ON A PRODUCT-SYMMETRIC RECURRENT-METRIC
CONNECTION IN AN ALMOST HERMITIAN
MANIFOLD

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Abstract. In the present paper, we define a product-symmetric recurrent-metric connection in an almost Hermitian manifold and study some properties of this connection, in particular, its curvature properties.

1. Introduction

Let $M^n = (M^n, g)$ be a Riemannian manifold of dimension $n$ with a metric tensor $g$. A linear connection $\nabla$ on $M^n$ satisfies

$(i) \ \nabla_{fX + gY} = f\nabla_X + g\nabla_Y, \quad (ii) \ \nabla_X(fY) = (Xf)Y + f\nabla_X Y,$

where $f, g$ are smooth functions on $M^n$ and $X, Y$ are smooth vector fields on $M^n$. The torsion tensor $T$ of $\nabla$ is given by

$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$

If the torsion tensor $T$ vanishes, then $\nabla$ is said to be symmetric, otherwise it is nonsymmetric. If the metric tensor $g$ of $M^n$ satisfies $\nabla g = 0$, then $\nabla$ is said to be a metric connection, otherwise it is nonmetric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In 1930, Hayden [9] introduced a metric connection with a nonzero torsion on a Riemannian manifold.

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In particular, a nonsymmetric connection $\nabla$ is called semi-symmetric if the torsion tensor $T$ of $\nabla$ satisfies

$$T(X,Y) = u(Y)X - u(X)Y,$$

where $u$ is a 1-form on $M^n$. In 1924, Friedmann and Schouten [7] introduced the idea of a semi-symmetric linear connection in a differential manifold. In case of a semi-symmetric metric connection, Yano [19] considered such a connection and studied some of its properties. In fact, Yano proved that a Riemannian manifold is conformally flat if and only if it admits a semi-symmetric metric connection whose curvature tensor vanishes. Later on such a connection on a Riemannian manifold equipped with certain geometric structures was extensively studied by several authors [4,5,10,14,20]. In 1975, Golab [8] defined and studied quarter-symmetric linear connections in manifolds. A linear connection is said to be a quarter-symmetric connection if its torsion tensor $T$ is of the form

$$T(X,Y) = u(Y)\phi X - u(X)\phi Y,$$

where $u$ is a 1-form and $\phi$ is a tensor of type (1,1). In case of quarter-symmetric metric connections, several authors investigate their properties in [6,11,12,13,15,16,17,21] among others. In 2003, Sengupta and Biswas [18] defined quarter-symmetric nonmetric connection in a Sasakian manifold and studied its properties. Recently Chaubey and Ojha [3] defined a quarter-symmetric nonmetric connection on an almost Hermitian manifold and studied its geometric properties. In series, the properties of quarter-symmetric nonmetric connection in a Kähler manifold have been studied in [2]. The purpose of this paper is to introduce a different kind of nonsymmetric nonmetric connection (namely, product-symmetric recurrent-metric connection) into an almost Hermitian manifold and to study some properties of such a connection. In particular, we investigate the various symmetries and properties of the curvature with respect to the product-symmetric recurrent-metric connection under certain conditions.

2. A product-symmetric recurrent-metric connection in an almost Hermitian manifold

Let $M^n = (M^n, g, J)$ be an almost Hermitian manifold of dimension $n(= 2m)$ with almost complex structure $J$ and compatible Riemannian metric $g$, i.e., $J^2X = -X$ and $g(JX, JY) = g(X, Y)$. We define a linear
connection $\nabla$ in an almost Hermitian manifold $M^n$ as follows:
\begin{equation}
\nabla_X Y = \nabla_X Y - u(X)Y - u(JX)JY,
\end{equation}
where $\nabla$ denotes the Levi-Civita connection. Using (2.1), the torsion tensor $\bar{T}$ of $\nabla$ is given by
\[ \bar{T}(X,Y) = u(Y)X - u(X)Y + u(JY)JX - u(JX)JY. \]
Further, using (2.1), we have
\[ (\nabla_X g)(Y,Z) = 2u(X)g(Y,Z). \]

The linear connection $\bar{\nabla}$ of the type defined by (2.1) is said to be a product-symmetric recurrent-metric connection (briefly, PSRM connection) in an almost Hermitian manifold. For instance, we find a non-trivial PSRM connection in a product almost Hermitian manifold as follows:

**Example.** Let $(M^n, g_{M^n}, J_{M^n})$ be an almost Hermitian manifold and $(T^2, g_{T^2}, J_{T^2})$ a flat torus with standard complex structure. Then it is easy to see that the product space $(M^{n+2}, g, J) = (M^n \times T^2, g_{M^n} + g_{T^2}, J_{M^n} + J_{T^2})$ is an almost Hermitian manifold. Since $T^2$ has a nowhere vanishing vector field, we can choose such a vector field $U$ tangent to $T^2$ at each point in $M^n \times T^2$ and so we obtain a non-trivial 1-form $u$ associated with $U$ on $M^{n+2}$ by $g(U, X) = u(X)$. Now we can define a non-trivial PSRM connection $\nabla$ in $(M^{n+2}, g, J)$ by using the 1-form $u$ mentioned above as follows:
\[ \nabla_X Y = \nabla_X Y - u(X)Y - u(JX)JY. \]

The Riemannian curvature tensor $R$ has the following well known $SO(n)$-decomposition [1]:
\[ R(X,Y,Z,V) = \frac{s}{2n(n-1)} g \bullet g(X,Y;Z,V) + \frac{1}{n-2} (r - \frac{s}{n} g) \bullet g(X,Y,Z,V) + W(X,Y,Z,V), \]
where $s,r,W$ are the scalar, Ricci, Weyl curvature tensor, respectively. Here the symbol $\bullet$ is the Nomizu-Kulkarni product of symmetric (0,2)-tensors generating a curvature type tensor:
\[ h \bullet k(X,Y,Z,W) = h(X,Z)k(Y,W) + h(Y,W)k(X,Z) - h(X,W)k(Y,Z) - h(Y,Z)k(X,W). \]
Note that $W = 0$ if and only if $(M^n, g)$ is conformally flat. The Weyl curvature tensor depends only on the conformal class of $(M^n, g)$. Moreover, it satisfies the curvature symmetries and so we can treat it as a conformal curvature tensor. In particular, the Weyl curvature tensor is traceless. Analogous to the definition of curvature tensor $R$, Ricci tensor $r$, scalar curvature $s$ and Weyl curvature tensor $W$, we define the curvature tensor $\bar{R}$, Ricci tensor $\bar{r}$, scalar curvature $\bar{s}$ and Weyl curvature tensor $\bar{W}$ with respect to $\bar{\nabla}$ by

$$\bar{R}(X,Y,Z,V) = g(\bar{R}(X,Y)Z,V) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,V),$$

(2.2)

$$\bar{r}(Y,Z) = \sum_{i=1,...,n} \bar{R}(e_i,Y,Z,e_i),$$

(2.3)

$$\bar{s} = \sum_{i=1,...,n} \bar{r}(e_i,e_i),$$

and

$$\bar{W}(X,Y,Z,V) = \bar{R}(X,Y,Z,V) - \frac{\bar{s}}{2n(n-1)} g \bullet g(X,Y,Z,V) - \frac{1}{n-2} (\bar{r} - \frac{\bar{s}}{n} g) \bullet g(X,Y,Z,V),$$

where $\{e_i\}_{i=1,...,n}$ is an orthonormal frame. The Riemannian manifold $(M^n, g)$ is called Einstein if the Ricci tensor $r$ of $M^n$ is proportional to the metric tensor $g$, i.e., $r = \frac{s}{n} g$. A 1-form $v$ on $M^n$ is said to be double-recurrent if it satisfies

$$\nabla_X v = v(X) v.$$

It is easy to see that a double-recurrent 1-form $v$ is closed. Concerning the PSRM connection in an almost Hermitian manifold, we have the following result:

**Theorem 2.1.** Let $M^n = (M^n, g, J)$ be an almost Hermitian manifold equipped with PSRM connection $\nabla$.

(i) Then we have


(ii) If the 1-form $u$ of $\nabla$ is double-recurrent, then we have

$$\bar{R}(X,Y,Z,W) = -\bar{R}(X,Y,W,Z).$$

(iii) If the 1-form $u$ of $\nabla$ is double-recurrent and in addition, $M^n$ is Kähler, then we have

$$\bar{r}(Y,Z) = \bar{r}(Z,Y).$$
(iv) If the 1-form $u$ of $\bar{\nabla}$ is double-recurrent and $M^n$ is Kähler and in addition $\bar{s} \geq s$, then the connections $\bar{\nabla}$ and $\nabla$ coincide.

Proof. Taking account of (2.1) and (2.2), we have
\begin{align}
\bar{R}(X,Y,Z,W) &= R(X,Y,Z,W) - g(\nabla_X U,Y)g(Z,W) \\
&\quad + g(\nabla_Y U,X)g(Z,W) - g(\nabla_X U,Y)g(JZ,W) \\
&\quad + g(\nabla_Y U,JX)g(JZ,W) + g(U,JX)g((\nabla_Y J)Z,W) \\
&\quad - g(U,JY)g((\nabla_X J)Z,W),
\end{align}
(2.4)
where $U$ is a vector field associated with the 1-form $u$ by $u(X) = g(U,X)$.

From (2.4), it follows that
\[ \bar{R}(X,Y,Z,W) = -\bar{R}(Y,X,Z,W) \]
holds true. If we further assume that the 1-form $u$ of $\bar{\nabla}$ is double-recurrent, then we have
\begin{align}
\bar{R}(X,Y,Z,W) &= R(X,Y,Z,W) - u(X)u(JY)g(JZ,W) \\
&\quad + u(Y)u(JX)g(JZ,W) - u((\nabla_X J)Y)g(JZ,W) \\
&\quad + u((\nabla_Y J)X)g(JZ,W) + u(JX)g((\nabla_Y J)Z,W) \\
&\quad - u(JY)g((\nabla_X J)Z,W),
\end{align}
(2.5)
It is easy to see that when $M^n = (M^n, g, J)$ is an almost Hermitian manifold, the identity
\begin{align}
 g((\nabla_X J)Y, Z) &= -g(Y, (\nabla_X J)Z)
\end{align}
(2.6)
holds. Therefore it follows from (2.5) and (2.6) that
\[ \bar{R}(X,Y,Z,W) = -\bar{R}(X,Y,W,Z) \]
holds. If we further assume that $M^n$ is Kähler, then (2.5) yields
\begin{align}
\bar{R}(X,Y,Z,W) &= R(X,Y,Z,W) - u(X)u(JY)g(JZ,W) \\
&\quad + u(Y)u(JX)g(JZ,W),
\end{align}
from which follows that
\begin{align}
\bar{r}(Y,Z) &= r(Y,Z) - u(JY)u(JZ) - u(Y)u(Z).
\end{align}
(2.7)
The identity (2.7) immediately yields
\[ \bar{r}(Y,Z) = \bar{r}(Z,Y). \]
Taking account of (2.7), we have
\begin{align}
\bar{s} &= s - 2||u||^2.
\end{align}
(2.8)
If we assume $\bar{s} \geq s$ on $M^n$, then from (2.8) we obtain $u = 0$, which tells us that the connections $\bar{\nabla}$ and $\nabla$ coincide. This completes the proof of Theorem 2.1.

Concerning Weyl curvature tensors, we have

**Corollary 2.2.** Let $M^n = (M^n, g, J)$ be a Kähler manifold equipped with PSRM connection $\bar{\nabla}$. If the associated 1-form $u$ of $\bar{\nabla}$ is double-recurrent and $\bar{s} \geq s$ on $M^n$, then the Weyl curvature tensors $W$ and $\bar{W}$ coincide.

On the other hand, the Kähler form $\Omega$ and the Nijenhuis tensor $N$ of an almost Hermitian manifold $M^n$ are defined by $\Omega(X,Y) = g(JX,Y)$ and $N(X,Y) = (\nabla_X J)Y - (\nabla_Y J)X - J((\nabla_X J)Y) + J((\nabla_Y J)X)$, respectively. Recall that $M^n$ is Kähler if and only if $\nabla J = 0$; $M^n$ is Hermitian if and only if $N = 0$ (in fact, it follows from the Nirenberg-Newlander theorem that the vanishing of $N$ is equivalent to the integrability of the almost complex structure $J$); $M^n$ is almost Kähler if and only if $(\nabla_X \Omega)(Y,Z) + (\nabla_Y \Omega)(Z,X) + (\nabla_Z \Omega)(X,Y) = 0$ (i.e., the Kähler form $\Omega$ of $M^n$ is closed). We call an almost Hermitian manifold $M^n = (M^n, g, J)$ Einstein with respect to $\bar{\nabla}$ if the Ricci tensor $\bar{\rho}$ with respect to $\bar{\nabla}$ is proportional to the metric tensor $g$ on $M^n$, i.e., $\bar{\rho} = \bar{s} / n g$.

Analogous to the definition of the Nijenhuis tensor $N$, we define the Nijenhuis tensor $\bar{N}$ with respect to $\bar{\nabla}$ by

$$\bar{N}(X,Y) = (\bar{\nabla}_X J)Y - (\bar{\nabla}_Y J)X - J((\bar{\nabla}_X J)Y) + J((\bar{\nabla}_Y J)X).$$

Then we obtain the following:

**Theorem 2.3.** On an almost Hermitian manifold $M^n = (M^n, g, J)$ with PSRM connection $\bar{\nabla}$, the Nijenhuis tensors $N$ and $\bar{N}$ coincide.

**Proof.** Covariant derivative of $JY$ with respect to $\bar{\nabla}$ gives

$$(\bar{\nabla}_X J)Y + J(\bar{\nabla}_X Y) = \bar{\nabla}_X (JY).$$

Taking account of (2.1) and the above identity, we have

$$\bar{\nabla}_X J)Y = (\bar{\nabla}_X J)Y.$$ (2.9)

From (2.9) and the definition of Nijenhuis tensor, it follows that

$$\bar{N}(X,Y) = N(X,Y),$$

for arbitrary vector fields $X, Y$ on $M^n$. This completes the proof of Theorem 2.3.

Since an almost Hermitian manifold with $N = 0$ is Hermitian, we have
Corollary 2.4. Let $M^n = (M^n, g, J)$ be an almost Hermitian manifold with PSRM connection $\bar{\nabla}$. If the Nijenhuis tensor $\bar{N}$ with respect to $\bar{\nabla}$ vanishes, then $M^n$ is Hermitian.

Since the Nijenhuis tensor $N$ vanishes on the Kähler manifold, we get

Corollary 2.5. On a Kähler manifold $M^n = (M^n, g, J)$ with PSRM connection $\bar{\nabla}$, the Nijenhuis tensor $\bar{N}$ with respect to $\bar{\nabla}$ vanishes.

On the other hand, we have the following:

Theorem 2.6. Let $M^n = (M^n, g, J) \ (n > 2)$ be an almost Kähler manifold equipped with PSRM connection $\bar{\nabla}$. Then the Kähler form $\Omega$ of $M^n$ is closed with respect to $\bar{\nabla}$ if and only if the connections $\bar{\nabla}$ and $\nabla$ coincide.

Proof. We have

$$X(\Omega(Y, Z)) = (\bar{\nabla}_X \Omega)(Y, Z) - \Omega(\bar{\nabla}_X Y - \nabla_X Y, Z) - \Omega(Y, \bar{\nabla}_X Z - \nabla_X Z).$$

Taking account of (2.1), the last identity becomes

$$(\bar{\nabla}_X \Omega)(Y, Z) = (\nabla_X \Omega)(Y, Z) - \Omega(-u(X)Y - u(JX)JY, Z)$$

Taking cyclic sum of (2.10) in $X, Y, Z$, we have

$$= (\nabla_X \Omega)(X, Z) + 2u(X)\Omega(Y, Z).$$

(2.11)

Considering the almost Kähler condition, we have from the last identity

$$= (\nabla_X \Omega)(X, Z) + 2u(Y)\Omega(Z, X) + 2u(Z)\Omega(X, Y).$$

(2.12)

If we assume that the Kähler form $\Omega$ of $M^n$ is closed with respect to $\bar{\nabla}$, then putting $X = e_i, Y = e_j$ and $Z = Je_j \ (e_i \notin \text{span}\{e_j, Je_j\})$ in (2.12), we get $u(e_i) = 0$ for each $i = 1, 2, ..., n (= 2m)$. Hence we have $\nabla = \bar{\nabla}$. Conversely, suppose that $\nabla = \bar{\nabla}$, i.e., $u = 0$, then it follows from (2.12) that

$$(\nabla_X \Omega)(Y, Z) + (\nabla_Y \Omega)(Z, X) + (\nabla_Z \Omega)(X, Y) = 0.$$
This completes the proof of Theorem 2.6.

Since a Kähler manifold is almost Kähler, we immediately have

**Corollary 2.7.** On a Kähler manifold $M^n = (M^n, g, J)(n > 2)$ with PSRM connection $\nabla$, the Kähler form $\Omega$ of $M^n$ is closed with respect to $\nabla$ if and only if the connections $\nabla$ and $\nabla$ coincide.

### 3. Concurrent vector field and PSRM connection

A vector field $V$ on $M^n$ is said to be concurrent if it satisfies

$$\nabla_X V = fX,$$

where $f$ is a function on $M^n$. Then we have the following:

**Theorem 3.1.** Let $M^n = (M^n, g, J)$ be an almost Hermitian manifold equipped with PSRM connection $\nabla$. Then we have


If the vector field $U$ associated with the 1-form $u$ by $u(X) = g(U, X)$ is concurrent, then we obtain

$$\check{\mathcal{R}}(X, Y, Z, W) = -\check{\mathcal{R}}(X, Y, W, Z).$$

If we further assume that $M^n$ is Kähler, then we have the following results:

(a) $\check{\mathcal{R}}(X, Y, Z, W) = \check{\mathcal{R}}(Z, W, X, Y),$

(b) $\check{\mathcal{R}}(X, Y, Z, W) + \check{\mathcal{R}}(Y, Z, X, W) + \check{\mathcal{R}}(Z, X, Y, W) = 0$ if and only if the function $f$ of a concurrent vector field $U$ vanishes, that is, the vector field $U$ is parallel,

(c) $\check{\mathcal{R}}(Y, Z) = \check{\mathcal{R}}(Z, Y),$

(d) $M^n$ is Einstein with respect to $\nabla$ if and only if $M^n$ is Einstein.

**Proof.** It follows from (2.4) that

$$\check{\mathcal{R}}(X, Y, Z, W) = -\check{\mathcal{R}}(Y, X, Z, W)$$

holds true. Suppose that the vector field $U$ associated with the 1-form $u$ is concurrent. Then we have from (2.4)

$$\check{\mathcal{R}}(X, Y, Z, W) = R(X, Y, Z, W) + 2fg(JX, Y)g(JZ, W)$$

$$- g(U, (\nabla_X J)Y)g(JZ, W) + g(U, (\nabla_Y J)X)g(JZ, W)$$

$$+ g(U, JX)g((\nabla_Y J)Z, W) - g(U, JY)g((\nabla_J J)Z, W).$$

(3.2)

It follows from (2.6) and (3.2) that

$$\check{\mathcal{R}}(X, Y, Z, W) = -\check{\mathcal{R}}(X, Y, W, Z).$$
On a product-symmetric recurrent-metric connection holds true. If we further suppose that $M^n$ is Kähler, then we have from (3.2)

\[ \bar{R}(X,Y,Z,W) = R(X,Y,Z,W) + 2fg(JX,Y)g(JZ,W), \]

from which follows that

\[ \bar{r}(Y,Z) = r(Y,Z) - 2fg(Y,Z) \]

and

\[ \bar{s} = s - 2fn. \]

With the help of (3.3), it is obvious that $\bar{R}(X,Y,Z,W) = \bar{R}(Z,W,X,Y)$ holds true. On the other hand, from (3.3) and the first Bianchi identity, it follows that $\bar{R}(X,Y,Z,W) + \bar{R}(Y,Z,X,W) + \bar{R}(Z,X,Y,W) = 0$ implies that the function $f$ vanishes. Conversely, according to (3.3) and the first Bianchi identity, it is clear that $f = 0$ implies $\bar{R}(X,Y,Z,W) + \bar{R}(Y,Z,X,W) + \bar{R}(Z,X,Y,W) = 0$. On the other hand, by the aid of (3.4), it is easy to see that $\bar{r}(Y,Z) = \bar{r}(Z,Y)$ holds. And by virtue of (3.4), we also conclude that if $M^n$ is Einstein with respect to $\bar{\nabla}$, then $M^n$ is Einstein, and vice versa. This completes the proof of Theorem 3.1.

References


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