THE LATTICE OF INTERVAL-VALUED FUZZY IDEALS
OF A RING

Keon Chang Lee, Kul Hur and Pyung Ki Lim*  

Abstract. We investigate the lattice structure of various sublattices of the lattice of interval-valued fuzzy subrings of a given ring. We prove that a special class of interval-valued fuzzy ideals of a ring. Finally, we show that the lattice of interval-valued fuzzy ideals of $R$ is not complemented[resp. has no atoms(dual atoms)].

1. Introduction and Preliminaries


In this paper, we investigate the lattice of various sublattices of the lattice of interval-valued fuzzy subgroups of a given ring. We prove that a special class of interval-valued fuzzy ideals forms a modular sublattice of the lattice of interval-valued fuzzy ideals of a ring. Finally, we show that the lattice of interval-valued fuzzy ideals of $R$ is not complemented[resp. has no atoms(dual atoms)].

Now, we will list some basic concepts and two results which are needed in the later sections.

Received May 24, 2012. Accepted August 1, 2012.
2000 Mathematics Subject Classification. 03F55, 06C05, 16D25.
Key words and phrases. Lattice, modularity, interval-valued fuzzy subring, interval-valued fuzzy left(right)ideal, (strong) level subset.
* Corresponding Author
This paper was supported by Wonkwang University in 2012.
Let $D(I)$ be the set of all closed subintervals of the unit interval $I = [0, 1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where $M^L$ and $M^U$ are the lower and the upper end points respectively. Especially, we denoted $\emptyset = [0, 0], 1 = [1, 1]$, and $a = [a, a]$ for every $a \in (0, 1)$. We also note that 

(i) $(\forall M, N \in D(I)) (M = N \iff M^L = N^L, M^U = N^U)$, 
(ii) $(\forall M, N \in D(I)) (M \leq N \iff M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the complement of $M$, denoted by $M^C$, is defined by $M^C = 1 - M = [1 - M^U, 1 - M^L]$ (See [6]).

**Definition 1.1** [6, 8]. A mapping $A : X \to D(I)$ is called an interval-valued fuzzy set (in short, IVS) in $X$ and is denoted by $A = [A^L, A^U]$.

Thus for each $x \in X$, $A(x) = [A^L(x), A^U(x)]$, where $A^L(x)$[resp. $A^U(x)$] is called the lower[resp. upper] end point of $x$ to $A$. For any $[a, b] \in D(I)$, the interval-valued fuzzy set $A$ in $X$ defined by $A(x) = [a, b]$ for each $x \in X$ is denoted by $[a, b]$ and if $a = b$, then the IVS $[a, b]$ is denoted by simply $\tilde{a}$. In particular, $\tilde{0}$ and $\tilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy whole set in $X$, respectively.

We will denote the set of all IVSs in $X$ as $D(I)^X$. It is clear that $[A, A] \in D(I)^X$ for each $A \in I^X$.

**Definition 1.2** [6]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then

(a) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.
(b) $A = B$ iff $A \subset B$ and $B \subset A$.
(c) $A^C = [1 - A^U, 1 - A^L]$.
(d) $A \cup B = [A^L \lor B^L, A^U \lor B^U]$.
(d') $\bigcup_{\alpha \in \Gamma} A_\alpha = [\bigvee_{\alpha \in \Gamma} A^L_\alpha, \bigvee_{\alpha \in \Gamma} A^U_\alpha]$.
(e) $A \cap B = [A^L \land B^L, A^U \land B^U]$.
(e') $\bigcap_{\alpha \in \Gamma} A_\alpha = [\bigwedge_{\alpha \in \Gamma} A^L_\alpha, \bigwedge_{\alpha \in \Gamma} A^U_\alpha]$.

**Result 1.A** [6, Theorem 1]. Let $A, B, C \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then

(a) $\emptyset \subset A \subset \tilde{1}$.
(b) $A \cup B = B \cup A$, $A \cap B = B \cap A$.
(c) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.
(d) $A, B \subset A \cup B$, $A \cap B \subset A, B$. 

Keon Chang Lee, Kul Hur and Pyung Ki Lim
The Lattice of Interval-Valued Fuzzy Ideals of a Ring

(e) \( A \cap \left( \bigcup_{\alpha \in \Gamma} A_{\alpha} \right) = \bigcup_{\alpha \in \Gamma} (A \cap A_{\alpha}) \).

(f) \( A \cup \left( \bigcap_{\alpha \in \Gamma} A_{\alpha} \right) = \bigcap_{\alpha \in \Gamma} (A \cup A_{\alpha}) \).

(g) \( (\tilde{0})^c = \tilde{1}, \ (\tilde{1})^c = \tilde{0} \).

(h) \( (A_{\alpha})^c = A \).

(i) \( (\bigcup_{\alpha \in \Gamma} A_{\alpha})^c = \bigcap_{\alpha \in \Gamma} A_{\alpha}^c ; \ (\bigcap_{\alpha \in \Gamma} A_{\alpha})^c = \bigcup_{\alpha \in \Gamma} A_{\alpha}^c \).

It is obvious that \((D(I)^X, \cup, \cap)\) is complete lattice satisfying the De-Morgan’s Laws.

Definition 1.3 [2]. Let \( A \) be an IVS in a group \( G \). Then \( A \) is called an interval-valued fuzzy subgroup (in short, IVG) in \( G \) if it satisfies the conditions : For any \( x, y \in G \),

(a) \( A^L(xy) \geq A^L(x) \land A^L(y) \) and \( A^U(xy) \geq A^U(x) \land A^U(y) \).

(b) \( A^L(x^{-1}) \geq A^L(x) \) and \( A^U(x^{-1}) \geq A^U(x) \).

We will denote the set of all IVGs of \( G \) as \( \text{IVG}(G) \).

Result 1.B [2, Proposition 3.1]. Let \( A \) be an IVG in a group \( G \).

(a) \( A(x^{-1}) = A(x), \forall x \in G \).

(b) \( A^L(e) \geq A^L(x) \) and \( A^U(e) \geq A^U(x), \forall x \in G \), where \( e \) is the identity of \( G \).

Throughout this paper, \( L = (L, +, \cdot) \) denotes a lattice, where “+” and “\( \cdot \)” denote the sup and inf, respectively. For a general background of lattice theory, we refer to [1]. Moreover, we will denote by \( R \) a ring having the zero “0”, with respect to binary operations “+” and “\( \cdot \)”.

2. Lattice of interval-valued fuzzy subrings

Definition 2.1 [5]. Let \( R \) be a ring and let \( A \in D(I)^R \). Then \( A \) is called an interval-valued fuzzy subring (in short, IVR) of \( R \) if it satisfies the following conditions: For any \( x, y \in R \),

(i) \( A^L(x + y) \geq A^L(x) \land A^L(y) \) and \( A^U(x + y) \geq A^U(x) \land A^U(y) \).

(ii) \( A^L(-x) \geq A^L(x) \) and \( A^U(-x) \geq A^U(x) \).

(iii) \( A^L(xy) \geq A^L(x) \land A^L(y) \) and \( A^U(xy) \geq A^U(x) \land A^U(y) \).

We will denote the set of all IVRs of \( R \) as \( \text{IVR}(R) \).
From Result 1.B, it can be easily verified that if \( A \in \text{IVR}(R) \), \( A^L(x) \leq A^L(0) \), \( A^U(x) \leq A^U(0) \) and \( A(x) = A(-x) \) for each \( x \in R \). We shall call \( A(0) \) as the *tip* of the interval-valued fuzzy subring \( A \).

**Result 2.A** [5, Proposition 6.2]. Let \( A \in D(I)^R \). Then \( A \in \text{IVR}(R) \) if and only if for any \( x, y \in R \),

(a) \( A^L(x - y) \geq A^L(x) \wedge A^L(y) \) and \( A^U(x - y) \geq A^U(x) \wedge A^U(y) \).

(b) \( A^L(xy) \geq A^L(x) \wedge A^L(y) \) and \( A^U(xy) \geq A^U(x) \wedge A^U(y) \).

**Proposition 2.2.** Let \( \{A_\alpha\}_{\alpha \in \Gamma} \subseteq \text{IVR}(R) \). Then \( \bigcap_{\alpha \in \Gamma} A_\alpha \in \text{IVR}(R) \).

**Proof.** Let \( A = \bigcap_{\alpha \in \Gamma} A_\alpha \) and let \( x, y \in R \). Then

\[
A^L(x - y) = \bigwedge_{\alpha \in \Gamma} A^L_\alpha(x - y) \geq \bigwedge_{\alpha \in \Gamma} (A^L_\alpha(x) \wedge A^L_\alpha(y)) \quad \text{(Since \( A^\alpha \in \text{IVR}(R) \))}
\]

\[
= \bigwedge_{\alpha \in \Gamma} A^L_\alpha(x) \wedge \bigwedge_{\alpha \in \Gamma} A^L_\alpha(y) = \bigwedge_{\alpha \in \Gamma} A_\alpha(x) \wedge \bigwedge_{\alpha \in \Gamma} A_\alpha(y) = A^L(x) \wedge A^L(y).
\]

By the similar arguments, we have that \( A^U(x - y) \geq A^U(x) \wedge A^U(y) \). Similarly, we have \( A^L(xy) \geq A^L(x) \wedge A^L(y) \) and \( A^U(xy) \geq A^U(x) \wedge A^U(y) \). Hence, by Result 2.A, \( A = \bigcap_{\alpha \in \Gamma} A_\alpha \in \text{IVR}(R) \).

**Definition 2.3.** Let \( A \in \text{IVR}(R) \). Then an interval-valued fuzzy subring *generated* by \( A \) is the least interval-valued fuzzy subring of \( R \) containing \( A \) and denoted by \( (A) \).

Here, we construct the lattice of interval-valued fuzzy subrings such as interval-valued fuzzy (left, right) ideals. The common feature of all these constructions is that the intersection of an arbitrary family of interval-valued fuzzy subrings is always an interval-valued fuzzy subring (See proposition 2.2). Therefore, we consider the \( \text{inf} \) of a family of interval-valued fuzzy subrings to be their intersection, whereas the union of two interval-valued fuzzy subrings may not be an interval-valued fuzzy subring. Hence, we shall always be talking the \( \text{sup} \) of a family of interval-valued fuzzy subrings to be the interval-valued fuzzy subring generated by the union of that family. The outcome of the above discussion can be described can be described by the following propositions.
Proposition 2.4. IVR(R) forms a complete lattice under the ordering of interval-valued fuzzy set inclusion ⊆.

Definition 2.5 [5]. Let \(A \in \text{IVR}(R)\). Then \(A\) is called an:

1. interval-valued fuzzy left ideal (in short, IVLI) in \(R\) if \(A^L(xy) \geq A^L(y)\) and \(A^U(xy) \geq A^U(y)\) for any \(x, y \in R\).

2. interval-valued fuzzy right ideal (in short, IVRI) in \(X\) if \(A^L(xy) \geq A^L(x)\) and \(A^U(xy) \geq A^U(x)\) for any \(x, y \in R\).

3. interval-valued fuzzy ideal (in short, IVI) in \(X\) if it both an IVLI and an IVRI in \(R\).

We will denote the set of all IVIs [resp. IVLIs and IVRIs] of \(R\) as \(\text{IVI}(R)\) [resp. \(\text{IVLI}(R)\) and \(\text{IVRI}(R)\)]. In particular, \(\text{IVI}([\lambda_0, \mu_0])\) denotes the set of all IVIs with the same tip "[\(\lambda_0, \mu_0]\)". It is clear that \(\text{IVI}(R) = \text{IVLI}(R) \cap \text{IVRI}(R)\).

Result 2.B [5, Proposition 6.5]. Let \(A \in D(I)^R\). Then \(A \in \text{IVI}(R)\) [resp. \(\text{IVLI}(R)\) and \(\text{IVRI}(R)\)] if and only if for any \(x, y \in R\),

(a) \(A^L(x - y) \geq A^L(x) \land A^L(y)\) and \(A^U(x - y) \geq A^U(x) \land A^U(y)\).

(b) \(A^L(xy) \geq A^L(x) \lor A^L(y)\) and \(A^U(xy) \geq A^U(x) \lor A^U(y)\) [resp. \(A^L(xy) \geq A^L(y), A^U(xy) \geq A^U(y)\) and \(A^L(xy) \geq A^L(x), A^U(xy) \geq A^U(x)\)].

The proof of the following result is similar to Proposition 2.2.

Proposition 2.6. Let \(\{A_\alpha\}_{\alpha \in \Gamma} \subset \text{IVI}(R)\) [resp. \(\text{IVLI}(R)\) and \(\text{IVRI}(R)\)] if and only if for any \(x, y \in R\),

(a) \(A^L(x - y) \geq A^L(x) \land A^L(y)\) and \(A^U(x - y) \geq A^U(x) \land A^U(y)\).

(b) \(A^L(xy) \geq A^L(x) \lor A^L(y)\) and \(A^U(xy) \geq A^U(x) \lor A^U(y)\) [resp. \(A^L(xy) \geq A^L(y), A^U(xy) \geq A^U(y)\) and \(A^L(xy) \geq A^L(x), A^U(xy) \geq A^U(x)\)].

Definition 2.7. Let \(A \in D(I)^R\). Then the IVI [resp. IVLI and IVRI] generated by \(A\) is the least IVI [resp. IVLI and IVRI] of \(R\) containing \(A\) and denoted by \((A)\).

The following is easily verified.

Proposition 2.8. (a) \(\text{IVI}(R)\) [resp. \(\text{IVLI}(R)\) and \(\text{IVRI}(R)\)] is a meet complete sublattice of \(\text{IVR}(R)\).

(b) \(\text{IVI}([\lambda, \mu])\) is a complete sublattice of \(\text{IVR}(R)\).

Definition 2.9[2,5]. Let \(X\) be a set and let \(A \in D(I)^X\). Then \(A\) is said to have the sup-property if for each \(Y \in P(X)\), there exists \(y_0 \in Y\) such that \(A(y_0) = [\bigvee_{x \in Y} A^L(x), \bigvee_{x \in Y} A^U(x)]\), where \(P(X)\) denotes the
power set of $X$.

**Definition 2.10.** Let $A, B \in D(I)^R$. Then the *sum* $A + B$ and the *product* $A \circ B$ of $A$ and $B$ are defined as follows, respectively: For each $z \in R$,

(i) $(A + B)(z) = \bigvee_{z=x+y} (A^L(x) \land B^L(y)), \bigvee_{z=x+y} (A^L(x) \land B^U(y))$,

(ii) $(A \circ B)(z) = \begin{cases} \bigvee_{z=xy} (A^L(x) \land B^L(y)), \bigvee_{z=xy} (A^U(x) \land B^U(y)) & \text{if } z = xy; \\ [0,0] & \text{otherwise.} \end{cases}$

**Definition 2.11.** Let $X$ be a set, let $A \in D(I)^X$ and let $[\lambda, \mu] \in D(I)$.

(i) [5] The set $A_{[\lambda, \mu]} = \{ x \in X : A^L(x) \geq \lambda \text{ and } A^U(x) \geq \mu \}$ is called a $[\lambda, \mu]$-level-subset of $A$.

(ii) The set $A^*_{[\lambda, \mu]} = \{ x \in X : A^L(x) > \lambda \text{ and } A^U(x) > \mu \}$ is called a strong $[\lambda, \mu]$-level-subset of $A$.

The following is the immediate result of Propositions 4.16 and 4.17 in [5].

**Theorem 2.12.** Let $A \in D(I)^R$. Then $A \in IVI(R)$ if and only if $A_{[\lambda, \mu]}$ is an ideal for each $[\lambda, \mu] \in D(I)$ with $\lambda \leq A^L(0)$ and $\mu \leq A^U(0)$.

**Lemma 2.13.** Let $A, B \in IVI(R)$. If $A$ and $B$ have the sup-property, then $(A + B)_{[\lambda, \mu]} = A_{[\lambda, \mu]} + B_{[\lambda, \mu]}$ for each $[\lambda, \mu] \in D(I)$.

**Proof.** Let $z \in (A + B)_{[\lambda, \mu]}$. Then

$$(A + B)^L(z) = \bigvee_{z=x+y} (A^L(x) \land B^L(y))$$

and

$$(A + B)^U(z) = \bigvee_{z=x+y} (A^U(x) \land B^U(y)).$$

For each decomposition $z = x + y$, we have either $A^L(x) \leq B^L(y)$ and $A^U(x) \leq B^U(y)$ or $A^L(x) \geq B^L(y)$ and $A^U(x) \geq B^U(y)$. This contradiction leads as to define the following subsets of $R$:
\[ X(z) = \{ x \in R : z = x + y \text{ for some } y \in R \text{ such that } A^L(x) \leq B^L(y) \text{ and } A^U(x) \leq B^U(y) \}, \]

\[ Y(z) = \{ x \in R : z = x + y \text{ for some } y \in R \text{ such that } A^L(x) \geq B^L(y) \text{ and } A^U(x) \geq B^U(y) \}, \]

\[ X^*(z) = \{ x \in R : z = x + y \text{ for some } y \in R \text{ such that } A^L(x) \geq B^L(y) \text{ and } A^U(x) \geq B^U(y) \}. \]

Then clearly \( R = X(z) \cup X^*(z) \). Since \( A \) and \( B \) have the sup-property, there exist \( x_0 \in X(z) \) and \( y_0 \in Y(z) \) such that

\[ A^L(x_0) = \bigvee_{x \in X(z)} A^L(x), \quad A^U(x_0) = \bigvee_{x \in X(z)} A^U(x) \]

and

\[ B^L(y_0) = \bigvee_{y \in Y(z)} B^L(y), \quad B^U(y_0) = \bigvee_{y \in Y(z)} B^U(y). \] (2.2)

Since \( x_0 \in X(z) \), there exists \( y_0' \in R \) with \( z = x_0 + y_0' \) such that \( A^L(x_0) \leq B^L(y_0') \) and \( A^U(x_0) \leq B^U(y_0') \).

Since \( y_0 \in Y(z) \), there exists \( x_0' \in R \) with \( z = x_0' + y_0 \) such that \( A^L(x_0') \geq B^L(y_0) \) and \( A^U(x_0') \geq B^U(y_0) \).

But for \( A(x_0) \) and \( B(y_0) \), we have either \( A^L(x_0) \geq B^L(y_0) \) and \( A^U(x_0) \geq B^U(y_0) \) or \( A^L(x_0) \leq B^L(y_0) \) and \( A^U(x_0) \leq B^U(y_0) \).

Case (i): Suppose \( A^L(x_0) \geq B^L(y_0) \) and \( A^U(x_0) \geq B^U(y_0) \). Then

\[
\bigvee_{z=x+y} (A^L(x) \land B^L(y)) = \bigvee_{x \in R} (A^L(x) \land B^L(z - x)) \quad \text{(Since } y = z - x) \\
= ( \bigvee_{x \in X(z)} (A^L(x) \land B^L(z - x))) \lor ( \bigvee_{x \in X^*(z)} (A^L(x) \land B^L(z - x))) \quad \text{(Since } R = X(z) \cup X^*(z)) \\
= ( \bigvee_{x \in X(z)} (A^L(x) \land B^L(y))) \lor ( \bigvee_{x \in Y(z)} (A^L(x) \land B^L(z - x))) \\
= ( \bigvee_{x \in X(z)} A^L(x)) \lor ( \bigvee_{x \in Y(z)} B^L(y)) \\
= A^L(x_0) \land B^L(y_0) \quad \text{(By (2.2))} \\
= A^L(x_0). \quad \text{(By the hypothesis)}
\]

Similarly, we have that \( \bigvee_{z=x+y} (A^U(x) \land B^U(y)) = A^U(x_0) \). Thus, by (2.1), \( A^L(x_0) = (A + B)^L(z) \geq \lambda \) and \( A^U(x_0) = (A + B)^U(z) \geq \mu \). So \( x_0 \in A_{[\lambda,\mu]} \). Since \( B^L(y_0') \geq A^L(x_0) \) and \( B^U(y_0') \geq A^U(x_0) \), \( B^L(y_0') \geq \lambda \) and \( B^U(y_0') \geq \mu \). Then \( y_0' \in B_{[\lambda,\mu]} \). Thus \( z = x_0 + y_0 \in A_{[\lambda,\mu]} + B_{[\lambda,\mu]} \).
Thus there exist $x$ and $B$. Proof. (Lemma 2.14.) Let $A, B \in D(I)^R$. Then as in Case (i), it follows that $x_0 \in A_{[\lambda, \mu]}$ and $y_0 \in B_{[\lambda, \mu]}$. Thus $z = x_0 + y_0 \in A_{[\lambda, \mu]} + B_{[\lambda, \mu]}$. So, in either case, $z \in A_{[\lambda, \mu]} + B_{[\lambda, \mu]}$. Hence $(A + B)_{[\lambda, \mu]} \subseteq A_{[\lambda, \mu]} + B_{[\lambda, \mu]}$. Now let $z \in A_{[\lambda, \mu]} + B_{[\lambda, \mu]}$. Then there exist $x_0 \in A_{[\lambda, \mu]}$ and $y_0 \in B_{[\lambda, \mu]}$ such that $z = x_0 + y_0$. Thus $A_L(x_0) \geq \lambda, A_U(x_0) \geq \mu$ and $B_L(x_0) \geq \lambda, B_U(x_0) \geq \mu$. So

$$(A + B)^L(z) = \bigvee_{z=x+y} (A^L(x) \land B^L(y)) \geq \lambda$$

and

$$(A + B)^U(z) = \bigvee_{z=x+y} (A^U(x) \land B^U(y)) \geq \mu.$$  

Thus $z \in (A + B)_{[\lambda, \mu]}$. Hence $A_{[\lambda, \mu]} + B_{[\lambda, \mu]} \subseteq (A + B)_{[\lambda, \mu]}$. Therefore $(A + B)_{[\lambda, \mu]} = A_{[\lambda, \mu]} + B_{[\lambda, \mu]}$. This completes the proof. \hfill \Box

**Lemma 2.14.** Let $A, B \in D(I)^R$. and let $[\lambda, \mu] \in D(I)$. Then $(A + B)_{[\lambda, \mu]} = A^*_{[\lambda, \mu]} + B^*_{[\lambda, \mu]}$.

**Proof.** Suppose $(A + B)_{[\lambda, \mu]} = \emptyset$. Then clearly $(A + B)_{[\lambda, \mu]}^* \subseteq A^*_{[\lambda, \mu]} + B^*_{[\lambda, \mu]}$. Suppose $(A + B)_{[\lambda, \mu]}^* \neq \emptyset$ and let $z \in (A + B)_{[\lambda, \mu]}^*$. Then

$$(A + B)^L(z) = \bigvee_{z=x+y} (A^L(x) \land B^L(y)) > \lambda$$

and

$$(A + B)^U(z) = \bigvee_{z=x+y} (A^U(x) \land B^U(y)) > \mu.$$  

Thus there exist $x_0, y_0 \in R$ with $z = x_0 + y_0$ such that $A^L(x_0) \land B^L(y_0) > \lambda$ and $A^U(x_0) \land B^U(y_0) > \mu$. So $A^L(x_0) > \lambda, A^U(x_0) > \mu$ and $B^L(y_0) > \lambda, B^U(y_0) > \mu$. Then $x_0 \in A^*_{[\lambda, \mu]}$ and $y_0 \in B^*_{[\lambda, \mu]}$. Thus $z = x_0 + y_0 \in A^*_{[\lambda, \mu]} + B^*_{[\lambda, \mu]}$. So $(A + B)_{[\lambda, \mu]}^* \subseteq A^*_{[\lambda, \mu]} + B^*_{[\lambda, \mu]}$.

Now for each $[\lambda, \mu] \in D(I)$, suppose

$$\left( \bigvee_{x \in R} A^L(x) \right) \land \left( \bigvee_{y \in R} B^L(y) \right) \leq \lambda$$

and

$$\left( \bigvee_{x \in R} A^U(x) \right) \land \left( \bigvee_{y \in R} B^U(y) \right) \leq \mu.$$
Then one of $A^*_{[\lambda,\mu]}$ and $B^*_{[\lambda,\mu]}$ is $\emptyset$. Thus $A^*_{[\lambda,\mu]} + B^*_{[\lambda,\mu]} = \emptyset \subset (A+B)^*_{[\lambda,\mu]}$. Otherwise, $A^*_{[\lambda,\mu]} \neq \emptyset$ and $B^*_{[\lambda,\mu]} \neq \emptyset$. Then $A^*_{[\lambda,\mu]} + B^*_{[\lambda,\mu]} \neq \emptyset$. Let $z \in A^*_{[\lambda,\mu]} + B^*_{[\lambda,\mu]}$. Then it exist $x_0 \in A^*_{[\lambda,\mu]}$ and $y_0 \in B^*_{[\lambda,\mu]}$ such that $z = x_0 + y_0$ and

$$(A+B)^L(z) = \bigvee_{z=x+y} (A^L(x) \wedge B^L(y)) \geq A^L(x_0) \wedge B^L(y_0) > \lambda$$

and

$$(A+B)^U(z) = \bigvee_{z=x+y} (A^U(x) \wedge B^U(y)) \geq A^U(x_0) \wedge B^U(y_0) > \mu.$$  

Thus $z \in (A+B)_{[\lambda,\mu]}$. So $A^*_{[\lambda,\mu]} + B^*_{[\lambda,\mu]} \subset (A+B)_{[\lambda,\mu]}$. Hence $(A+B)_{[\lambda,\mu]} = A^*_{[\lambda,\mu]} + B^*_{[\lambda,\mu]}$. This completes the proof. \hfill \Box

**Theorem 2.15.** Let $A \in D(I)^R$. Then $A \in I(VI(R))$ if and only if $A^*_{[\lambda,\mu]} = \emptyset$ or $A^*_{[\lambda,\mu]} \in I(R)$[resp. $LI(R)$ and $RI(R)$] for each $[\lambda,\mu] \in D(I)$, where $I(R)$[resp. $LI(R)$ and $RI(R)$] denotes the set of all ideals[resp. left ideals and right ideals] of $R$.

**Proof.** We prove this lemma for left ideal, since other cases are similar. It is clear that $A = 0$ if and only if $A^*_{[\lambda,\mu]} = \emptyset$ for each $[\lambda,\mu] \in D(I)$. Now we assume that $A \neq 0$. 

$(\Rightarrow)$: Suppose $A \in I(VI(R))$ and let $[\lambda,\mu] \in D(I)$. Let $x, y \in A^*_{[\lambda,\mu]}$ and let $z \in R$. Then

$$A^L(x - y) \geq A^L(x) \wedge A^L(y) \quad \text{(Since } A \in I(VI(R))\text{)}$$

$$> \lambda \quad \text{(Since } x, y \in A^*_{[\lambda,\mu]}\text{)}$$

and

$$A^L(x - y) \geq A^L(x) \wedge A^L(y) > \mu.$$  

Also,

$$A^L(zx) \geq A^L(x) \quad \text{(Since } A \in I(VI(R))\text{)}$$

$$> \lambda \quad \text{(Since } x \in A^*_{[\lambda,\mu]}\text{)}$$

and

$$A^U(zx) \geq A^U(x) > \mu.$$ 

Thus $x - y \in A^*_{[\lambda,\mu]}$ and $zx \in A^*_{[\lambda,\mu]}$. Hence $A^*_{[\lambda,\mu]} \in LI(R)$.

$(\Leftarrow)$: Suppose the necessary condition holds. For any $x, y \in R$, let $A(x) = [\lambda,\mu]$ and let $A(y) = [s, t]$ such that $\lambda \leq s$ and $\mu \leq t$. 

Case (i): Suppose $[\lambda, \mu] = [0, 0]$. Then
\[ A^L(x - y) \geq \lambda = A^L(x) \land A^L(y), A^U(x - y) \geq \mu = A^U(x) \land A^U(y) \]
and
\[ A^L(zx) \geq \lambda = A^L(x) \land A^U(zx) \geq \mu = A^U(x), \text{ for each } z \in R. \]
Thus $A \in \text{IVLI}(R)$.

Case (ii): Suppose $[\lambda, \mu] \neq [0, 0]$. For each $\epsilon > 0$, let $\epsilon < \lambda$. Then we have
\[ A^L(y) > s - \epsilon \geq \lambda - \epsilon, A^U(y) > t - \epsilon \geq \mu - \epsilon. \]
and
\[ A^L(x) > \lambda - \epsilon, A^U(x) > \mu - \epsilon. \]
Thus $x, y \in A^*_{[\lambda - \epsilon, \mu - \epsilon]}$. By the hypothesis, $A^*_{[\lambda - \epsilon, \mu - \epsilon]} \in \text{I}(R)$. So $x - y \in A^*_{[\lambda - \epsilon, \mu - \epsilon]}$ and $zx \in A^*_{[\lambda - \epsilon, \mu - \epsilon]}$ for each $z \in R$. Then $A^L(x - y) \geq \lambda - \epsilon, A^U(x - y) > \mu - \epsilon$ and $A^L(zx) \geq \lambda - \epsilon, A^U(zx) > \mu - \epsilon$ for each $z \in R$. Since $\epsilon$ is an arbitrary, $A^L(x - y) \geq \lambda = A^L(x) \land A^L(y)$, $A^U(x - y) \geq \mu = A^U(x) \land A^U(y)$ and $A^L(zx) \geq \lambda = A^L(x)$, $A^U(zx) \geq \mu = A^U(x)$. Hence $A \in \text{IVLI}(R)$. This completes the proof.

**Proposition 2.16.** IVLI$_{[\lambda_0, \mu_0]}(R)$, IVRI$_{[\lambda_0, \mu_0]}(R)$ and IVI$_{[\lambda_0, \mu_0]}(R)$ are sublattices of IVR(R) and for any $A, B \in \text{IVLI}_{[\lambda_0, \mu_0]}(R)$[resp. IVRI$_{[\lambda_0, \mu_0]}(R)$ and IVI$_{[\lambda_0, \mu_0]}(R)$], $A \lor B = A + B$.

**Proof.** It is easy to see that IVLI$_{[\lambda_0, \mu_0]}(R)$, IVRI$_{[\lambda_0, \mu_0]}(R)$ and IVI$_{[\lambda_0, \mu_0]}(R)$ are sublattices of IVR(R). We do only prove that $A \lor B = A + B$ for any $A, B \in \text{IVLI}_{[\lambda_0, \mu_0]}(R)$ (For IVRI$_{[\lambda_0, \mu_0]}(R)$ and IVI$_{[\lambda_0, \mu_0]}(R)$, the proofs are similar). Let $z \in R$. Then
\[ (A + B)^L(z) = \bigvee_{z = x + y} (A^L(x) \land B^L(y)) \leq A^L(0) \land B^L(0) = \lambda_0 \]
and
\[ (A + B)^U(z) = \bigvee_{z = x + y} (A^U(x) \land B^U(y)) \leq A^U(0) \land B^U(0) = \mu_0. \]
Thus $\bigvee_{z = x + y} (A + B)^L(z) \leq \lambda_0$ and $\bigvee_{z = x + y} (A + B)^U(z) \leq \mu_0$. On the other hand, $\bigvee_{z \in R}(A + B)^L(z) \geq A^L(0) \land B^L(0) = \lambda_0$. By the similar arguments, we have that $\bigvee_{z \in R}(A + B)^U(z) \geq \mu_0$. So
\[
\bigvee_{z \in R} (A + B)^L(z), \bigvee_{z \in R} (A + B)^U(z) = (A + B)(0) = [\lambda_0, \mu_0]. (2.3)
\]

For each \([\lambda_0, \mu_0] \in D(I_1)\) with \(\lambda < \lambda_0\) and \(\mu < \mu_0\), \((A + B)^*_{[\lambda, \mu]} \neq \emptyset\).

By Lemma 2.14, \((A + B)^*_{[\lambda, \mu]} = A_{[\lambda, \mu]}^* + B_{[\lambda, \mu]}^*\). Since \(A, B \in \text{IVLI}(R)\), by Theorem 2.15, \(A_{[\lambda, \mu]}^*, B_{[\lambda, \mu]}^* \in \text{LI}(R)\). Thus \((A + B)^*_{[\lambda, \mu]} \in \text{LI}(R)\). So, by Theorem 2.15, \(A + B \in \text{IVLI}(R)\). Moreover,

\[
A + B \in \text{IVLI}_{[\lambda_0, \mu_0]}(R). (2.4)
\]

Let \(z \in R\). Then

\[
(A + B)^L(z) = \bigvee_{z = x + y} (A^L(x) \land B^L(y)) \geq A^L(z) \land B^L(0) = A^L(z)
\]

and

\[
(A + B)^U(z) = \bigvee_{z = x + y} (A^U(x) \land B^U(y)) \geq A^U(z) \land B^U(0) = B^U(z).
\]

Thus \(A \subseteq A + B\). By the similar arguments, we have \(B \subseteq A + B\). So

\[
A \subset A + B \text{ and } B \subset A + B. \quad (2.5)
\]

Now let \(C \in \text{IVLI}(R)\) such that \(A \subseteq C\) and \(B \subseteq C\) and let \(z \in R\). Then

\[
(A + B)^L(z) = \bigvee_{z = x + y} (A^L(x) \land B^L(y)) \leq \bigvee_{z = x + y} (C^L(x) \land C^L(y)) \\
\leq \bigvee_{z = x + y} C^L(z) \text{ (Since } C^L(z) = C^L(x + y) \geq C^L(x) \land C^L(y) \text{)} \\
= C^L(z).
\]

Similarly, we have that \((A + B)^U(z) \geq C^U(z)\). Thus

\[
A + B \subset C. \quad (2.6)
\]

Hence, by (2.3),(2.4), (2.5) and (2.6), \(A + B = A \lor B\). This completes the proof.

\textbf{Remark 2.16.} (a) \(A \lor B = A + B\) is not true in \(\text{IVR}(R)\), \(\text{IVLI}(R)\), \(\text{IVRI}(R)\) and \(\text{VI}(R)\) (See Example 2.17).

(b) As well-known, \(S + T\) is not subring in general, where \(S\) and \(T\) are subrings of \(R\). Hence \(A \lor B = A + B\) is not true in \(\text{IVR}_{[\lambda_0, \mu_0]}(R)\) (See Example 2.18).
Example 2.17. We define two mappings \( A : \mathbb{R} \to D(I) \) and \( B : \mathbb{R} \to D(I) \) as follows, respectively: For each \( x \in \mathbb{R} \),
\[
A(x) = [0.4, 0.5] \quad \text{and} \quad B(x) = [0.3, 0.6].
\]
Then clearly \( A, B \in \text{IVR}(\mathbb{R}) \) [resp. IVLI(\mathbb{R}), IVRI(\mathbb{R}) and IVI(\mathbb{R})]. Moreover, it is easy to see that \( (A + B)(0) = [0.3, 0.6] \) and \( (A \lor B)(0) = [0.4, 0.5] \).

Example 2.18. Let \( R = \{(a, b) : a, b \in \mathbb{Z}\} \), where \( \mathbb{Z} \) is the ring of integers. We define the additive operation and the multiplicative operation on \( R \) as follows, respectively: For any \( (a, b) \), \((c, d) \) \( \in R \),
\[
(a, b) + (c, d) = (a + c, b + d) \quad \text{and} \quad (a, b) \cdot (c, d) = (0, 0).
\]
Then \( (R, +, \cdot) \) forms a ring with zero \((0, 0)\). Now we define three mappings \( A, B, C : \mathbb{R} \to D(I) \) as follows, respectively: For each \( (x, y) \in R \),
\[
A(x, y) = \begin{cases}
(\frac{1}{3}, \frac{3}{5}], & \text{if } y = 0; \\
[0, 0], & \text{if } y \neq 0.
\end{cases}
\]
\[
B(x, y) = \begin{cases}
(\frac{1}{3}, \frac{3}{5}], & \text{if } x = 0; \\
[0, 0], & \text{if } x \neq 0.
\end{cases}
\]
\[
C(x, y) = \begin{cases}
(\frac{1}{3}, \frac{3}{5}], & \text{if } x = y; \\
[0, 0], & \text{if } x \neq y.
\end{cases}
\]
Then it is easy to see that \( A, B, C \in \text{IVI}(\mathbb{R}) \). Let \((x, y) \in R \). Then
\[
(A + B)^L(x, y) = \bigvee_{(x,y)=(x_1,y_1)+(x_2,y_2)} (A^L(x_1, y_1) \land B^L(x_2, y_2))
\]
\[
\leq A^L(x, 0) \land B^L(0, y) = \frac{1}{3}
\]
and
\[
(A + B)^U(x, y) = \bigvee_{(x,y)=(x_1,y_1)+(x_2,y_2)} (A^U(x_1, y_1) \land B^U(x_2, y_2))
\]
\[
\leq A^U(x, 0) \land B^U(0, y) = \frac{3}{5}
\]
Thus \((C \land (A + B))^L(x, y) = C^L(x, y) \land (A + B)^L(x, y) = C^L(x, y) \land (A + B)^L(x, y) = C^L(x, y) \) and \((C \land (A + B))^U(x, y) = C^U(x, y) \land (A + B)^U(x, y) = C^U(x, y) \). So \( C \land (A + B) = C \). On the other hand,
\[(C \land A)^L(x, y) = C^L(x, y) \land A^L(x, y) = \begin{cases} 
\frac{1}{3}, & \text{if } (x, y) = (0, 0); \\
0, & \text{if } (x, y) \neq (0, 0). 
\end{cases}\]

and

\[(C \land A)^U(x, y) = C^U(x, y) \land A^U(x, y) = \begin{cases} 
\frac{3}{5}, & \text{if } (x, y) = (0, 0); \\
0, & \text{if } (x, y) \neq (0, 0). 
\end{cases}\]

Also,

\[(C \land B)^L(x, y) = C^L(x, y) \land B^L(x, y) = \begin{cases} 
\frac{1}{3}, & \text{if } (x, y) = (0, 0); \\
0, & \text{if } (x, y) \neq (0, 0). 
\end{cases}\]

and

\[(C \land B)^U(x, y) = C^U(x, y) \land B^U(x, y) = \begin{cases} 
\frac{3}{5}, & \text{if } (x, y) = (0, 0); \\
0, & \text{if } (x, y) \neq (0, 0). 
\end{cases}\]

Thus

\[((C \land A) + (C \land B))^L(x, y) = \begin{cases} 
\frac{1}{3}, & \text{if } (x, y) = (0, 0); \\
0, & \text{if } (x, y) \neq (0, 0). 
\end{cases}\]

and

\[((C \land A) + (C \land B))^U(x, y) = \begin{cases} 
\frac{3}{5}, & \text{if } (x, y) = (0, 0); \\
0, & \text{if } (x, y) \neq (0, 0). 
\end{cases}\]

So \(C \land (A+B) \neq (C \land A) + (C \land B)\). Hence \(\text{IVI}(R)\) is not distributive. \(\square\)

**Lemma 2.19.** Let \(A, B \in \text{IVI}(R)\). If \(A\) and \(B\) have the sup-property, then the following holds:

(a) \(A + B\) has the sup-property.

(b) \(A \cap B\) has the sup-property.
Proof. (a) Let $S$ be any subset of $R$. Then

$$\bigvee_{z \in S} (A + B)^L(z) = \bigvee_{z \in S} (\bigvee_{z = x + y} (A^L(x) \wedge B^L(y)))$$

$$= \bigvee_{z \in S, z = x + y} (A^L(x) \wedge B^L(y))$$

Similarly, we have that

$$\bigvee_{z \in S} (A + B)^U(z) = \bigvee_{z \in S} (\bigvee_{z = x + y} (A^U(x) \wedge B^U(y)))$$

$$= \bigvee_{z \in S, z = x + y} (A^U(x) \wedge B^U(y)).$$

Let us define two subsets $X(S)$ and $Y(S)$ of $R$ by

$X(S) = \{x \in R : z \in S, z = x + y \text{ for some } y \in R \text{ such that } A^L(x) \leq B^L(y) \text{ and } A^U(x) \leq B^U(y)\}$,

$Y(S) = \{y \in R : z \in S, z = x + y \text{ for some } x \in R \text{ such that } A^L(x) \geq B^L(y) \text{ and } A^U(x) \geq B^U(y)\}$.

Since $A$ and $B$ have the sup-property, there exist $x' \in X(S)$ and $y'' \in Y(S)$ such that

$$A^L(x') = \bigvee_{x \in X(S)} (A^L(x), A^U(x')) = \bigvee_{x \in X(S)} A^U(x)$$

and

$$B^L(y'') = \bigvee_{y \in Y(S)} (B^L(y), B^U(y'')) = \bigvee_{y \in Y(S)} B^U(y).$$

(2.7)

Since $x' \in X(S)$, there exists $z_1 \in S$ such that $z_1 = x' + y'$ for some $y' \in R$ satisfying $A^L(x') \leq B^L(y')$ and $A^U(x') \leq B^U(y')$. Also, since $y'' \in Y(S)$, there exists $z_2 \in S$ such that $z_2 = x'' + y''$ for some $x'' \in R$ satisfying $A^L(x'') \geq B^L(y'')$ and $A^U(x'') \geq B^U(y'')$.

On the other hand, we have either $A^L(x') \geq B^L(y'')$, $A^U(x') \geq B^U(y'')$ or $A^L(x') \leq B^L(y'')$, $A^U(x') \leq B^U(y'')$. 

Similarly, we have that \( A^L(x') \geq B^L(y'') \) and \( A^U(x') \geq B^U(y'') \). Then
\[
\bigvee_{x \in S, z = x + y} (A^L(x) \land B^L(y)) = \bigvee_{x \in X(S)} (A^L(x) \land B^L(y)) \cup \bigvee_{y \in Y(S)} (A^L(x) \land B^L(y))
\]
(As in Lemma 2.13)
\[
= ( \bigvee_{x \in X(S)} A^L(x)) \cup ( \bigvee_{y \in Y(S)} B^L(y))
\]
\[
= A^L(x') \lor B^L(y') \quad \text{(By (2.7))}
\]
\[
= A^L(x'). \quad \text{(By the hypothesis)}
\]
Similarly, we have that \( \bigvee_{x \in S, z = x + y} (A^U(x) \land B^U(y)) = A^U(x'). \)

and
\[
\bigvee_{x \in S} (A + B)^L(z) = A^L(x') \quad \text{(2.8)}
\]

Now we show that
\[
\bigvee_{x \in S} (A + B)^L(z) = (A + B)^L(z_1) \quad \text{and} \quad \bigvee_{x \in S} (A + B)^U(z) = (A + B)^U(z_1).
\]
For decompositions \( z_1 = x'_i + y'_i \), we have
\[
(A + B)^L(z_1) = \bigvee_{x = x'_i + y'_i} (A^L(x'_i) \land B^L(y'_i))
\]
and
\[
(A + B)^U(z_1) = \bigvee_{x = x'_i + y'_i} (A^U(x'_i) \land B^U(y'_i)).
\]
Again, we construct subset \( X(z_1) \) and \( Y(z_1) \) of \( R \) as follows: \( X(z_1) = \{x'_i \in R : z_1 = x'_i + y'_i \text{ for some } y'_i \in R \text{ such that } A^L(x'_i) \leq B^L(y'_i) \text{ and } A^U(x'_i) \leq B^U(y'_i) \} \),
\[
Y(z_1) = \{y'_i \in R : z_1 = x'_i + y'_i \text{ for some } x'_i \in R \text{ such that } A^L(x'_i) \geq B^L(y'_i) \text{ and } A^U(x'_i) \geq B^U(y'_i) \}.
\]
Then
\[
\bigvee_{x = x'_i + y'_i} (A^L(x'_i) \land B^L(y'_i)) = ( \bigvee_{x'_i \in X(z_1)} A^L(x'_i) \land B^L(y'_i)) \cup ( \bigvee_{x'_i \in X(z_1)} A^L(x'_i) \land B^L(y'_i)) \quad \text{(As in Lemma 2.13)}
\]
\[
= ( \bigvee_{x'_i \in X(z_1)} A^L(x'_i)) \cup ( \bigvee_{y'_i \in Y(z_1)} B^L(y'_i)).
\]
By the similar arguments, we have that

\[
\bigvee_{z_1 = x'_i + y'_i} (A^U(x'_i) \land B^U(y'_i)) = (\bigvee_{x'_i \in X(z_1)} A^U(x'_i)) \land (\bigvee_{y'_i \in Y(z_1)} (B^U(y'_i))).
\]

Since \(X(z_1) \subset X(S)\) and \(x'_i \in X(z_1)\),

\[
A^L(x') \leq \bigvee_{x'_i \in X(z_1)} A^L(x'_i) \leq \bigvee_{x \in X(S)} A^L(x) = A^L(x'_i)
\]

and

\[
A^U(x') \geq \bigvee_{x'_i \in X(z_1)} A^U(x'_i) \geq \bigvee_{x \in X(S)} A^U(x) = A^U(x'_i).
\]

Thus

\[
\bigvee_{x'_i \in X(z_1)} A^L(x'_i) = A^L(x'_i) \text{ and } \bigvee_{x'_i \in X(z_1)} A^U(x'_i) = A^U(x'_i).
\]

Also, since \(Y(z_1) \subset Y(S)\) and \(y'' \in Y(z_1)\), we have

\[
\bigvee_{y'_i \in Y(z_1)} B^L(y'_i) = B^L(y''_i) \text{ and } \bigvee_{y'_i \in Y(z_1)} B^U(y'_i) = B^U(x''_i).
\]

By the hypothesis,

\[
\bigvee_{x'_i \in X(z_1)} A^L(x'_i) \geq B^L(y''_i) = \bigvee_{y'_i \in Y(z_1)} B^L(y'_i)
\]

and

\[
\bigvee_{x'_i \in X(z_1)} A^U(x'_i) \geq B^U(y''_i) = \bigvee_{y'_i \in Y(z_1)} B^U(y'_i).
\]

Thus

\[
(A + B)^L = (\bigvee_{x'_i \in X(z_1)} A^L(x'_i)) \lor (\bigvee_{y'_i \in Y(z_1)} B^L(y'_i)) = A^L(x')
\]

and

\[
(A + B)^U = (\bigvee_{x'_i \in X(z_1)} A^U(x'_i)) \lor (\bigvee_{y'_i \in Y(z_1)} B^U(y'_i)) = A^U(x').
\]

So, by (2.8) and (2.9),

\[
\bigvee_{z \in S} (A + B)^L(z) = (A + B)^L(z_1) \text{ and } \bigvee_{z \in S} (A + B)^U(z) = (A + B)^U(z_1).
\]
Case (ii): Suppose $A^L(x') \leq B^L(y'')$ and $A^U(x') \leq B^U(y'')$. By proceeding in a similar way as in Case (i), we can verify that
\[
\bigvee_{z \in S} (A + B)^L(z) = (A + B)^L(z_2) \quad \text{and} \quad \bigvee_{z \in S} (A + B)^U(z) = (A + B)^U(z_2)
\]
for some $z_2 \in S$. Hence, in all, $A + B$ has the sup-property.

(b) The proof is left as an exercise for the reader. This completes the proof. $\square$

**Proposition 2.20.** Let $\text{IVI}_{[\lambda_0, \mu_0]}(R)$ be the set of all IVIs with the sup-property and same tip $"[\lambda_0, \mu_0]"$. Then $\text{IVI}_{[\lambda_0, \mu_0]}(R)$ forms a sub-lattice of $\text{IVI}_{[\lambda_0, \mu_0]}(R)$ and hence of $\text{IVI}(R)$.

**Proof.** Let $A, B \in \text{IVI}_{[\lambda_0, \mu_0]}(R)$. We show that $A \lor B = A + B$. Since $A, B \in \text{IVI}(R)$, by Lemma 2.13, $A_{[\lambda, \mu]}$ and $B_{[\lambda, \mu]}$ are ideals for each $[\lambda, \mu] \in D(I)$ with $\lambda \leq (A + B)^L(0) = \lambda_0$ $\mu \leq (A + B)^U(0) = \mu_0$. Then $A_{[\lambda, \mu]} + B_{[\lambda, \mu]}$ is an ideal of $R$. Since $A$ and $B$ have the sup-property, by Lemma 2.13, $A_{[\lambda, \mu]} + B_{[\lambda, \mu]} = (A + B)_{[\lambda, \mu]}$. Thus $(A + B)_{[\lambda, \mu]}$ is an ideal of $R$. So, by Theorem 2.12, $A + B \in \text{IVI}(R)$. Since $A$ and $B$ have the same tip $"[\lambda_0, \mu_0]"$, we have
\[
(A + B)^L(z) = \bigvee_{z = x + y} A^L(x) \land B^L(y) \geq A^L(z) \land B^L(0) = A^L(z)
\]
and
\[
(A + B)^U(z) = \bigvee_{z = x + y} A^U(x) \land B^U(y) \geq A^U(z) \land B^U(0) = A^U(z)
\]
for each $z \in S$. Then $A \subset A + B$. By the similar arguments, we have $B \subset A + B$.

Now let $C \in \text{IVI}(R)$ contain $A$ and $B$ and let $z \in S$ such that $z = x + y$. Then
\[
C^L(z) = C^L(x + y) \geq C^L(x) \land C^L(y) \quad \text{and} \quad C^U(z) = C^U(x + y) \geq C^U(x) \land C^U(y).
\]

Thus
\[
(A + B)^L(z) = \bigvee_{z = x + y} (A^L(x) \land B^L(y)) \leq \bigvee_{z = x + y} (C^L(x) \land C^L(y)) \quad \text{(Since } A \subset C \text{ and } B \subset C) = C^L(z).
\]
Similarly, we have that \((A + B)^U(z) \leq C^U(z)\). So \(A + B \subseteq C\). Hence \(A + B\) is the least interval-valued fuzzy ideal containing \(A\) and \(B\). Therefore \(A + B = A \lor B\). On the other hand, by Lemma 2.19(a), \(A \lor B\) has the sup-property. Thus \(A \lor B \in IVI_{\lambda_0, \mu_0}(R)\). So \(IVI_{\lambda_0, \mu_0}(R)\) forms a sublattice of \(IVI_{\lambda_0, \mu_0}(R)\) and hence of \(IVI(R)\). This completes the proof.

The following lattice diagram is the interrelationship of different sublattices of the lattice \(IVR(R)\):

![Lattice Diagram](image)

**Figure 1**

Now we obtain an interval-valued fuzzy analog of a well-known result that the set of ideals of a ring forms a modular lattice.

### 3. Interval-valued fuzzy ideals and modularity

In the previous section, we discussed various sublattices of the lattice of interval-valued fuzzy ideals of a ring. Hence, we obtain an interval-valued fuzzy analog of a well known result that the set of ideals of a ring forms a modular lattice.
Lemma 3.1. Let $A \in \text{IVR}(R)$. If $A^L(x) < A^L(y)$ and $A^U(x) < A^U(y)$ for some $x, y \in R$, then $A(x + y) = A(x)$.

Proof. Since $A \in \text{IVR}(R)$,

$$A^L(x + y) \geq A^L(x) \land A^L(y) = A^L(x)$$

and

$$A^U(x + y) \geq A^U(x) \land A^U(y) = A^U(x).$$

Assume that $A^L(x + y) > A^L(x)$ and $A^U(x + y) > A^U(x)$. Then

$$A^L(x) = A^L(y + x - y) \geq A^L(x + y) \land A^L(y) > A^L(x)$$

and

$$A^U(x) = A^U(y + x - y) \geq A^U(x + y) \land A^U(y) > A^U(x).$$

This contradicts the fact that $A(x) = A(x)$. Hence $A(x + y) = A(x)$. □

Proposition 3.2. The sublattice $\text{IVI}_{[\lambda_0, \mu_0]}(R)$ is modular.

Proof. Since the modular inequality is valid for every lattice, for any $A, B, C \in \text{IVI}_{[\lambda_0, \mu_0]}(R)$ with $B \subset A$, we have that $B \lor (A \land C) \subset A \land (B \lor C)$.

Assume that $A \land (B \lor C) \neq B \lor (A \land C)$. Then there exists $z \in R$ such that

$$(A \land (B \lor C))^L(z) > (B \lor (A \land C))^L(z)$$

and

$$(A \land (B \lor C))^U(z) > (B \lor (A \land C))^U(z).$$

Thus, by Proposition 2.20,

$$A^L(z) \land (B + C)^L(z) > (B + (A \land C))^L(z)$$

and

$$A^U(z) \land (B + C)^U(z) > (B + (A \land C))^U(z).$$

So

$$A^L(z) > (B + (A \land C))^L(z), A^U(z) > (B + (A \land C))^U(z)$$

and

$$B^L(z) \land C^L(z) > (B + (A \land C))^L(z)$$

Then there exist $x_0, y_0 \in R$ with $z = x_0 + y_0$ such that

$$B^L(x_0) \land C^L(y_0) > (B + (A \land C))^L(z)$$

and

$$B^U(x_0) \land C^U(y_0) > (B + (A \land C))^U(z).$$
Thus
\[ B_L(x_0) > (B + (A ∩ C))^L(z), B_U(x_0) > (B + (A ∩ C))^U(z) \] (3.2)

and
\[ C_L(y_0) > (B + (A ∩ C))^L(z), C_U(y_0) > (B + (A ∩ C))^U(z). \] (3.3)

On the other hand,
\[ (B + (A ∩ C))^L(z) = \bigvee_{x=z+y} (B^L(x) \land (A ∩ C)^L(y)) \]
\[ \geq (B^L(x_0) \land (A ∩ C)^L(y_0)) \]
\[ = B^L(x_0) \land A^L(y_0) \land C^L(y_0). \]

Similarly, we have that \((B + (A ∩ C))^U(z) \geq B_U(x_0) \land A_U(y_0) \land C_U(y_0).\)

Then, by (3.1),(3.2),(3.3),
\[ A^L(z), B^L(x_0), C^L(y_0) > B^L(x_0) \land A^L(y_0) \land C^L(y_0) \]

and
\[ A^U(z), B^U(x_0), C^U(y_0) > B^U(x_0) \land A^U(y_0) \land C^U(y_0). \]

Thus
\[ B^L(x_0) \land A^L(y_0) \land C^L(y_0) = A^L(y_0) \]

and
\[ B^U(x_0) \land A^U(y_0) \land C^U(y_0) = A^U(y_0). \]

So
\[ A^L(-y_0) = A^L(y_0) < A^L(x_0 + y_0) = A^L(z) \]

and
\[ A^U(-y_0) = A^U(y_0) < A^U(x_0 + y_0) = A^U(z). \]

By Lemma 3.1,
\[ A^L(y_0) = A^L(x_0 + y_0 - y_0) = A^L(x_0) \]

and
\[ A^U(y_0) = A^U(x_0 + y_0 - y_0) = A^U(x_0). \]

Then
\[ B^L(x_0) > A^L(y_0) = A^L(x_0) \]

and
\[ B^U(x_0) > A^U(y_0) = A^U(x_0). \]

This contradicts the fact that \(B \subset A\). Hence \(A \land (B \lor C) = B \lor (A \land C)\). Therefore IVIs\(x_{[\lambda_0,\mu_0]}(R)\) is modular. This completes the proof. \qed
Remark 3.3. As a special case, $\text{IVI}_{[1,1]}(R)$ is a complete sublattice of $\text{IVI}(R)$ and $\text{IVI}_{[1,1]}(R)$ is a modular sublattice of $\text{IVI}(R)$.

Proposition 3.4. (The generalization of Proposition 3.2) $\text{IVLI}_{[\lambda_0,\mu_0]}(R)$, $\text{IVRI}_{[\lambda_0,\mu_0]}(R)$ and $\text{IVI}_{[\lambda_0,\mu_0]}(R)$ are all modular.

Proof. The proofs are similar to Proposition 3.2.

Proposition 3.5. $\text{IVI}(R)$ is bounded.

Proof. It is clear that $\mathbf{0} \in \text{IVI}(R)$ and $\mathbf{1} \in \text{IVI}(R)$. Moreover, $\mathbf{0} \subset A \subset \mathbf{1}$ for each $A \in \text{IVI}(R)$. Hence $\text{IVI}(R)$ is bounded.

Proposition 3.6. (a) $\text{IVI}(R)$ is not complemented.

(b) $\text{IVI}(R)$ has no atoms.

(c) $\text{IVI}(R)$ has no dual atoms.

Proof. (a) We define a mapping $A : R \rightarrow D(I)$ as follows: For each $x \in R, A(x) = \left[ \frac{1}{2}, \frac{1}{2} \right]$. Then clearly $[A, A^c] \in \text{IVI}(R)$. But $A \cup A^c \neq \mathbf{1}$ and $A \cap A^c \neq \mathbf{0}$. Thus $A$ has no complement in $\text{IVI}(R)$. Hence $\text{IVI}(R)$ is not complemented.

(b) Suppose $A \in \text{IVI}(R)$ with $A \neq \mathbf{0}$. We define a mapping $B : R \rightarrow D(I)$ as follows: For each $x \in R, B^L(x) = \frac{1}{2} A^L(x)$ and $B^U(x) = \frac{1}{2} A^U(x)$. Then clearly $B \in \text{IVI}(R)$. Moreover, $\mathbf{0} \subsetneq B \subsetneq A$. Hence $\text{IVI}(R)$ has no atoms.

(c) Suppose $A \in \text{IVI}(R)$ with $A \neq \mathbf{1}$. We define a mapping $B : R \rightarrow D(I)$ as follows: For each $x \in R$,

$$B^L(x) = \frac{1}{2} + \frac{1}{2} A^L(x) \quad \text{and} \quad B^U(x) = \frac{1}{2} + \frac{1}{2} A^U(x).$$

Then clearly $A \subsetneq B \subsetneq \mathbf{1}$. 

Now let $x, y \in R$. Then
\[
B^L(xy) = \frac{1}{2} + \frac{1}{2} A^L(xy) \\
\geq \frac{1}{2} + \frac{1}{2} (A^L(x) \lor A^L(y)) \quad \text{(Since $A \in IVI(R)$)} \\
= \left(\frac{1}{2} + \frac{1}{2} A^L(x)\right) \lor \left(\frac{1}{2} + \frac{1}{2} A^L(y)\right). \\
= B^L(x) \lor B^L(y).
\]

By the similar arguments, we have that $B^U(xy) \geq B^U(x) \lor B^U(y)$. Also,
\[
B^L(x - y) = \frac{1}{2} + \frac{1}{2} A^L(x - y) \\
\geq \frac{1}{2} + \frac{1}{2} (A^L(x) \land A^L(y)) \quad \text{(Since $A \in IVI(R)$)} \\
= \left(\frac{1}{2} + \frac{1}{2} A^L(x)\right) \land \left(\frac{1}{2} + \frac{1}{2} A^L(y)\right). \\
= B^L(x) \land B^L(y).
\]

By the similar arguments, we have that $B^U(x - y) \geq B^U(x) \land B^U(y)$. So $B \in IVI(R)$. Hence IVI(R) has no dual atoms.

References

Keon Chang Lee
Department of Computer Science, Dongshin University,
Naju 520-714, Korea.
E-mail: kclee@dsu.ac.kr

Kul Hur
Division of Mathematics and Informational Statistics, and Nanoscale Science and Technology Institute, Wonkwang University,
Iksan 570-749, Korea.
E-mail: kulhur@wonkwang.ac.kr

Pyung Ki Lim
Division of Mathematics and Informational Statistics, and Nanoscale Science and Technology Institute, Wonkwang University,
Iksan 570-749, Korea.
E-mail: pklim@wonkwang.ac.kr