) \) and each \( R \in \text{RI}(S) \), \( L \cap R \cap B \subset LR \).

We give a characterization of a semigroup that is both intraregular and left quasiregular in terms of interval-valued fuzzy sets.

Theorem 6.3. Let \( S \) be a semigroup. Then the following are equivalent:

(a) \( S \) is both intraregular and left quasiregular.
(b) \( B \cap C \cap A \subseteq B \circ C \circ A \) for each \( A \in \text{IVB}(S) \), each \( B \in \text{IVL}(S) \) and each \( C \in \text{IVR}(S) \).
(c) \( B \cap C \cap A \subseteq B \circ C \circ A \) for each \( A \in \text{IVGB}(S) \), each \( B \in \text{IVL}(S) \) and each \( C \in \text{IVR}(S) \).

Proof. (b) \( \Rightarrow \) (c): It is clear.
(c) \( \Rightarrow \) (a): It can be seen as in the proof of Theorem 5.5\([e) \text{ implies (a)}]\).

(a) \( \Rightarrow \) (b): Suppose the condition (a) holds. Let \( A \in \text{IVB}(S) \), let \( B \in \text{IVL}(S) \) and let \( C \in \text{IVR}(S) \). Let \( a \in S \). Since \( S \) is left quasiregular, by Result 6.A, there exist \( u, v \in S \) such that \( a = uava \). Then

\[
a = uava = u(xa^2y)va = ((ux)a)((a(yv))a).
\]
Thus
\[(B \circ C \circ A)^L(a) = \bigvee_{a=pq} (B^L(p) \land (C \circ A)^L(q))\]
\[\geq B^L((ux)a) \land (C \circ A)^L((avy)a)\]
\[\geq B^L(a) \land \left( \bigvee_{a=pq} (C^L \land A^L(q)) \right) \quad \text{(Since } B \in \text{IVLI}(S))\]
\[\geq B^L(a) \land C^L(a(yv)) \land A^L(a)\]
\[\geq B^L(a) \land C^L(a) \land A^L(a) \quad \text{(Since } C \in \text{IVRI}(S))\]
\[= (B \cap C \cap A)^L(a).\]

Similarly, we have \((B \circ C \circ A)^U(a) \geq (B \cap C \cap A)^U(a).\) Hence \(B \cap C \cap A \subset B \circ C \circ A.\) This completes the proof.

\[\square\]

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