A NOTE ON AXIOMATIC FEYNMAN OPERATIONAL
CALCULUS

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Abstract. In this note we prove the space \((A, \| \cdot \|)\) is a Banach space and \(\|ab\| \leq \|a\|\|b\|\) for \(a, b \in A\) where \(A := \{a := (a_t)_{t \in G} : \sum_{t \in G} \|a_t\|_t < \infty\}, G = \mathbb{N}^*\). Also we show some property in \((A, \| \cdot \|)\).

1. Introduction and Preliminaries

In 1988, Johnson, G.W. and Lapidus, M.L. [2] presented the definition and the theories of noncommutative operations on Wiener functionals. They introduced a general axiomatic framework for Feynman’s operational calculus and disentangle algebras \(\{A_t : t > 0\}\) [3,4].

In this note we introduce the algebraic and analytic axioms and we prove the space \((A, \| \cdot \|)\) is a Banach space and \(\|ab\| \leq \|a\|\|b\|\) for \(a, b \in A\) where \(A := \{a := (a_t)_{t \in G} : \sum_{t \in G} \|a_t\|_t < \infty\}, G = \mathbb{N}^*\). Also we show that \(\|a + b\| \leq \sum_{n \in G}(\sum_{t=1}^{n} \|a_t\|_t) + \sum_{n \in G}(\sum_{t=1}^{n} \|b_t\|_t)\) and \(\|ab\| \leq \|a\|\|b\|\) for \(a, b \in A\).

In this section we present some notations, algebraic and analytic axioms, definitions and Theorems from [3,4].

Let \(G = \mathbb{N}^* = \{1, 2, \cdots\}\) be the additive semigroup of positive integers. Let \(G = (0, \infty)\) be the additive semigroup of positive real numbers.

(Axiom 1)

For each \(t \in G\), let \(A_t\) be a (real or complex) unital algebras, with unit element denoted \(1_t\) and with zero denoted \(0_t\). Given any \(t_1, t_2 \in G\), we assume that there exist two operations \(*\) and \(\oplus\) such that \(*\) (resp., \(\oplus\)): \(A_{t_1} \times A_{t_2} \to A_{t_1 + t_2}\) is bilinear (resp., linear) and associative in the following

Received May 7, 2012. Accepted May 29, 2012.
2000 Mathematics Subject Classification. 28C20.
Key words and phrases. Disentangle algebra, Banach algebra.
sense: if \( t_j \in G \) and \( a_j \in A_{t_j} (j = 1, 2, 3) \), then \((a_1 * a_2) * a_3 = a_1 * (a_2 * a_3)\) [resp., \((a_1 + a_2) + a_3\)], in \( A_{t_1 + t_2 + t_3} \).

(Axiom 2)

For \( t_j \in G \) and \( a_j, b_j \in A_{t_j} (j = 1, 2) \), we have

\[(1.1) \quad (a_1 + a_2) (b_1 + b_2) = (a_1 b_1) * 1_{t_2} + a_1 * b_2 + b_1 * a_2 + 1_{t_1} * (a_2 b_2) \]

\[(1.2) \quad (a_1 * a_2) (b_1 + b_2) = (a_1 b_1) * a_2 + a_1 * (a_2 b_2) \]

\[(1.3) \quad (a_1 + a_2) (b_1 * b_2) = (a_1 b_1) * b_2 + b_1 * (a_2 b_2) \]

\[(1.4) \quad (a_1 * a_2) (b_1 * b_2) = (a_1 b_1) * (a_2 b_2) \]

(Axiom 3)

For each \( t \in G \), \( A_t = (A_t, \| \cdot \|_t) \) is a Banach algebra, equipped with the norm \( \| \cdot \|_t \). Moreover, for \( t_j \in G \) and \( a_j \in A_{t_j} (j = 1, 2) \), we have

\[\| a_1 * a_2 \|_{t_1 + t_2} \leq \| a_1 \|_{t_1} \| a_2 \|_{t_2} \quad \text{and} \quad \| a_1 + a_2 \|_{t_1 + t_2} \leq \| a_1 \|_{t_1} + \| a_2 \|_{t_2} \]

(Axiom 4)

For all \( t_1, t_2 \in G \), the set \( \Delta := \{ a_1 * a_2 : a_1 \in A_{t_1}, a_2 \in A_{t_2} \} \) is total in \( A_{t_1 + t_2} \); that is, the space of (finite) linear combinations of elements of \( \Delta \) is dense in \((A_{t_1 + t_2}, \| \cdot \|_{t_1 + t_2})\).

In order to state our axiom (5), we introduce a Banach algebra \( B \) (independent of \( t \in G \)) and also assume that \( G \) is abelian.

(Axiom 5)

For every \( t \in G \), there exists a continuous linear map \( K^t : A_t \rightarrow B \) such that for all \( t_j \in G \) and \( a_j \in A_{t_j} (j = 1, 2) \), we have

\[(1.5) \quad K^{t_1 + t_2} ([a_1, a_2]) = [K^{t_1} (a_1), K^{t_2} (a_2)] \]

where \([a_1, a_2] = a_1 * a_2 - a_2 * a_1 \in A_{t_1 + t_2}\) and

\[ [K^{t_1} (a_1), K^{t_2} (a_2)] = K^{t_1} (a_1) K^{t_2} (a_2) - K^{t_2} (a_2) K^{t_1} (a_1) \in B \]

Moreover, we assume that \( sup_{t \in G} \| K^t \| < \infty \), where \( \| \cdot \| \) denotes the norm in \( \mathcal{L} (A_t, B) \).

**Definition 1.1.** We will say that \( K^t \) is multiplicative if instead of requiring (1.5) in Axiom 5, we assume that \( K^{t_1 + t_2} (a_1 * a_2) = K^{t_1} (a_1) K^{t_2} (a_2) \), for all \( t_j \in G \) and \( a_j \in A_{t_j} (j = 1, 2) \).

We now assume that \((A_t)_{t \in G}, +, *)\) satisfies axioms (1) ~ (4) and that \(((K^t)_{t \in G}, B)\) satisfies axiom (5).

**Lemma 1.2.** If \( u, v \in A_{t_1 + t_2} \) are such that \( u(a_1 * a_2) = v(a_1 * a_2) \), for all \( a_j \in A_{t_j} (j = 1, 2) \), then \( u = v \).
Proof. Since $\Delta := \{a_1 \ast a_2 : a_1 \in A_{t_1}, a_2 \in A_{t_2}\}$ is total in $A_{t_1 + t_2}$, for any given $\epsilon > 0$, there exists a linear combination $\sum_{i=1}^{n} c_i(a_1 \ast a_2_i)$ of elements of $\Delta$ such that $\|1_{t_1 + t_2} - \sum_{i=1}^{n} c_i(a_1 \ast a_2_i)\|_{t_1 + t_2} < \epsilon$. Then by the triangle inequality,

$$\|u - v\|_{t_1 + t_2} = \|(u - v)1_{t_1 + t_2}\|_{t_1 + t_2}$$

$$\leq \|(u - v)[1_{t_1 + t_2} - \sum_{i=1}^{n} c_i(a_1 \ast a_2_i)]\|_{t_1 + t_2} + \|(u - v)\sum_{i=1}^{n} c_i(a_1 \ast a_2_i)\|_{t_1 + t_2},$$

where $a_1 \in A_{t_1}$ and $a_2 \in A_{t_2}$ ($i = 1, \cdots, n$). Since $(A_{t_1 + t_2}, \|\cdot\|_{t_1 + t_2})$ is a Banach algebra, $\|(u - v)[1_{t_1 + t_2} - \sum_{i=1}^{n} c_i(a_1 \ast a_2_i)]\|_{t_1 + t_2} \leq \|u - v\|_{t_1 + t_2}\epsilon$ and by assumption

$$\|\sum_{i=1}^{n} c_i(a_1 \ast a_2_i)\|_{t_1 + t_2} \leq \sum_{i=1}^{n} c_i\|u - v)(a_1 \ast a_2_i)\|_{t_1 + t_2} = 0.$$ Hence we obtain $\|u - v\|_{t_1 + t_2} \leq \|u - v\|_{t_1 + t_2}\epsilon$. But since $\epsilon > 0$ was arbitrary, we have $\|u - v\|_{t_1 + t_2} = 0$, that is, $u = v$. \hfill $\square$

**Theorem 1.3.** We have the following equalities:

1. $a \ast 1_{t_2} = a \ast 0_{t_2}, 1_{t_1} \ast b = 0_{t_1} \ast b$, for all $a \in A_{t_1}$ and $b \in A_{t_2}$.
2. $0_{t_1} + 0_{t_2} = 0_{t_1 + t_2}, 1_{t_1} \ast 1_{t_2} = 1_{t_1 + t_2}$.
3. The map $L$ defined by $L(a) = a \ast 1_{t_2}$ for $a \in A_{t_1}$ is a continuous algebra homomorphism from $A_{t_1}$ to $A_{t_1 + t_2}$.
4. The map $R$ defined by $R(b) = 1_{t_1} \ast b$ for $b \in A_{t_2}$ is a continuous algebra homomorphism from $A_{t_2}$ to $A_{t_1 + t_2}$.

*Proof.* (1),(2) proved by [3,4]. (3) Since $L(ab) = ab \ast 1_{t_2} = ab \ast (1_{t_2}1_{t_2}) = (a \ast 1_{t_2})(b \ast 1_{t_2}) = L(a)L(b)$ and $\|L(a)\|_{t_1 + t_2} = \|a \ast 1_{t_2}\|_{t_1 + t_2} \leq \|a\|_{t_1}\|1_{t_2}\|_{t_2} = \|a\|_{t_1}$, $L$ is a continuous algebra homomorphism. (4) Since $R(ab) = 1_{t_1} \ast ab = (1_{t_1}1_{t_2})(1_{t_1}1_{t_2}) = (1_{t_1} \ast a)(1_{t_1} \ast b) = R(a)R(b)$ and $\|R(b)\|_{t_1 + t_2} = \|1_{t_1} \ast b\|_{t_1 + t_2} \leq \|1_{t_1}\|_{t_1}\|b\|_{t_2} = \|b\|_{t_2}$, $R$ is a continuous algebra homomorphism. \hfill $\square$

**Theorem 1.4.** Let $L$ and $R$ be defined as in Theorem 1.3. Then for all $a \in A_{t_1}$ and $b \in A_{t_2}$, we have the following equalities in $A_{t_1 + t_2}$:

$$a \ast b = L(a)R(b), a \ast b = L(a) + R(b).$$

Moreover, for all $a \in A_{t_1}$ and $b \in A_{t_2}$, $L(a)R(b) = R(b)L(a)$.

*Proof.* See [3,4]. \hfill $\square$

**Theorem 1.5.** For $a \in A_{t_1}$ and $b \in A_{t_2}$, we have the following equalities in $A_{t_1 + t_2}$.

1. $\exp(a \ast b) = \exp(a) \ast \exp(b)$. 

*Proof.* See [3,4]. \hfill $\square$
(2) \((a+b)^n = \sum_{p+q=n} \frac{n!}{p!q!} a^p b^q\).

(Here, the exponential of \(u \in A_t\) is defined by means of the analytic fuctional calculus in the Banach algebra \(A_t\).)

Proof. (1) See [3,4]. \((2)(a+b)^n = (L(a) + R(b))^n = \sum_{p+q=n} \frac{n!}{p!q!} L(a)^p R(b)^q = \sum_{p+q=n} \frac{n!}{p!q!} a^p * b^q = \sum_{p+q=n} \frac{n!}{p!q!} a^p * b^q.\)

Corollary 1.6. Let \(a \in A_{t_1}\) and \(b \in A_{t_2}\). Then \(K^{t_1+t_2}(\exp(a+b)) - K^{t_1+t_2}(\exp(b+a)) = [K^{t_1}(\exp(a)), K^{t_2}(\exp(b))].\)

Proof. By theorem 1.5 and axiom (5), \(K^{t_1+t_2}(\exp(a+b)) - K^{t_1+t_2}(\exp(b+a)) = K^{t_1+t_2}(\exp(a)*\exp(b)) - K^{t_1+t_2}(\exp(b)*\exp(a)) = K^{t_1}(\exp(a))K^{t_2}(\exp(b)) - K^{t_2}(\exp(b))K^{t_1}(\exp(a)) = [K^{t_1}(\exp(a)), K^{t_2}(\exp(b))].\)

\(\Box\)

2. Some Banach Algebras

In this section we consider the family of Banach algebras \((A_t, \| \cdot \|_t)_{t \in \mathbb{G}}\). Let \(\mathcal{A} := \{a := (a_t)_{t \in \mathbb{G}} : \sum_{t \in \mathbb{G}} \|a_t\|_t < \infty\}\), equipped with the norm \(\|a\| = \sum_{t \in \mathbb{G}} \|a_t\|_t\) and the natural vector space operations.

Theorem 2.1. When \(G = \mathbb{N}^* = \{1, 2, \cdots\}\), the space \((\mathcal{A}, \| \cdot \|)\) is a Banach space and \(\|ab\| \leq \|a\| \|b\|\) for \(a, b \in \mathcal{A}\).

Proof. Since \(\|a\| = \sum_{t \in \mathbb{G}} \|a_t\|_t \geq 0, \|a\| \geq 0\). Moreover, \(\|a\| = 0 \iff \sum_{t \in \mathbb{G}} \|a_t\|_t = 0 \iff \|a\| = 0\) for all \(t\). Since \(\| \cdot \|_t\) is norm, \(a_0 = 0\) for all \(t\). So \(a = 0\). Certainly, \(\|a\| = \sum_{t \in \mathbb{G}} \|a_t\|_t = \|a_0\| = \|a\|\) and \(\|a + b\| = \sum_{t \in \mathbb{G}} \|a_t + b_t\|_t \leq \sum_{t \in \mathbb{G}} \|a_t\|_t + \|b_t\|_t = \|a\| + \|b\|\).

Next, to show \(\mathcal{A}\) is complete, consider \((a^{(n)})\) is a Cauchy sequence in \(\mathcal{A}\), where \(a^{(n)} = (a^{(n)}_1, a^{(n)}_2, \cdots)\). Then for every \(\epsilon > 0\) there exists a \(K\) such that for all \(i, j \geq K\), \(\|a^{(i)} - a^{(j)}\| = \sum_{t \in \mathbb{G}} \|a^{(i)}_t - a^{(j)}_t\|_t < \epsilon\). Thus for every \(t = 1, 2, \cdots\), \(\|a^{(i)}_t - a^{(j)}_t\|_t < \epsilon\) for all \(i, j \geq K\). We choose a fixed \(t\), \((a^{(1)}_t, a^{(2)}_t, \cdots)\) is a Cauchy sequence. Since \(A_t\) is Banach space, \(a^{(j)}_t \to a_t\) as \(j \to \infty\). Using these limits, we define \(a = (a_1, a_2, \cdots)\).

For all \(i, j \geq K\) and \(p = 1, 2, \cdots\), \(\sum_{t = 1}^{p} \|a^{(i)}_t - a^{(j)}_t\|_t < \epsilon\). Letting \(j \to \infty\), we obtain for \(i \geq K\), \(\sum_{t = 1}^{p} \|a^{(i)}_t - a_t\|_t < \epsilon\) for \(p = 1, 2, \cdots\). Letting \(p \to \infty\), then for \(i \geq K\), \(\|a^{(i)}_t - a_t\| = \sum_{t \in \mathbb{G}} \|a^{(i)}_t - a_t\|_t \leq \epsilon\). So \(a^{(n)}\) converges to \(a \in \mathcal{A}\). Finally we show that \(\|ab\| \leq \|a\| \|b\|\) for \(a, b \in \mathcal{A}\). For \(a, b \in \mathcal{A}\), \(a \cdot b = (a_t) \cdot (b_t)_t = (a_t \cdot b_t)_t\). Since \(a_t, b_t \in A_t\) and \(A_t\) is Banach
algebra, \( a_t b_t \in A_t \) and \( \|a_t b_t\|_t \leq \|a_t\|_t \|b_t\|_t \). So \( \|a \cdot b\| = \sum_{t \in \mathbb{G}} \|a_t b_t\|_t \leq \sum_{t \in \mathbb{G}} \|a_t\|_t \|b_t\|_t = \|a\| \|b\| \).

For \( a = (a_t)_{t \in \mathbb{G}} \) and \( b = (b_t)_{t \in \mathbb{G}} \) in \( A \), we define \( a \ast b \) and \( a + b \) in \( A \) by

\[
(2.1) \quad (a \ast b)_t = \sum_{t_1 + t_2 = t} (a_{t_1} \ast b_{t_2})
\]

and

\[
(2.2) \quad (a + b)_t = \sum_{t_1 + t_2 = t} (a_{t_1} \ast b_{t_2});
\]

Also, we define \( K : A \rightarrow \mathcal{B} \) by

\[
(2.3) \quad K(a) = \sum_{t \in \mathbb{G}} K^t(a_t).
\]

**Theorem 2.2.** When \( \mathbb{G} = \mathbb{N}^* = \{1, 2, \ldots \} \), the operations \( \ast \) and \( \dot{+} \) given by (2.1) and (2.2) and the integral \( \mathcal{K} \) by (2.3) are well defined and satisfies \( \|a + b\| \leq \sum_{n \in \mathbb{G}} (\sum_{t=1}^n \|a_t\|_t) + \sum_{n \in \mathbb{G}} (\sum_{t=1}^n \|b_t\|_t) \) and \( \|a \ast b\| \leq \|a\| \|b\| \) for \( a, b \) in \( A \).

**Proof.** We assume that \( a = c \) and \( b = d \) for \( a, b, c, d \in A \). Then \( a = (a_t)_{t \in \mathbb{G}} \), \( b = (b_t)_{t \in \mathbb{G}} \), \( c = (c_t)_{t \in \mathbb{G}} \) and \( d = (d_t)_{t \in \mathbb{G}} \). So for all \( t \), \( a_t = c_t \) and \( b_t = d_t \) and \( a_t, b_t, c_t, d_t \in A_t \) and so \( \sum_{t_1 + t_2 = t} (a_{t_1} \ast b_{t_2}) = \sum_{t_1 + t_2 = t} (c_{t_1} \ast d_{t_2}) \). Thus \( a \ast b = c \ast d \). Therefore \( \ast \) given by (2.1) is well defined. Similarly \( \dot{+} \) given by (2.2) is well defined. Assume that \( a = b \) for \( a, b \in A \). Then for all \( t \) and \( a_t, b_t \in A_t \), \( a_t = b_t \). Since \( \mathcal{K}^t(a_t) = \mathcal{K}^t(b_t) \) for all \( t \in \mathbb{G} \), \( \mathcal{K}(a) = \mathcal{K}(b) \). Also the following the inequalities.

\[
\|a + b\| = \sum_{t \in \mathbb{G}} \|a + b\|_t = \sum_{t \in \mathbb{G}} \|a_{t_1} \ast b_{t_2}\|_t \\
\leq \sum_{t \in \mathbb{G}} \left( \sum_{t_1 + t_2 = t} \|a_{t_1}\|_{t_1} + \|b_{t_2}\|_{t_2} \right) = \sum_{n \in \mathbb{G}} (\sum_{t=1}^n \|a_t\|_t) + \sum_{n \in \mathbb{G}} (\sum_{t=1}^n \|b_t\|_t).
\]

And

\[
\|a \ast b\| = \sum_{t \in \mathbb{G}} \|a \ast b\|_t = \sum_{t \in \mathbb{G}} \|a_{t_1} \ast b_{t_2}\|_t \\
\leq \sum_{t \in \mathbb{G}} \left( \sum_{t_1 + t_2 = t} \|a_{t_1}\|_{t_1} \|b_{t_2}\|_{t_2} \right) = \left( \sum_{t_1 \in \mathbb{G}} \|a_{t_1}\|_{t_1} \right) \left( \sum_{t_2 \in \mathbb{G}} \|b_{t_2}\|_{t_2} \right) = \|a\| \|b\|.
\]
Let $G = (0, \infty)$ be the additive semigroup of positive real numbers. For $t > 0$, let $A_t$ be the commutative Banach algebra of (equivalence classes of) Wiener functionals in [1] and [2]. Let $C^t = C([0, t], \mathbb{R}^d)$ be the space of continuous paths $x : [0, t] \to \mathbb{R}^d$. In [2] we define $F*G$ and $F+G$ as $\mathbb{C}$-valued functions on $C^{t_1+t_2}$ via the formulas $(F*G)(x) = F(x_1) \cdot G(x_2)$ and $(F+G)(x) = F(x_1) + G(x_2)$ where $x \in C^{t_1+t_2}$, $x_1 \in C^{t_1}$ and $x_2 \in C^{t_2}$ with $x_1(s) = x(s), s \in [0, t_1]$ and $x_2(s) = x(t_1 + s), s \in [0, t_2]$. Finally, given $\lambda \in \mathbb{C}_+$, the nonzero complex numbers with nonnegative real part, and $t > 0$, $K^\lambda := K^\lambda_t : A_t \to \mathcal{L}(L^2(\mathbb{R}^d))$ be the operator valued functional integral defined in [1,2].

**Theorem 2.3.** For every fixed $\lambda \in \mathbb{C}_+, \{\{A_t\}_{t>0}, *, +\}$ and $(K^\lambda)_{t>0}$ in [1] and [2] satisfies axioms (1), (2), (3), (5) where $G = (0, \infty)$ and $\mathcal{B} := \mathcal{L}(L^2(\mathbb{R}^d))$.

**Proof.** Axiom(1) proved in Proposition 5.1 in [2]. In axiom(2) we show that (1, 1). For $a_1, b_1 \in A_{t_1}$, and $a_2, b_2 \in A_{t_2}$, and $x \in C^{t_1+t_2}$, $x_1 \in C^{t_1}$, and $x_2 \in C^{t_2}$ with $x_1(s) = x(s), s \in [0, t_1]$ and $x_2(s) = x(t_1 + s), s \in [0, t_2]$, 

\[
(a_1 + a_2)(b_1 + b_2)(x) = ((a_1(x_1) + a_2(x_2)))(b_1(x_1) + b_2(x_2))
\]

\[
= a_1(x_1)b_1(x_1) + (a_1(x_1)b_2(x_2) + a_2(x_2)b_1(x_1) + a_2(x_2)b_2(x_2)
\]

\[
= (a_1b_1)(x_1) + (a_1 * b_2)(x) + (b_1 * a_2)(x) + (a_2b_2)(x)
\]

\[
= (a_1b_1 + 1_{t_2})(x) + (a_1 + b_2)(x) + (b_1 * a_2)(x) + (1_{t_1} + a_2b_2)(x).
\]

Since

\[
(a_1 * a_2)(b_1 + b_2)(x) = (a_1(x_1)a_2(x_2))(b_1(x_1) + b_2(x_2))
\]

\[
= (a_1(b_1)x_1a_2(x_2) + a_1(x_1)(a_2b_2)(x_2))(b_1(x_1) + b_2(x_2))
\]

\[
= (a_1b_1)(x_1)(a_2b_2)(x_2) + b_1(x_1)(a_2b_2)(x_2) = (a_1b_1 * a_2)(x) + (1_{t_1} * a_2b_2)(x),
\]

we prove the (1.2). Now we show that (1.3).

\[
(a_1 + a_2)(b_1 * b_2)(x) = (a_1(x_1) + a_2(x_2))(b_1(x_1)b_2(x_2)
\]

\[
= (a_1b_1)(x_1)b_2(x_2) + b_1(x_1)(a_2b_2)(x_2) = ((a_1b_1) * b_2)(x) + (b_1 * (a_2b_2))(x).
\]

Next we prove the (1.4).

\[
(a_1 * a_2)(b_1 + b_2)(x) = a_1(x_1)a_2(x_2)b_1(x_1)b_2(x_2)
\]

\[
= (a_1b_1)(x_1)(a_2b_2)(x_2) = (a_1b_1) * (a_2b_2)(x).
\]

Axiom(3) established in Theorem 6.1 in [1] and Theorem 5.2 in [2]. Axiom(5) proved by Corollary 5.2 in [2].
A Note on axiomatic Feynman operational calculus

References


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