STABILITY OF A QUADRATIC-ADDITIVE TYPE FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN NORMED SPACES

CHANG-JU LEE AND YANG-HI LEE

Abstract. In this paper, we investigate the stability for the functional equation

\[ 2f(x + y) + f(x - y) + f(y - x) - f(2x) - f(2y) = 0 \]

in non-Archimedean normed spaces.

1. Introduction

We introduce some terminologies and notations used in the theory of non-Archimedean spaces (see [3]).

**Definition 1.1.** A field \( K \) equipped with a function (valuation) \(| \cdot |\) from \( K \) into \([0, \infty)\) is called a *non-Archimedean field* if the function \(| \cdot | : K \to [0, \infty)\) satisfies the following conditions:

(i) \(|r| = 0\) if and only if \(r = 0\);
(ii) \(|rs| = |r||s|\);
(iii) \(|r + s| \leq \max\{|r|, |s|\}\) for all \(r, s \in K\).

Clearly, \(|1| = |1|\) and \(|n| \leq 1\) for all \(n \in \mathbb{N}\).

**Definition 1.2.** Let \( X \) be a vector space over a scalar field \( K \) with a non-Archimedean nontrivial valuation \(| \cdot |\). A function \( \| \cdot \| : X \to \mathbb{R} \) is a *non-Archimedean norm* if it satisfies the following conditions:

(i) \(\|x\| = 0\) if and only if \(x = 0\);
(ii) \(\|rx\| = |r|\|x\|\) (\(r \in K, x \in X\)).

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(iii) the strong triangle inequality, namely,
\[ \|x + y\| \leq \max\{\|x\|, \|y\|\} \]
for all \(x, y \in X\). In this case, the pair \((X, \|\cdot\|)\) is called a non-Archimedean space. Due to the fact that
\[ \|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\} \quad (n > m) \]
a sequence \(\{x_n\}\) is Cauchy if and only if \(\{x_{n+1} - x_n\}\) converges to zero in a non-Archimedean space. By a complete non-Archi- medean we mean one in which every Cauchy sequence is convergent.


\begin{equation}
(1.1) \quad f(x + y) = f(x) + f(y)
\end{equation}
and the quadratic functional equation
\begin{equation}
(1.2) \quad f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0
\end{equation}
in non-Archimedean normed spaces.

Now we consider a quadratic-additive type functional equation
\begin{equation}
(1.3) \quad 2f(x + y) + f(x - y) + f(y - x) - f(2x) - f(2y) = 0
\end{equation}
which solution is called a quadratic-additive mapping. We remark that the general solution of the equation (1.3) is of the form \(f(x) = Q(x) + A(x)\), where \(Q\) is a quadratic mapping and \(A\) is an additive mapping.

In this paper, we get a general stability result of the quadratic-additive type functional equation (1.3) in non-Archimedean normed spaces.

2. Stability of a quadratic-additive type functional equation

Throughout this section, we assume that \(X\) is a real linear space and \(Y\) is a complete non-Archimedean space with \(|2| < 1\).

For a given mapping \(f : X \to Y\), we use the abbreviation
\[ Df(x, y) := 2f(x + y) + f(x - y) + f(y - x) - f(2x) - f(2y) \]
for all \(x, y \in X\). Now, we will prove the stability of the quadratic-additive type functional equation (1.3).

**Theorem 2.1.** Let \(\varphi : X^2 \to [0, \infty)\) be a function such that
\[
\lim_{n \to \infty} \frac{\varphi(2^n x, 2^n y)}{|4^n|} = 0, \quad (x, y \in X)
\]
and let for each \(x \in X\) the limit
\[
\lim_{n \to \infty} \max \left\{ \frac{\varphi(2^j x, 0)}{|2| \cdot |4|^{j+1}}, \frac{\varphi(-2^j x, 0)}{|2| \cdot |4|^{j+1}} : 0 \leq j < n \right\}
\]
denoted by \(\tilde{\varphi}(x)\), exists. Suppose that \(f : X \to Y\) is a mapping satisfying
\[
\|Df(x, y)\| \leq \varphi(x, y), \quad (x, y \in X).
\]
Then there exists a unique quadratic-additive mapping \(T : X \to Y\) such that
\[
\|f(x) - T(x)\| \leq \tilde{\varphi}(x), \quad (x \in X).
\]

**Proof.** It follows from \(|2| < 1\) and (2.1) that
\[
\varphi(0, 0) \leq \lim_{n \to \infty} \frac{\varphi(0, 0)}{|4^n|} = 0.
\]
So we have
\[
\|2f(0)\| = \|Df(0, 0)\| \leq \varphi(0, 0) = 0,
\]
i.e., \(f(0) = 0\). Let \(J_n f : X \to Y\) be a function defined by
\[
J_n f(x) = \frac{f(2^n x) + f(-2^n x)}{2 \cdot |4^n|} + \frac{f(2^n x) - f(-2^n x)}{2n+1}
\]
for all \(x \in X\) and \(n \in \mathbb{N}\). Notice that \(J_0 f(x) = f(x)\) and
\[
\|J_n f(x) - J_{n+1} f(x)\|
\]
\[
= \left\| \frac{Df(2^n x, 0)}{2 \cdot |4^n|} + \frac{Df(-2^n x, 0)}{2 \cdot |4^n|} + \frac{Df(2^{n+1} x, 0)}{2^{n+2}} - \frac{Df(-2^{n+1} x, 0)}{2^{n+2}} \right\|
\]
\[
\leq \max \left\{ \left\| \frac{Df(2^n x, 0)}{|2| \cdot |4|^{n+1}}, \frac{Df(-2^n x, 0)}{|2| \cdot |4|^{n+1}}, \frac{Df(2^{n+1} x, 0)}{|2| \cdot |4|^{n+2}}, \frac{Df(-2^{n+1} x, 0)}{|2| \cdot |4|^{n+2}} \right\} \right\}
\]
(2.5) ≤ \max \left\{ \frac{\varphi(2^n x, 0)}{|2| \cdot |4|^{n+1}}, \frac{\varphi(-2^n x, 0)}{|2| \cdot |4|^{n+1}} \right\}
for all \( x \in X \) and \( j \geq 0 \). It follows from (2.1) and (2.5) that the sequence \( \{ J_n f(x) \} \) is Cauchy. Since \( Y \) is complete, we conclude that \( \{ J_n f(x) \} \) is convergent. Set

\[
T(x) := \lim_{n \to \infty} J_n f(x).
\]

Using induction one can show that

\[
\| J_n f(x) - f(x) \| \leq \max \left\{ \frac{\varphi(2^j x, 0)}{|2|^{2j+3}}, \frac{\varphi(-2^j x, 0)}{|2|^{2j+3}} : 0 \leq j < n \right\}
\]

for all \( x \in X \) and \( n \in \mathbb{N} \). By taking \( n \) to approach infinity in (2.6) and using (2.2), one obtains (2.4). Replacing \( x \) and \( y \) by \( 2^nx \) and \( 2^ny \), respectively, in (2.3), we get

\[
\| DJ_n f(x, y) \| = \left\| \frac{Df(2^nx, 2^ny) - Df(-2^nx, -2^ny)}{2^{n+1}} + \frac{Df(2^nx, 2^ny) + Df(-2^nx, -2^ny)}{2^{2n+3}} \right\|
\]

\[
\leq \max \left\{ \frac{\varphi(2^n x, 2^n y)}{|2|^{n+1}}, \frac{\varphi(-2^n x, -2^n y)}{|2|^{n+1}} \right\}
\]

for all \( x, y \in X \) and \( n \in \mathbb{N} \). Taking the limit as \( n \to \infty \) and using (2.1), we get \( DT(x, y) = 0 \). Thus, the mapping \( T \) is a quadratic-additive mapping satisfying (2.4). Now, to prove that the quadratic-additive mapping \( T \) is unique. If \( T' \) is another quadratic-additive mapping satisfying (2.4), then

\[
T'(x) - J_k T'(x) = \sum_{j=0}^{k-1} \left( \frac{(1 + 2^{j+1}) DT'(2^j x, 0)}{2 \cdot 4^{j+1}} + \frac{(1 - 2^{j+1}) DT'(-2^j x, 0)}{2 \cdot 4^{j+1}} \right)
\]

\[
= 0
\]

for any \( k \in \mathbb{N} \) and so...
\[
\|T(x) - T'(x)\| = \lim_{k \to \infty} \|J_k T(x) - J_k T'(x)\| \\
\leq \lim_{k \to \infty} \max\{\|J_k T(x) - J_k f(x)\|, \|J_k f(x) - J_k T'(x)\|\} \\
\leq \lim_{k \to \infty} \max\{\|T(2^k x) - f(2^k x)\|, \|T(-2^k x) - f(-2^k x)\|, \\
\|T'(2^k x) - T'(2^k x)\|, \|f(-2^k x) - T'(-2^k x)\|\} \\
\leq \lim_{k \to \infty} \lim_{n \to \infty} \max\{\frac{\varphi(2^j x, 0)}{|4|^j + 2}, \frac{\varphi(-2^j x, 0)}{|4|^j + 2} : 0 \leq j < n + k\} \\
= 0 \quad (x \in X).
\]

Therefore \(T = T'\). This completes the proof of the uniqueness of \(T\).

**Corollary 2.2.** Let \(X\) and \(Y\) be non-Archimedean normed spaces over \(K\) with \(|2| < 1\). If \(Y\) is complete and for some \(2 < r\), \(f : X \to Y\) satisfies the condition

\[
\|Df(x, y)\| \leq \theta(\|x\|^r + \|y\|^r)
\]

for all \(x, y \in X\). Then there exists a unique quadratic-additive mapping \(T : X \to Y\) such that

\[
(2.7) \quad \|f(x) - T(x)\| \leq |2|^{-3} \theta \|x\|^r.
\]

**Proof.** Let \(\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)\). Since \(|2| < 1\) and \(r - 2 > 0\),

\[
\lim_{n \to \infty} |4|^{-n} \varphi(2^n x, 2^n y) = \lim_{n \to \infty} |2|^{n(r-2)} \varphi(x, y) = 0
\]

for all \(x, y \in X\). Therefore \(f\) and \(\varphi\) satisfy the conditions of Theorem 2.1. It is easy to see that \(\varphi(x) = |2|^{-3}\theta \|x\|^r\). By Theorem 2.1 there is a unique quadratic-additive mapping \(T : X \to Y\) satisfying (2.7).

**Theorem 2.3.** Let \(\varphi : X^2 \to [0, \infty)\) be a function such that

\[
(2.8) \quad \lim_{n \to \infty} |2|^n \varphi(2^{-n} x, 2^{-n} y) = 0, \quad (x, y \in X)
\]

and let for each \(x \in X\) the limit

\[
(2.9) \quad \lim_{n \to \infty} \max\{\|2|^{-1} \varphi\left(\frac{x}{2^{j+1}}, 0\right), \|2|^{-1} \varphi\left(\frac{-x}{2^{j+1}}, 0\right) : 0 \leq j < n\}
\]

denoted by \(\tilde{\varphi}(x)\), exists. Suppose that \(f : X \to Y\) is a mapping satisfying

\[
(2.10) \quad \|Df(x, y)\| \leq \varphi(x, y), \quad (x, y \in X)
\]
with \( f(0) = 0 \). Then there exists a unique quadratic-additive mapping \( T : X \to Y \) such that
\[
\| f(x) - T(x) \| \leq \tilde{\varphi}(x), \quad (x \in X).
\]
In particular, \( T \) is given by
\[
T(x) = \lim_{n \to \infty} \left( 2^{2n} - 1 \left( f\left( \frac{x}{2^n} \right) + f\left( -\frac{x}{2^n} \right) \right) + 2^{n-1} \left( f\left( \frac{x}{2^n} \right) - f\left( \frac{-x}{2^n} \right) \right) \right)
\]
for all \( x \in X \).

Proof. Let \( J_n : X \to Y \) be a function defined by
\[
J_n f(x) = 2^{2n-1} \left( f\left( \frac{x}{2^n} \right) + f\left( -\frac{x}{2^n} \right) \right) + 2^{n-1} \left( f\left( \frac{x}{2^n} \right) - f\left( \frac{-x}{2^n} \right) \right)
\]
for all \( x \in X \) and \( n \in \mathbb{N} \). Notice that \( J_0 f(x) = f(x) \) and
\[
\| J_j f(x) - J_{j+1} f(x) \| = \left\| - \frac{4^j + 2^j}{2} Df\left( \frac{x}{2^{j+1}}, 0 \right) - \frac{4^j - 2^j}{2} Df\left( \frac{-x}{2^{j+1}}, 0 \right) \right\|
\]
(2.12)
\[
\leq \max \left\{ |2|^{j-1} \varphi \left( \frac{x}{2^{j+1}}, 0 \right), |2|^{j-1} \varphi \left( \frac{-x}{2^{j+1}}, 0 \right) \right\}
\]
for all \( x \in X \) and \( j \geq 0 \). It follows from (2.8) and (2.12) that the sequence \( \{J_n f(x)\} \) is Cauchy. Since \( Y \) is complete, we conclude that \( \{J_n f(x)\} \) is convergent. Set
\[
T(x) := \lim_{n \to \infty} J_n f(x).
\]
Using induction one can show that
\[
\|J_n f(x) - f(x)\| \leq \max \left\{ |2|^{j-1} \varphi \left( \frac{x}{2^{j+1}}, 0 \right), |2|^{j-1} \varphi \left( \frac{-x}{2^{j+1}}, 0 \right) : 0 \leq j < n \right\}
\]
(2.13)
for all \( x \in X \) and \( n \in \mathbb{N} \). By taking \( n \) to approach infinity in (2.13) and using (2.9), one obtains (2.11). Replacing \( x \) and \( y \) by \( 2^{-n}x \) and \( 2^{-n}y \), respectively, in (2.10), we get
\[
\| D J_n f(x, y) \| = \left\| 2^{n-1} Df\left( \frac{x}{2^n}, \frac{y}{2^n} \right) - 2^{n-1} Df\left( \frac{-x}{2^n}, \frac{-y}{2^n} \right) \right\|
\]
\[
+ 2^{2n-1} Df\left( \frac{x}{2^n}, \frac{y}{2^n} \right) + 2^{2n-1} Df\left( \frac{-x}{2^n}, \frac{-y}{2^n} \right) \right\|
\]
\[
\leq \max \left\{ |2|^{n-1} \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right), |2|^{n-1} \varphi \left( \frac{-x}{2^n}, \frac{-y}{2^n} \right) \right\}
\]
Stability of a quadratic-additive type functional equation for all $x, y \in X$ and $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ and using (2.8), we get $DT(x, y) = 0$. Thus, the mapping $T$ is a quadratic-additive mapping satisfying (2.11).

If $T'$ is another quadratic-additive mapping satisfying (2.11), then

$$T'(x) - J_k T'(x) = \sum_{j=0}^{k-1} \left( -\frac{4^j + 2^j}{2} D f \left( \frac{x}{2^{j+1}}, 0 \right) - \frac{4^j - 2^j}{2} D f \left( \frac{-x}{2^{j+1}}, 0 \right) \right) = 0$$

for any $k \in \mathbb{N}$ and so

$$\| T(x) - T'(x) \| = \lim_{k \to \infty} \| J_k T(x) - J_k T'(x) \| \leq \lim_{k \to \infty} \max \{ \| J_k T(x) - J_k f(x) \|, \| J_k f(x) - J_k T'(x) \| \} \leq \lim_{k \to \infty} 2^{k-1} \max \left\{ \left\| f \left( \frac{x}{2^k} \right) - T \left( \frac{x}{2^k} \right) \right\|, \left\| f \left( -\frac{x}{2^k} \right) - T' \left( -\frac{x}{2^k} \right) \right\| \right\} \leq \lim_{k \to \infty} 2^{k-1} \tilde{\varphi} \left( \frac{x}{2^k} \right) = 0 \quad (x \in X).$$

Therefore $T = T'$. This completes the proof of the uniqueness of $T$. □

**Corollary 2.4.** Let $X$ and $Y$ be non-Archimedean normed spaces over $\mathbb{K}$ with $|2| < 1$. If $Y$ is complete and for some $0 \leq r < 1$, $f : X \to Y$ satisfies the condition

$$\| Df(x, y) \| \leq \theta (\| x \|^r + \| y \|^r)$$

for all $x, y \in X$. Then there exists a unique quadratic-additive mapping $T : X \to Y$ such that

$$(2.14) \quad \| f(x) - T(x) \| \leq \begin{cases} |2|^{-1-r} \theta \| x \|^r & \text{if } 0 < r < 1, \\ 2 |2|^{-1} \theta & \text{if } r = 0. \end{cases}$$

**Proof.** Let $\varphi(x, y) = \theta (\| x \|^r + \| y \|^r)$. Since $|2| < 1$ and $1 - r > 0$,

$$\lim_{n \to \infty} 2^n \varphi(2^{-n} x, 2^{-n} y) = \lim_{n \to \infty} 2^{n(1-r)} \varphi(x, y) = 0$$

for all $x, y \in X$. Therefore $f$ and $\varphi$ satisfy the conditions of Theorem 2.3. It is easy to see that $\tilde{\varphi}(x) = |2|^{-1-r} \theta \| x \|^r$ if $0 < r < 1$ and $\tilde{\varphi}(x) = 2 |2|^{-1} \theta$ if $r = 0$. By Theorem 2.3 there is a unique quadratic-additive mapping $T : X \to Y$ satisfying (2.14). □
References


Chang-Ju Lee
Department of Mathematics Education, Gongju National University of Education,
Gongju 314-060, Republic of Korea.
E-mail: chjlee@gjue.ac.kr

Yang-Hi Lee
Department of Mathematics Education, Gongju National University of Education,
Gongju 314-711, Republic of Korea.
E-mail: yanghi2@hanmail.net