VOLUME FORMULAS OF A EUCLIDEAN TETRAHEDRON

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Abstract. We survey and study several volume formulas of a Euclidean tetrahedron.

1. Introduction

In two- and three-dimensional Euclidean geometry, triangles and tetrahedra are elementary objects. Just as there are many well-known area formulas for triangles, there exist many volume formulas for tetrahedra (see [10], [9], [11], [2], [4] and [3]).

We survey several volume formulas of a Euclidean tetrahedron in Section 2, and derive the following two theorems for volume formulas in Section 3. For the definitions of notations at the following theorems, we explain them at the next section.

Theorem 1 For three edges $a_{12}, a_{23}, a_{31}$ of the facial triangle $A_0$ with adjacent (to $A_0$) dihedral angles $\alpha_{12}, \alpha_{23}, \alpha_{31}$, the volume $V$ of the tetrahedron is given by

$$V = \frac{2}{3} A_0 \frac{\sin \alpha_{12} \sin \alpha_{23} \sin \alpha_{31}}{a_{12} \cos \alpha_{12} \sin \alpha_{23} \sin \alpha_{31} + a_{23} \sin \alpha_{12} \cos \alpha_{23} \sin \alpha_{31} + a_{31} \sin \alpha_{12} \sin \alpha_{23} \cos \alpha_{31}}.$$

Theorem 2 For a tetrahedron with dihedral angles $\alpha_{01}, \ldots, \alpha_{23}$ and inradius $r$, its volume $V$ is expressed as

$$V(\alpha_{01}, \ldots, \alpha_{23}, r) = \frac{1}{6} \frac{(\sqrt{N_0} + \sqrt{N_1} + \sqrt{N_2} + \sqrt{N_3})^3}{\sqrt{N_0 N_1 N_2 N_3}} r^3,$$

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where \( N_l = 1 - \cos^2 \alpha_l - \cos^2 \alpha_{lj} - \cos^2 \alpha_{lk} - 2 \cos \alpha_l \cos \alpha_{lj} \cos \alpha_{lk} \) with \( \{i, j, k, l\} = \{0, 1, 2, 3\} \).

### 2. Known volume formulas of a Euclidean tetrahedron

In order to study the subject, we need the following notations. Let \( v_0, v_1, v_2, v_3 \) be independent points in three-dimensional Euclidean space \( \mathbb{E}^3 \). Then, \( V_3 \) or \( V_3 \) denotes the volume of the tetrahedron \( T = [v_0, v_1, v_2, v_3] \); \( A_i \), a triangle face of \( T \), where \( v_i \notin A_i \) for \( i = 0, 1, 2, 3 \) (for simplicity, \( A_i \) also denotes the area of the triangle face); \( a_{ij} \), the edge length of \([v_i, v_j]\); \( b_{ij} \), the edge length of \( A_i \cap A_j \); \( \alpha_{ij} \), the dihedral angle at a given edge \([v_i, v_j]\); \( \beta_{ij} \), the dihedral angle at a given edge \( A_i \cap A_j \); and \( \theta_{ij} \), the facial angle corresponding to the vertex \( v_i \) and face \( A_j \), where \( j \in \{0, 1, 2, 3\} \setminus \{i\} \). Note that \( a_{ij} = a_{ji} \), \( b_{ij} = b_{ji} \), \( \alpha_{ij} = \alpha_{ji} \), \( \beta_{ij} = \beta_{ji} \); however, in general, \( \theta_{ij} \neq \theta_{ji} \), and \( a_{ij} = b_{kl}, \alpha_{ij} = \beta_{kl} \), where \( \{i, j, k, l\} = \{0, 1, 2, 3\} \).

The volume of an \( n \)-dimensional simplex \([v_0, v_1, \ldots, v_n]\), denoted by \( V_n \), can be expressed in terms of the edge length \( a_{ij} \) (see §9.7 in [1]). Thus, the volume \( V_3 \) is given by

\[
V_3^2 = \frac{(-1)^{3+1}}{2^3(3!)^2} \Gamma(v_0, \ldots, v_3) = \frac{1}{288} \Gamma(v_0, \ldots, v_3),
\]

where

\[
\Gamma(v_0, \ldots, v_3) = \begin{vmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & a_{01}^2 & a_{02}^2 & a_{03}^2 \\
1 & a_{10}^2 & 0 & a_{12}^2 & a_{13}^2 \\
1 & a_{20}^2 & a_{21}^2 & 0 & a_{23}^2 \\
1 & a_{30}^2 & a_{31}^2 & a_{32}^2 & 0
\end{vmatrix} = 0.
\]

There is another volume formula based on two facial areas \( A_i, A_j \), the common edge length \( b_{ij} (= a_{kl}) \), and the dihedral angle \( \beta_{ij} (= \alpha_{kl}) \), where \( \{i, j, k, l\} = \{0, 1, 2, 3\} \). Then, the volume of the tetrahedron is given by

\[
V = \frac{2}{3b_{ij}} A_i A_j \sin \beta_{ij} = \frac{2}{3a_{kl}} A_i A_j \sin \alpha_{kl},
\]

since we know that \( V = \frac{1}{3} A_0 h = \frac{1}{3} A_0 h' \sin \alpha_{23} = \frac{1}{4} A_0 (\frac{a_{kl}}{a_{ki}}) \sin \alpha_{23} \), where \( h' \) is the height of \( A_1 \) for the base \([v_2, v_3]\). Note that (2) is similar to the triangle area formula, \( V_2 = \frac{1}{2} ab \sin \gamma \), and it is essentially represented by six terms comprising the edge lengths and angles of \( T \).
From (2), we derive
\[ V = \frac{2}{3} \left( \frac{1}{2} a_{03} a_{23} \sin \theta_{31} \right) \left( \frac{1}{2} a_{12} a_{23} \sin \theta_{20} \right) \sin \alpha_{23} \]
\[ = \frac{1}{6} a_{03} a_{21} a_{32} \sin \theta_{31} \sin \theta_{20} \sin \alpha_{23}. \]

Thus, we obtain third volume formula
\[ (3) \quad V = \frac{1}{6} a_{ij} a_{jk} a_{kl} \sin \theta_{jl} \sin \theta_{ki} \sin \alpha_{jk}, \]

and this formula is also similar to the triangle area formula.

Now, let us derive another volume formula.

From the formula (2), we have
\[ V = \frac{2}{3 a_{02}} A_1 A_3 \sin \alpha_{02} \]
\[ = \frac{2}{3 a_{02}} \left( \frac{1}{2} a_{01} a_{02} \sin \theta_{03} \right) \left( \frac{1}{2} a_{02} a_{03} \sin \theta_{01} \right) \sin \alpha_{02} \]
\[ = \frac{a_{01} a_{02} a_{03}}{6} \sin \theta_{03} \sin \theta_{01} \sin \alpha_{02}. \]

Then, by a small sphere truncation around the vertex \( v_0 \) and spherical cosine law, we know
\[ \cos \theta_{03} = \frac{\cos \alpha_{01} \cos \alpha_{02} + \cos \alpha_{03}}{\sin \alpha_{01} \sin \alpha_{02}} \quad \text{and} \quad \cos \theta_{01} = \frac{\cos \alpha_{02} \cos \alpha_{03} + \cos \alpha_{01}}{\sin \alpha_{02} \sin \alpha_{03}}. \]

So \( \sin \theta_{01} \) and \( \sin \theta_{03} \) are obtained. In particular,
\[ \sin \theta_{03} = \sqrt{1 - \cos^2 \alpha_{01} - \cos^2 \alpha_{02} - \cos^2 \alpha_{03} - 2 \cos^2 \alpha_{01} \cos^2 \alpha_{02} \cos^2 \alpha_{03}} \]
\[ \sin \alpha_{01} \sin \alpha_{02} \]

Hence we have the following volume formula:
\[ (4) \quad V = \frac{a_{01} a_{02} a_{03}}{6} \frac{1 - \cos^2 \alpha_{01} - \cos^2 \alpha_{02} - \cos^2 \alpha_{03} - 2 \cos \alpha_{01} \cos \alpha_{02} \cos \alpha_{03}}{\sin \alpha_{01} \sin \alpha_{02} \sin \alpha_{03}}. \]

In (4), if we replace the dihedral angles by facial angles around the vertex \( v_0 \), we can derive useful volume formulas (see [3] and [6] for the first one, and [8] and [7] for the second one in (5)) similar to a triangle
area formula and Heron’s formula:

\[ V = \frac{1}{3} a_{01}a_{02}a_{03}\sqrt{\sin \theta \sin(\theta - \theta_{01}) \sin(\theta - \theta_{02}) \sin(\theta - \theta_{03})}, \]

where \( \theta = \frac{\theta_{01} + \theta_{02} + \theta_{03}}{2} \)

\[ = \frac{1}{6} a_{01}a_{02}a_{03}\sqrt{1 - \cos^2 \theta_{01} - \cos^2 \theta_{02} - \cos^2 \theta_{03} + 2 \cos \theta_{01} \cos \theta_{02} \cos \theta_{03}}. \]

From now, we consider volume formulas with a special condition.

In the case of a rectangular triangle, the rectangular condition for a tetrahedron, i.e., \( \alpha_{01} = \alpha_{02} = \alpha_{03} = \frac{\pi}{2} \), gives a simple volume formula:

\[ (6) \quad V(a_{01}, a_{02}, a_{03}) = \frac{a_{01}a_{02}a_{03}}{6}. \]

If a tetrahedron has the three consecutive rectangular dihedral angles condition, i.e., \( \alpha_{20} = \alpha_{03} = \alpha_{31} = \frac{\pi}{2} \), then we call it an orthoscheme or a birectangular tetrahedron \( T \). Let \( x = a_{03} \) and \( y = a_{02} \); then, from the orthoscheme condition, we get 4 rectangular facial triangles of \( T \). Therefore, we have \( a_{01}^2 + y^2 = a_{12}^2 \) and \( x^2 + a_{23}^2 = y^2 \); hence, we get \( x = \sqrt{a_{12}^2 - a_{01}^2 - a_{23}^2} \). In addition, we know that \( V = \frac{1}{3} a_{01}A_1 = \frac{1}{3} a_{01}(\frac{1}{2}xa_{23}) \).

Therefore, we can simply represent its volume as

\[ (7) \quad V(a_{01}, a_{12}, a_{23}) = \frac{a_{01}a_{23}}{6} \sqrt{a_{12}^2 - a_{01}^2 - a_{23}^2}. \]

There is another volume formula [2] with the condition that \( T \) can be embedded as a face of a rectangular 4-simplex. This formula is expressed as

\[ (8) \quad V(a_{01}, \ldots, a_{23}) = \frac{1}{6} \sqrt{4u \cdot p(p - a_{12})(p - a_{23})(p - a_{31}) - q^2}, \]

where

\[ u = a_{12}^2 + a_{30}^2 = a_{13}^2 + a_{20}^2 = a_{10}^2 + a_{23}^2, \]

\[ p = \frac{a_{12} + a_{23} + a_{31}}{2}, \quad q = a_{12}a_{23}a_{31}. \]

We know that the three angles of any Euclidean triangle do not determine the triangle, since we have the \( \alpha + \beta + \gamma = \pi \) angle condition. Similarly, any Euclidean tetrahedron has a dihedral angle condition, i.e.,
the determinant of the Gram matrix is 0:

\[
\begin{vmatrix}
1 & -\cos \beta_{01} & -\cos \beta_{02} & -\cos \beta_{03} \\
-\cos \beta_{10} & 1 & -\cos \beta_{12} & -\cos \beta_{13} \\
-\cos \beta_{20} & -\cos \beta_{21} & 1 & -\cos \beta_{23} \\
-\cos \beta_{30} & -\cos \beta_{31} & -\cos \beta_{32} & 1
\end{vmatrix}
= \begin{vmatrix}
1 & -\cos \alpha_{23} & -\cos \alpha_{13} & -\cos \alpha_{12} \\
-\cos \alpha_{02} & 1 & -\cos \alpha_{03} & -\cos \alpha_{01} \\
-\cos \alpha_{12} & -\cos \alpha_{02} & -\cos \alpha_{01} & 1
\end{vmatrix}
= 0.
\]

Thus, one dihedral angle is determined by the other 5 dihedral angles. Therefore, if we want to derive a volume formula with all dihedral angles, we have to add one variable.

We know an area formula \((V_2 = 2R^2 \sin \alpha \sin \beta \sin \gamma)\) of a triangle with three angles and circumradius \(R\), and a volume formula with all dihedral angles and circumradius \(R\) (see [4]), that is,

\[
V(\alpha_{01}, \ldots, \alpha_{23}, R) = \frac{32}{3} \frac{N_0 N_1 N_2 N_3}{M^2} R^3,
\]

where

\[
N_l = 1 - \cos^2 \alpha_{li} - \cos^2 \alpha_{lj} - \cos^2 \alpha_{lk} - 2 \cos \alpha_{li} \cos \alpha_{lj} \cos \alpha_{lk}
\]

with \(\{i, j, k, l\} = \{0, 1, 2, 3\}\)

and

\[
M = \begin{vmatrix}
0 & \sin^2 \alpha_{01} & \sin^2 \alpha_{02} & \sin^2 \alpha_{03} \\
\sin^2 \alpha_{10} & 0 & \sin^2 \alpha_{12} & \sin^2 \alpha_{13} \\
\sin^2 \alpha_{20} & \sin^2 \alpha_{21} & 0 & \sin^2 \alpha_{23} \\
\sin^2 \alpha_{30} & \sin^2 \alpha_{31} & \sin^2 \alpha_{32} & 0
\end{vmatrix}.
\]

Note that there was a mistake in the sign of the last term of the definition of \(N_l\) in [4].

### 3. two volume formulas of a Euclidean tetrahedron

In Section 2, we surveyed nine volume formulas (1), (2), \ldots, (9). Now we introduce two new volume formulas of a Euclidean tetrahedron.

Let \(a, b, c\) denote the sides of a Euclidean triangle, and let \(\alpha, \beta, \gamma\) denote the corresponding angles for three sides, respectively. Then, the
area \( (V_2) \) of the triangle is given by \( V_2 = \frac{1}{2}ab \sin \gamma \). In addition, we have

\[
V_2 = \frac{1}{2}c^2 \tan \alpha \tan \beta = \frac{1}{2}c^2 \frac{\sin \alpha \sin \beta}{\sin \alpha \cos \beta + \cos \alpha \sin \beta}
\]

by using \( V_2 = \frac{1}{2}c^2 \frac{\sin \alpha \sin \beta}{\sin \gamma} = \frac{1}{2}c^2 \frac{\sin \alpha \sin \beta}{\sin \gamma} = \frac{1}{2}c^2 \frac{\sin \alpha \sin \beta}{\sin (\alpha + \beta)} \).

There is a similar type volume formula for a tetrahedron corresponding to the above area formula.

**Theorem 1.** For three edges \( a_{12}, a_{23}, a_{31} \) of the facial triangle \( A_0 \) with adjacent (to \( A_0 \)) dihedral angles \( \alpha_{12}, \alpha_{23}, \alpha_{31} \), the volume \( V \) of the tetrahedron is given by

\[
V = \frac{2}{3}A_0 \frac{\tan \alpha_{12} \tan \alpha_{23} \tan \alpha_{31}}{a_{12} \tan \alpha_{23} \tan \alpha_{31} + a_{23} \tan \alpha_{12} \tan \alpha_{31} + a_{31} \tan \alpha_{12} \tan \alpha_{23}}
\]

\[
= \frac{2}{3}A_0 \sin \alpha_{12} \sin \alpha_{23} \sin \alpha_{31} + a_{23} \sin \alpha_{12} \cos \alpha_{23} \sin \alpha_{31} + a_{31} \sin \alpha_{12} \sin \alpha_{23} \cos \alpha_{31}.
\]

**Proof.** First, suppose that \( \alpha_{12}, \alpha_{23}, \) and \( \alpha_{31} \) are acute angles. Let us calculate the length \( \overline{v_0H} \), where \( \overline{v_0H} \) is a height with base \([v_1, v_2, v_3]\). If we denote the feet of perpendiculars from \( H \) to \([v_1, v_2], [v_2, v_3], \) and \([v_3, v_1]\) as \( E, F, \) and \( G \), respectively, then \( \angle v_0EH, \angle v_0FH, \) and \( \angle v_0GH \) are the same to \( \alpha_{12}, \alpha_{23}, \) and \( \alpha_{31}, \) respectively. Let \( \overline{EH} = x, \overline{FH} = y, \) and \( \overline{GH} = z. \) Then, \( \triangle v_0EH, \triangle v_0FH, \) and \( \triangle v_0GH \) are right triangles. Thus, we have

\[
\overline{v_0H} = x \tan \alpha_{12} = y \tan \alpha_{23} = z \tan \alpha_{31},
\]

\[
y = \frac{\tan \alpha_{12}}{\tan \alpha_{23}} x, \quad z = \frac{\tan \alpha_{12}}{\tan \alpha_{31}} x.
\]

In the plane \([v_1, v_2, v_3]\), its area \( A_0 \) is given by \( 2A_0 = a_{12}x + a_{23}y + a_{31}z. \) Here, we can obtain the value of \( x \):

\[
x = \frac{2A_0}{a_{12} + a_{23} \tan \alpha_{12} + a_{31} \tan \alpha_{12}},
\]

and by substitution, we get

\[
\overline{v_0H} = x \tan \alpha_{12}
\]

\[
= \frac{2A_0 \tan \alpha_{12} \tan \alpha_{23} \tan \alpha_{31}}{a_{12} \tan \alpha_{23} \tan \alpha_{31} + a_{23} \tan \alpha_{12} \tan \alpha_{31} + a_{31} \tan \alpha_{12} \tan \alpha_{23}}.
\]
Therefore, we obtain
\[ V = \frac{1}{3} A_0 \cdot \overline{v_0 H} \]
\[ = \frac{2}{3} A_0^2 \frac{\tan \alpha_{12} \tan \alpha_{23} \tan \alpha_{31}}{a_{12} \tan \alpha_{23} \tan \alpha_{31} + a_{23} \tan \alpha_{12} \tan \alpha_{31} + a_{31} \tan \alpha_{12} \tan \alpha_{23}}. \]
We considered only the case wherein \( \alpha_{12}, \alpha_{23}, \) and \( \alpha_{31} \) are acute angles. Furthermore, we can easily derive and verify the formulas for other cases.

In the expressions for \( V \) in Theorem 1, the first tangent expression requires all non-rectangular dihedral angles, i.e., \( \alpha, \beta, \gamma \neq \frac{\pi}{2} \); however, the second expression does not need to require the condition.

From now, we introduce another volume formula similar to the formula (9) with the inradius \( r \) instead of the circumradius \( R \). We can find an area formula of a triangle with three angles and inradius \( r \), and a volume formula with all dihedral angles and inradius \( r \).

For a triangle, we know that \( V_2 = Rr(\sin \alpha + \sin \beta + \sin \gamma) \) by \( V_2 = \frac{a+b+c}{2} r \) and the sine law, and we have already obtained
\[ V_2 = 2R^2 \sin \alpha \sin \beta \sin \gamma. \]
Hence, we obtain
\[ V_2 = \frac{1}{2} (\sin \alpha + \sin \beta + \sin \gamma)^2 \frac{r^2}{\sin \alpha \sin \beta \sin \gamma}. \]
Now, let us derive a volume formula with dihedral angles and inradius \( r \).

**Theorem 2.** For a tetrahedron with dihedral angles \( \alpha_{01}, \ldots, \alpha_{23} \) and inradius \( r \), its volume \( V \) is expressed as
\[ V(\alpha_{01}, \ldots, \alpha_{23}, r) = \frac{1}{6} \frac{\sqrt{N_0 + \sqrt{N_1 + \sqrt{N_2 + \sqrt{N_3}}}}^3}{\sqrt{N_0 N_1 N_2 N_3}} r^3. \]

**Proof.** From the formula (3), we define
\[ a(i, j, k, l) := a_{ij} a_{jk} a_{kl} = \frac{6V}{\sin \theta_{jl} \sin \theta_{kl} \sin \alpha_{ij}}, \]
also we obtain another formula from the formula (13):
\[ \frac{a(k, i, j, l)}{a(i, k, l, j)} = \frac{a_{ij}}{a_{kl}} = \frac{\sin \theta_{kj} \sin \theta_{il} \sin \alpha_{kl}}{\sin \theta_{il} \sin \theta_{jk} \sin \alpha_{ij}}. \]
So we can calculate \( a_{kl} \) by

\[
a_{kl} = \frac{(a_{ik}a_{kl}a_{lj})^2}{(a_{ik}a_{kl}a_{lj})^2} = \frac{6V \sin \theta_{il} \sin \theta_{jk} \sin^2 \alpha_{ij}}{\sin^2 \theta_{kl} \sin^2 \theta_{lj} \sin \alpha_{ik} \sin \alpha_{il}}.
\]

Also we can express \( \sin \theta_{ij} \) in terms of dihedral angles \( \alpha_{ij} \) by applying the spherical cosine law at the spherical triangle cut by a small sphere with center at a vertex of the tetrahedron.

\[
\sin^2 \theta_{ij} = 1 - \cos^2 \theta_{ij}
\]

\[
= 1 - \left( \frac{\cos \alpha_{ik} \cos \alpha_{il} + \cos \alpha_{ij}}{\sin \alpha_{ik} \sin \alpha_{il}} \right)^2
\]

\[
= 1 - \cos^2 \alpha_{ij} - \cos^2 \alpha_{ik} - \cos^2 \alpha_{il} - 2 \cos \alpha_{ij} \cos \alpha_{ik} \cos \alpha_{il}.
\]

Hence we get (see (10))

\[
\sin \theta_{ij} = \frac{\sqrt{N_i}}{\sin \alpha_{ik} \sin \alpha_{il}}.
\]

By substitution of (15) and (16) into an area formula, we derive that

\[
A_i = \frac{1}{2} a_{kl} a_{jl} \sin \theta_{li}
\]

\[
= \left( \frac{9V^2 \sin^2 \theta_{il} \sin \alpha_{ik} \sin \alpha_{lj}}{2 \sin \theta_{lj} \sin \theta_{ki} \sin \theta_{jk} \sin^2 \alpha_{il}} \right)^{\frac{1}{3}}
\]

\[
= \left( \frac{9V^2 N_i}{2 \sqrt{N_j N_k N_l}} \right)^{\frac{1}{3}}.
\]

Therefore we have

\[
V = \frac{r}{3} (A_0 + A_1 + A_2 + A_3)
\]

\[
= \frac{r}{3} \left( \frac{9V^2}{2} \right)^{\frac{1}{3}} \left( \frac{N_0}{\sqrt{N_1 N_2 N_3}} \right)^{\frac{1}{3}} + \left( \frac{N_1}{\sqrt{N_0 N_2 N_3}} \right)^{\frac{1}{3}}
\]

\[
+ \left( \frac{N_2}{\sqrt{N_0 N_1 N_3}} \right)^{\frac{1}{3}} + \left( \frac{N_3}{\sqrt{N_0 N_1 N_2}} \right)^{\frac{1}{3}}
\]

and finally we get

\[
V = \frac{r^3}{6} \left( \frac{N_0}{\sqrt{N_1 N_2 N_3}} \right)^{\frac{1}{3}} + \left( \frac{N_1}{\sqrt{N_0 N_2 N_3}} \right)^{\frac{1}{3}} + \left( \frac{N_2}{\sqrt{N_0 N_1 N_3}} \right)^{\frac{1}{3}} + \left( \frac{N_3}{\sqrt{N_0 N_1 N_2}} \right)^{\frac{1}{3}}.
\]
The condition of a regular tetrahedron implies that \( \cos \alpha = \frac{1}{3} \); hence, the volume of a regular tetrahedron with inradius \( r \) is \( 8\sqrt{3}r^3 \) by the above theorem.

A Coxeter diagram is a convenient means of describing tetrahedra. From [5], there are only three Euclidean Coxeter diagrams of dimension 3.

These three Euclidean tetrahedra \( T(\alpha_{01}, \alpha_{02}, \alpha_{03}, \alpha_{12}, \alpha_{13}, \alpha_{23}) \), are exactly \( T\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) \), \( T\left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) \) and \( T\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) \). Then, the volumes of the three Euclidean Coxeter tetrahedra are calculated by the above theorem.

**Example 3** The volumes of Coxeter tetrahedra with inradius \( r \) are expressed as

\[
V\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}, r\right) = \frac{32\sqrt{2}}{3}r^3,
\]
\[
V\left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, r\right) = \frac{28 + 20\sqrt{2}}{3}r^3,
\]
\[
V\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}, r\right) = \frac{25 + 22\sqrt{2}}{3}r^3.
\]

To the best of our knowledge, (11) and (12) have not been derived in other studies; however, there is a possibility that they are known results.

Now, we list and summarize the formulas with variables:

(1) \( = V(a_{01}, \ldots, a_{23}) \)
(2) \( = V(a_{01}, \ldots, \tilde{a}_{ij}, \ldots, a_{23}, \beta_{ij}) \)
(3) \( = V(a_{ij}, a_{jk}, a_{kl}, \theta_{jl}, \theta_{ki}, \alpha_{jk}) \)
(4) \( = V(a_{01}, a_{02}, a_{03}, a_{01}, a_{02}, a_{03}) \)
(5) \( = V(a_{01}, a_{02}, a_{03}, \theta_{01}, \theta_{02}, \theta_{03}) \)
(11) \( = V(a_{12}, a_{23}, a_{31}, \alpha_{12}, \alpha_{23}, a_{31}) \)
(9) \( = V(a_{01}, \ldots, a_{23}, R) \)
(12) \( = V(a_{01}, \ldots, a_{23}, r) \).

The uniqueness (up to isometry) of a tetrahedron with the fixed variables condition is trivial, and hence, we need not show it here.
There is a possibility that we don’t know the existence of other nice and symmetric volume formulas of a Euclidean tetrahedron with some geometric variables.

So we end this section with the following problem.

**Problem 4** Find nice and symmetric volume formulas of a Euclidean tetrahedron with the given geometric variables, and, if necessary, show the uniqueness (up to isometry) of the tetrahedron with the fixed variables condition.

**References**


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