ON THE EXTENDED $q$-EULER NUMBERS AND POLYNOMIALS OF HIGHER-ORDER WITH WEIGHT

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Abstract. The purpose of this paper is to give a new construction of the extended $q$-Euler numbers and polynomials of higher-order with weight by using $p$-adic $q$-integral on $\mathbb{Z}_p$.

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, the symbol $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}$, $\mathbb{C}$ and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of $p$-adic integers, the field of rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm of $\mathbb{C}_p$ is defined by $|p|_p = 1/p$. We assume that $\alpha \in \mathbb{Q}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

As an indeterminate, we consider that $q \in \mathbb{C}$ or $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, the we assume that $|1 - q|_p < 1$.

The $q$-number of $x$ is defined by $[x]_q = \frac{1 - q^x}{1 - q}$. Note that $\lim_{q \to 1}[x]_q = x$.

Recently, the $q$-Euler numbers with weight $\alpha$ are defined by

$$\tilde{E}_{0,q}^{(\alpha)} = 1, \quad \text{and} \quad q^{\alpha} \tilde{E}_q^{(\alpha)} + 1 + \tilde{E}_{n,q}^{(\alpha)} = 0 \text{ if } n > 0,$$

with the usual convention about replacing $\left(\tilde{E}_q^{(\alpha)}\right)^n$ by $\tilde{E}_n^{(\alpha)}$ (see[5]).

The $q$-Euler polynomials with weight $\alpha$ also defined by

$$\tilde{E}_{n,q}^{(\alpha)}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_{q}\alpha q^{alx} \tilde{E}_{l,q}^{(\alpha)} = \left([x]_q q^{\alpha} + q^{alx} \tilde{E}_q^{(\alpha)}\right)^n, \quad \text{for } n \geq 0.$$

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Let $f \in C(\mathbb{Z}_p)$ = the space of continuous functions on $\mathbb{Z}_p$. Then the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim as follows ([1-16]):

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x)$$

(3) $$= \lim_{N \to \infty} \frac{1 + q}{1 + q^{p_N}} \sum_{x=0}^{p_N-1} f(x)(-q)^x.$$ 

From (3), we have

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} f(l)q^l(-1)^{n-1-l},$$

where $f_n(x) = f(x + n)$ (see[1-16]).

From (2) and (3), we note that

$$\tilde{\mathcal{E}}_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} [y + x]_{q^\alpha}^n d\mu_{-q}(y).$$

(5)

Thus, by (5), we have

$$\tilde{\mathcal{E}}_{n,q}^{(\alpha)}(x) = \frac{[2]_q}{(1 - q)^n[\alpha]_q^n} \sum_{l=0}^{n} \binom{n}{l}(-1)^l q^{\alpha l x} \frac{1}{1 + q^{\alpha l + 1}}.$$ 

(6)

Note that $\lim_{q \to 1} \tilde{\mathcal{E}}_{n,q}^{(\alpha)}(x) = E_n(x)$ where $E_n(x)$ are the $n$-th ordinary Euler polynomial which are defined by $\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}$ (see[1-16]).

By using the fermionic multivariate $p$-adic $q$-integral on $\mathbb{Z}_p$, we give a new construction of the extended $q$-Euler numbers and polynomials of higher-order with weight $\alpha$.

From the extended $q$-Euler numbers and polynomials of higher-order with weight $\alpha$, we derive a new explicit formulae by those numbers and polynomials.

2. On the extended $q$-Euler numbers of higher-order with weight $\alpha$

In this section, we assume that $h_1, h_2, \cdots, h_k \in \mathbb{Z}_+$ and $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. Now we consider a sequence of $p$-adic rational numbers.
On the extended $q$-Euler numbers and polynomials as expansion of the $q$-Euler numbers and polynomials of order $k$ with weight $\alpha$ as follows:

\[
\tilde{E}_{n,q}^{(k,\alpha)}(h_1, h_2, \cdots, h_k)
\]

(7) = \[\frac{[2]^k_{n!}}{(1-q)^n[a]_{q^n}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \prod_{j=1}^{k} (1 + q^{\alpha l + h_j})\],

and

\[
\tilde{E}_{n,q}^{(k,\alpha)}(h_1, h_2, \cdots, h_k|x)
\]

(8) = \[\frac{[2]^k_{n!}}{(1-q)^n[a]_{q^n}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \prod_{j=1}^{k} (1 + q^{\alpha l + h_j})\]

By (7) and (8), we get

\[
\tilde{E}_{n,q}^{(k,\alpha)}(h_1, h_2, \cdots, h_k) = \left[\frac{2k}{1-q}\right]_{n!} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \prod_{j=1}^{k} (1 + q^{\alpha l + h_j}),
\]

and

\[
\tilde{E}_{n,q}^{(k,\alpha)}(h_1, h_2, \cdots, h_k|x) = \left[\frac{2k}{1-q}\right]_{n!} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \prod_{j=1}^{k} (1 + q^{\alpha l + h_j})
\]

From (9) we note that

\[
\sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{\prod_{j=1}^{k} (1 + q^{\alpha l + h_j})}
\]

(10) = \[\sum_{l=0}^{n} \binom{n}{l} [x]_{q^{\alpha l x}}^{-l} q^{\alpha l x} (1-q)^{n-l} [a]_{q^{n-l}}^{-l} \sum_{s=0}^{l} \binom{l}{s} (-1)^s \prod_{j=1}^{k} (1 + q^{\alpha s + h_j})^{-1} \cdot \]

Therefore, by (10), we obtain the following theorem.
Theorem 1. Let $h_1, h_2, \cdots h_k \in \mathbb{Z}_+$ and $k \in \mathbb{N}$. Then we have
\[
\sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha x} \left( \prod_{j=1}^{k} \left( 1 + q^{\alpha l + h_j} \right) \right)^{-1}
\]
\[
= \sum_{l=0}^{n} \binom{n}{l} [x]_{q^n}^{n-l} q^{\alpha x} (1-q)^{n-l}[\alpha]_{q^n}^{-1} \sum_{s=0}^{l} \binom{l}{s} (-1)^s \left( \prod_{j=1}^{k} \left( 1 + q^{\alpha s + h_j} \right) \right)^{-1}.
\]
By (4), we get
\[
q^{h_1} \int_{\mathbb{Z}_p} [x_1 + x + 1]_{q^n}^{n} q^{(h_1-1)x_1} d\mu_q(x_1)
\]
\[
= - \int_{\mathbb{Z}_p} [x_1 + x]_{q^n}^{n} q^{(h_1-1)x_1} d\mu_q(x_1) + [2]_{q^n} [x]_{q^n}^{n}.
\]
Therefore, by (11) we obtain the following theorem.

Theorem 2. For $n \in \mathbb{Z}_+$, we have
\[
q^{h_1} \tilde{E}_{n,q}^{(1,\alpha)}(h_1|x + 1) + \tilde{E}_{n,q}^{(1,\alpha)}(h_1|x) = [2]_{q^n} [x]_{q^n}^{n}.
\]
By (8) we get
\[
q^{\alpha x} \tilde{E}_{n,q}^{(k,\alpha)}(h_1 + \alpha, h_2 + \alpha, \cdots, h_k + \alpha|x)
\]
\[
= q^{\alpha x} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_{q^n}^{n} q^{\sum_{j=1}^{k} x_j (h_j + \alpha - 1)} d\mu_q(x_1) \cdots d\mu_q(x_k)
\]
\[
= (q^{\alpha} - 1) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_{q^n}^{n+1} q^{\sum_{j=1}^{k} x_j (h_j - 1)} d\mu_q(x_1) \cdots d\mu_q(x_k)
\]
\[
+ \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_{q^n}^{n} q^{\sum_{j=1}^{k} x_j (h_j - 1)} d\mu_q(x_1) \cdots d\mu_q(x_k).
\]
Thus, we obtain the following theorem.

Theorem 3. For $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, we have
\[
q^{\alpha x} \tilde{E}_{n,q}^{(k,\alpha)}(h_1 + \alpha, \cdots, h_k + \alpha|x)
\]
On the extended $q$-Euler numbers and polynomials

$$= (q^n - 1)\tilde{E}_{n+1,q}(h_1, \ldots, h_k|x) + \tilde{E}_{n,q}(h_1, \ldots, h_k|x).$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. Then we have

$$\tilde{E}_{n,q}(h_1, \ldots, h_k|x)$$

$$= \int_{\mathbb{Z}_p} \ldots \int_{\mathbb{Z}_p} [x + x_1 + \ldots + x_k]_{q^d} q^{\sum_{j=1}^{k} x_j(h_j-1)} d\mu_{-q}(x_1) \ldots d\mu_{-q}(x_k)$$

$$= [d]_{q^n} \sum_{a_1, \ldots, a_k=0}^{d-1} q^{\sum_{j=1}^{k} x_j a_j} (-1)^{\sum_{j=1}^{k} a_j} \tilde{E}_{n,q}^{(k,\alpha)}

\left( h_1, \ldots, h_k, \frac{\sum_{j=1}^{k} x_j + x}{d} \right).$$

Therefore, by (12), we obtain the following theorem.

**Theorem 4.** For $n \in \mathbb{Z}_{+}$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, we have

$$\tilde{E}_{n,q}(h_1, \ldots, h_k|x)$$

$$= [d]_{q^n} \sum_{a_1, \ldots, a_k=0}^{d-1} q^{\sum_{j=1}^{k} x_j a_j} (-1)^{\sum_{j=1}^{k} a_j} \tilde{E}_{n,q}^{(k,\alpha)}

\left( h_1, \ldots, h_k, \frac{\sum_{j=1}^{k} x_j + x}{d} \right).$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. For $N \in \mathbb{N}$, we get

$$X = X_d = \lim_{N} \mathbb{Z}/dp^N \mathbb{Z}, \quad X_1 = \mathbb{Z}_p, \quad X^* = \bigcup_{0 < a < dp, (a,p)=1} (a + dp\mathbb{Z}_p),$$

$$a + dp^N \mathbb{Z}_p = \{x \in X | x \equiv a \pmod{dp^N} \},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$.

Let $\chi$ be a primitive Dirichlet character with conductor $d \in \mathbb{N}$. Then we consider the generalized $q$-Bernoulli numbers of order $k$ with weight
\( \alpha \) as follows:

\[
\tilde{E}_{n,q}(k,\alpha)(h_1, \ldots, h_k)
= \int X \cdots \int X \left( \prod_{i=1}^{k} \chi(x_i) \right) [x_1 + \cdots + x_k]_q^{\sum_{i=1}^{k} x_i(h_i-1)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k).
\]

By (13), we note that

\[
\tilde{E}_{n,q}(k,\alpha)(h_1, \ldots, h_k)
= \left[ \frac{d}{d-1} \prod_{i=1}^{k} h_i \cdot (-1) \cdot \sum_{j=1}^{k} \chi(a_i) \right]^{q^{\sum_{i=1}^{k} a_j}}
\left( h_1, \ldots, h_k \right) \cdot \sum_{j=1}^{k} \frac{a_j}{d}.
\]

Let \( F_q^{(k,\alpha| h_1, \ldots, h_k)}(t, x) = \sum_{n=0}^{\infty} \tilde{E}_{n,q}(k,\alpha)(h_1, \ldots, h_k|x)^{\frac{t^n}{n!}}. \) Then, by (9), we get

\[
F_q^{(k,\alpha| h_1, \ldots, h_k)}(t, x) = \sum_{m_1, \ldots, m_k=0}^{\infty} q^{\sum_{j=1}^{k} h_j m_j} (-1)^{\sum_{j=1}^{k} m_j} \sum_{n=0}^{\infty} [m_1 + \cdots + m_k + x]_q^{\frac{t^n}{n!}}.
\]

Therefore, by (14), we obtain the following theorem.

**Theorem 5.** Let \( F_q^{(k,\alpha| h_1, \ldots, h_k)}(t, x) = \sum_{n=0}^{\infty} \tilde{E}_{n,q}(k,\alpha)(h_1, \ldots, h_k|x)^{\frac{t^n}{n!}}. \) Then we have

\[
F_q^{(k,\alpha| h_1, \ldots, h_k)}(t, x) = \sum_{m_1, \ldots, m_k=0}^{\infty} q^{\sum_{j=1}^{k} h_j m_j} (-1)^{\sum_{j=1}^{k} m_j} [m_1 + \cdots + m_k + x]_q^{\frac{t^n}{n!}}.
\]
3. Further Remark

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. As well known definition, the gamma function is defined by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt, \quad \text{where } s \in \mathbb{C} \text{ with } \Re(s) > 0. \quad (15)$$

From (15), we have

$$\Gamma(s + 1) = s\Gamma(s), \quad \text{and } \Gamma(n + 1) = n! \quad (n \in \mathbb{N}).$$

In $\mathbb{C}$, the extended $q$-Euler polynomials of order $k$ with weight $\alpha$ are given by

$$F_{q}^{(k,\alpha|h_1,\cdots,h_k)}(t,x) = \sum_{m_1,\ldots,m_k=0}^{\infty} q^{m_1+h_1}\cdots q^{m_k+h_k}(-1)^{\sum_{j=1}^{k} m_j \cdot (m_1+\cdots+m_k+x)} q^{\alpha} t^{n} \equiv \tilde{E}_{n,q}^{(k,\alpha)}(h_1,\cdots,h_k|x). \quad (16)$$

For $s \in \mathbb{C}$, it is easy to show that

$$\frac{1}{\Gamma(s)} \int_0^1 F_{q}^{(k,\alpha|h_1,\cdots,h_k)}(-t,x)t^{s-1} dt$$

$$= \sum_{m_1,\ldots,m_k=0}^{\infty} q^{m_1+h_1}\cdots q^{m_k+h_k}(-1)^{\sum_{j=1}^{k} m_j \cdot (m_1+\cdots+m_k+x)} q^{\alpha} \frac{t^{n}}{n!}, \quad (17)$$

where $x \neq 0, -1, -2, \cdots$.

From (17), we can define the multiple $q$-Euler Zeta function with weight $\alpha$ as follows: For $s \in \mathbb{C}$, define

$$\zeta_{q}^{(k,\alpha)}(h_1,\cdots,h_k|s,x) = \sum_{m_1,\ldots,m_k=0}^{\infty} q^{m_1+h_1}\cdots q^{m_k+h_k}(-1)^{\sum_{j=1}^{k} m_j \cdot (m_1+\cdots+m_k+x)} q^{\alpha} \frac{t^{n}}{n!}, \quad (18)$$

where $x \neq 0, -1, -2, \cdots$.

Note that $\zeta_{q}^{(k,\alpha)}(h_1,\cdots,h_k|s,x)$ is analytic function in whole complex $s$-plane. By using (16), (17), (18), and Laurent series, we obtain the following theorem.

**Theorem 6.** For $n \in \mathbb{Z_+}$, we have

$$\zeta_{q}^{(k,\alpha)}(h_1,\cdots,h_k|-n,x) = \tilde{E}_{n,q}^{(k,\alpha)}(h_1,\cdots,h_k|x).$$
References


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