MINIMAL CLOZ-COVERS OF $kX$

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Abstract. In this paper, we first show that $z_{kX} : E_{ce}(kX) \to kX$ is $z^\#$-irreducible and that if $G(E_{ce}(\beta X))$ is a base for closed sets in $\beta X$, then $E_{ce}(kX)$ is $C^*$-embedded in $E_{ce}(\beta X)$, where $kX$ is the extension of $X$ such that $eX \subseteq kX \subseteq \beta X$ and $kX$ is weakly Lindelöf. Using these, we will show that if $G(\beta X)$ is a base for closed sets in $\beta X$ and for any weakly Lindelöf space $Y$ with $X \subseteq Y \subseteq kX$, $kX = Y$, then $kE_{ce}(X) = E_{ce}(kX)$ if and only if $\beta E_{ce}(X) = E_{ce}(\beta X)$.

1. Introduction

All spaces in this paper are Tychonoff spaces and $\beta X(vX, \text{ resp.})$ denotes the Stone-Čech compactification(Hewitt realcompactification, resp.) of a space $X$.

Iliadis constructed the absolute of Hausdorff spaces, which is the minimal extremally disconnected cover of Hausdorff spaces and they turn out to be the perfect onto projective covers([5]). To generalize extremally disconnected spaces, basically disconnected spaces, quasi-F spaces and cloz-spaces have been introduced and their minimal covers have been studied by various authors([3]). In these ramifications, minimal covers of compact spaces can be nicely characterized.

In particular, Henriksen, Vermeer and Woods ([3]) introduced the notion of cloz-spaces and they showed that every compact space $X$ has a minimal cloz-cover $(E_{ce}(X), z_X)$. In [6], it was shown that every space has a minimal cloz-cover.

In this paper, we first show that $z_{kX} : E_{ce}(kX) \to kX$ is $z^\#$-irreducible and that if $G(E_{ce}(\beta X))$ is a base for closed sets in $\beta X$, then...
$E_{cc}(kX)$ is $C^*$-embedded in $E_{cc}(\beta X)$, where $kX$ is the extension of $X$ such that $vX \subseteq kX \subseteq \beta X$ and $kX$ is weakly Lindelöf. Using these, we will show that if $G(\beta X)$ is a base for closed sets in $\beta X$ and for any weakly Lindelöf space $Y$ with $X \subseteq Y \subseteq kX$, $kX = Y$, then $kE_{cc}(X) = E_{cc}(kX)$ if and only if $\beta E_{cc}(X) = E_{cc}(\beta X)$. For the terminology, we refer to [1] and [9].

2. Minimal cloz-covers of $kX$

The set $R(X)$ of all regular closed sets in a space $X$, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows: for any $A \in R(X)$ and any $\{A_i \mid i \in I\} \subseteq R(X)$,

- $\bigvee\{A_i \mid i \in I\} = cl_X(\bigcup\{A_i \mid i \in I\}),$
- $\bigwedge\{A_i \mid i \in I\} = cl_X(\bigcap\{A_i \mid i \in I\}),$
- $A' = cl_X(X - A)$

and a sublattice of $R(X)$ is a subset of $R(X)$ that contains $\emptyset$, $X$ and is closed under finite joins and meets.

Recall that a map $f : Y \longrightarrow X$ is called a covering map if it is a continuous, onto, perfect, and irreducible map.

**Lemma 2.1.** ([3])

1. Let $f : Y \longrightarrow X$ be a covering map. Then the map $\psi : R(Y) \longrightarrow R(X)$, defined by $\psi(A) = f(A)$, is a Boolean isomorphism and the inverse map $\psi^{-1}$ of $\psi$ is given by $\psi^{-1}(B) = cl_Y(f^{-1}(int_X(B))) = cl_Y(int_Y(f^{-1}(B)))$.
2. Let $X$ be a dense subspace of a space $K$. Then the map $\phi : R(K) \longrightarrow R(X)$, defined by $\phi(A) = A \cap X$, is a Boolean isomorphism and the inverse map $\phi^{-1}$ of $\phi$ is given by $\phi^{-1}(B) = cl_K(B)$.

**Definition 2.2.** Let $X$ be a space.

1. A cozero-set $C$ in $X$ is said to be a complemented cozero-set in $X$ if there is a cozero-set $D$ in $X$ such that $C \cap D = \emptyset$ and $C \cup D$ is a dense subset of $X$. In case, $\{C, D\}$ is called a complemented pair of cozero-sets in $X$.
2. Let $\mathcal{G}(X) = \{cl_X(C) \mid C$ is a complemented cozero-set in $X\}$.

Let $X$ be a space and $Z(X)^\# = \{cl_X(int_X(A)) \mid A$ is a zero-set in $X\}$. Suppose that $\{C, D\}$ is a complemented pair of cozero-sets in $X$. Then $cl_X(C) = cl_X(X - D)$ and since $cl_X(X - D) \in Z(X)^\#$, $cl_X(C) \in Z(X)^\#$. Hence $\mathcal{G}(X) = \{A \in Z(X)^\# \mid A' \in Z(X)^\#\}$ and $\mathcal{G}(X)$ is a Boolean subalgebra of $R(X)$.  


Since $X$ is $C^*$-embedded in $\beta X$, by Lemma 2.1., $G(X)$ and $G(\beta X)$ are Boolean isomorphic.

**Definition 2.3.** ([3]) A space $X$ is called a cloz-space if every element of $G(X)$ is a clopen set in $X$.

A space $X$ is a cloz-space if and only if $\beta X$ is a cloz-space([3]).

**Definition 2.4.** Let $X$ be a space.

1. A pair $(Y, f)$ is called a cloz-cover of $X$ if $Y$ is a cloz-space and $f : Y \to X$ is a covering map.
2. A cloz-cover $(Y, f)$ of $X$ is called a minimal cloz-cover of $X$ if for any cloz-cover $(Z, g)$ of $X$, there is a covering map $h : Z \to Y$ with $f \circ h = g$.

Let $B$ be a Boolean subalgebra of $R(X)$. Let $S(B) = \{\alpha \mid \alpha$ is a $B$-ultrafilter$\}$ and for any $B \in B$, let $\Sigma^B_\alpha = \{\alpha \in S(B) \mid B \in \alpha\}$. Then the space $S(B)$, equipped with the topology for which $\{\Sigma^B_\alpha \mid B \in B\}$ is a base, called the Stone-space of $B$. Then $S(B)$ is a compact zero-dimensional space([9]).

Henriksen, Vermeer and Woods showed that every compact space $X$ has the minimal cloz-cover $(E_{ec}(X), z_X)$. Let $X$ be a compact space, $S(G(X))$ the Stone-space of $G(X)$ and $E_{ec}(X) = \{(\alpha, x) \mid x \in \cap\{A \mid A \in \alpha\}\}$ the subspace of the product space $S(G(X)) \times X$. Then $(E_{ec}(X), z_X)$ is the minimal cloz-cover of $X$, where $z_X((\alpha, x)) = x([3])$. It was shown that every space has a minimal cloz-cover([7]).

For any space $X$, let $z_\beta = z_{\beta X}$.

**Lemma 2.5.** ([6]) Let $X$ be a space. If $z^{-1}_\beta(X)$ is a cloz-space, then $(z^{-1}_\beta(X), z_{\beta X})$ is the minimal cloz-cover of $X$.

A $z$-filter $\mathcal{F}$ on a space $X$ is called real if $\mathcal{F}$ is closed under countable intersections.

For any space $X$, let $kX = vX \cup \{p \in \beta X - vX \mid$ there is a real $z$-filter $\mathcal{F}$ on $X$ such that $\cap\{\text{cl}_{\omega X}(F) \mid F \in \mathcal{F}\} = \emptyset$ and $p \in \cap\{\text{cl}_{\beta X}(F) \mid F \in \mathcal{F}\}\}$. Then $kX$ is an extension of a space $X$ such that $vX \subseteq kX \subseteq \beta X([8])$.

A space $X$ is called a weakly Lindelöf space if for any open cover $\mathcal{U}$ of $X$, there is a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\cup\{V \mid V \in \mathcal{V}\}$ is a dense subset of $X$. It is well-known that a space $X$ is weakly Lindelöf if and only if for any $R(X)$-filter $\mathcal{F}$ with the countable meet property, $\cap\{F \mid F \in \mathcal{F}\} \neq \emptyset$. 
Lemma 2.6. ([8]) Let $X$ be a space. Then $kX$ is a weakly Lindelöf space and for any continuous map $f : X \to Y$, there is a continuous map $f_k : kX \to kY$ such that $f_k \circ kX = kY \circ f$.

Let $X$ be a space. For any $B \in \mathcal{G}(\beta X)$, let $\Sigma_B^{\mathcal{G}(\beta X)} = \Sigma_B$ and $(\Sigma_B \times X) \cap E^\alpha(kX) = \sigma_B$. Then for any $B \in \mathcal{G}(\beta X)$, $z_\beta(\Sigma_B \times X) = B$.

Let $z_k = z_{\beta_kX} : z_\beta^{-1}(kX) \to kX$ be the restriction and corestriction of $z_\beta$ with respect to $z_\beta^{-1}(kX)$ and $kX$, respectively. Clearly, we have the following lemma:

We recall that a covering map $f : Y \to X$ is called $z^\#$-irreducible if $f(Z(Y)^\#) = Z(X)^\#$. Let $f : Y \to X$ be a covering map and $Z$ a zero-set in $X$. By Lemma 2.1, $f(cl_Y(int_Y(f^{-1}(Z)))) = cl_X(int_X(Z))$ and $cl_Y(int_Y(f^{-1}(Z))) \subseteq Z(X)^\#$. Hence $Z(X)^\# \subseteq f(Z(Y)^\#)$ and so $f : Y \to X$ is $z^\#$-irreducible if and only if $f(Z(Y)^\#) \subseteq Z(X)^\#$.

Lemma 2.7. Let $f : Y \to X$ and $g : W \to Y$ be covering maps. Then $f \circ g : W \to X$ is $z^\#$-irreducible if and only if $f : Y \to X$ and $g : W \to Y$ are $z^\#$-irreducible.

Let $X$ be a space such that $\mathcal{G}(X)$ is a base for closed sets in $X$. Then $E^\alpha(X) = \{ \alpha \in S(\mathcal{G}(X)) \mid \cap \alpha \neq \emptyset \}$ is the subspace of $S(\mathcal{G}(X))(\{\emptyset\})$.

Theorem 2.8. Let $X$ be a space. Then we have the following:

1. $z_\beta^{-1}(kX)$ is a weakly Lindelöf space,
2. $z_k$ is $z^\#$-irreducible,
3. $z_k(\mathcal{G}(z_\beta^{-1}(kX))) = \mathcal{G}(kX)$,
4. $(z_\beta^{-1}(kX), z_k)$ is the minimal cloz-cover of $kX$, and
5. if $\mathcal{G}(E^\alpha(\beta X))$ is a base for closed sets in $\beta X$, then $E^\alpha(kX)$ is $C^*$-embedded in $E^\alpha(\beta X)$.

Proof. (1) Let $T = z_\beta^{-1}(kX)$. Suppose that there is an $R(T)$-filter $\mathcal{F}$ with the countable meet property such that $\cap \{ F \mid F \in \mathcal{F} \} = \emptyset$.

We first claim that $\cap \{ z_k(F) \mid F \in \mathcal{F} \} = \emptyset$. Suppose that $\cap \{ z_k(F) \mid F \in \mathcal{F} \} \neq \emptyset$. Pick $x \in \cap \{ z_k(F) \mid F \in \mathcal{F} \}$. Note that for any $A \subseteq R(T)$, $\cap A \subseteq \cap A$. Since $\mathcal{F}$ has the countable meet property, $\mathcal{F}$ has the finite intersection property. Hence $\{ F \cap z_k^{-1}(x) \mid F \in \mathcal{F} \}$ is a family of closed sets in $z_k^{-1}(x)$ with the finite intersection property. Since $z_k^{-1}(x)$ is a compact subset in $T$, $\cap \{ F \cap z_k^{-1}(x) \mid F \in \mathcal{F} \} = \emptyset$ and so $\cap \{ F \mid F \in \mathcal{F} \} = \emptyset$. This is a contradiction.

Hence $\cap \{ z_k(F) \mid F \in \mathcal{F} \} = \emptyset$. Since $kX$ is a weakly Lindelöf space, there is a sequence $(F_n)$ in $\mathcal{F}$ such that $cl_{kX}(\cup \{ kX - z_k(F_n) \mid n \in N \}) = kX$. Let $F \in \mathcal{F}$. Then $z_k^{-1}(z_k(T - F)) \supseteq T - F$ and hence
Minimal cloz-covers of $kX$

$z_k(F') \supseteq z_k(T - F) \supseteq kX - z_k(F)$. Thus $cl_{kX}(\cup\{z_k(F'_n) \mid n \in N\}) = kX$. Note that

$$kX = cl_{kX}(\cup\{z_k(F'_n) \mid n \in N\})$$

$$= cl_{kX}(z_k(\cup\{F'_n \mid n \in N\}))$$

$$= z_k(cl_{T}(\cup\{F'_n \mid n \in N\}))$$

$$= z_k(\vee\{F'_n \mid n \in N\}).$$

Since $z_k$ is an irreducible map, by Lemma 2.1, $\vee\{F'_n \mid n \in N\} = T$ and so $(\vee\{F'_n \mid n \in N\})' = \wedge\{F_n \mid n \in N\} = \emptyset$. Since $\mathcal{F}$ has the countable meet property, it is a contradiction. Hence $T$ is a weakly Lindelöf space.

(2) Take any $Z \in Z(T)^\#$. By (1), $T$ is a weakly Lindelöf space and hence there is a sequence $(A_n)$ in $Z(E_{cc}(\beta X))^{\#}$ such that $T - Z = cl_{E_{cc}(\beta X)}(\cup\{T - A_n \mid n \in N\}) \cap (T - Z)$ and for any $n \in N$, $T - A_n \subseteq T - Z$. Then clearly, $Z \subseteq cl_{E_{cc}(\beta X)}(int_{E_{cc}(\beta X)}(\cap\{A_n \mid n \in N\})) \cap T = \wedge\{A_n \mid n \in N\} \cap T$. Further,

$$Z = (T - cl_{E_{cc}(\beta X)}(\cup\{T - A_n \mid n \in N\})) \cup Z$$

$$= int_{E_{cc}(\beta X)}(\cap\{A_n \cap T \mid n \in N\}) \cup Z$$

and hence $\wedge\{A_n \cap T \mid n \in N\} \subseteq Z$. Thus $Z = (\wedge\{A_n \mid n \in N\}) \cap T$. Note that $z_k(Z) = z_\beta(\wedge\{A_n \mid n \in N\}) \cap kX = (\wedge\{z_\beta(A_n) \mid n \in N\}) \cap kX$. Since $\wedge\{A_n \mid n \in N\} \in Z(E_{cc}(\beta X))^{\#}$ and $z_\beta$ is $z^{\#}$-irreducible, $z_k(Z) \in kX^{\#}$.

(3) Clearly, $\mathcal{G}(kX) \subseteq z_k(\mathcal{G}(T))$. Let $B \in \mathcal{G}(T)$. Then $B, B' \in Z(T)^\#$. By (2) $z_k$ is $z^{\#}$-irreducible and so $z_k(B), z_k(B') \in Z(kX)^{\#}$. Hence $z_k(B) \in \mathcal{G}(kX)$ and thus $z_k(\mathcal{G}(T)) \subseteq \mathcal{G}(kX)$.

(4) Let $B \in \mathcal{G}(T)$. By (3), there is an $A \in \mathcal{G}(\beta X)$ such that $A \cap kX = z_k(B)$. Then $cl_{E_{cc}(\beta X)}(int_{E_{cc}(\beta X)}(z_\beta^{-1}(A)))$ is a clopen set in $E_{cc}(\beta X)$ and $cl_{E_{cc}(\beta X)}(int_{E_{cc}(\beta X)}(z_\beta^{-1}(A))) \cap T = B$. Hence $B$ is clopen in $T$ and so $T$ is a cloz-space. By Lemma 2.5, $(z_\beta^{-1}(kX), z_k)$ is the minimal cloz-cover of $kX$.

(5) Suppose that $\mathcal{G}(E_{cc}(\beta X))$ is a base for closed sets in $\beta X$. Then $E_{cc}(\beta X) = S(\mathcal{G}(\beta X))(3)$ and $E_{cc}(\beta X)$ is a zero-dimensional space. Since $z_\beta^{-1}(kX)$ is the minimal cloz-cover of $kX$, $\beta E_{cc}(kX)$ and $S(z_k(\mathcal{G}(T)))$
are homeomorphic ([7]). By (3), $S(z_k(G(T)))$ and $S(G(kX))$ are homeomorphic. By Lemma 2.1, $G(kX)$ and $G(\beta X)$ are Boolean isomorphic and so $\beta Ecc(kX)$ is homeomorphic to $Ecc(\beta X)$. 

Let $X$ be a space. Then there is a covering map $g : \beta Ecc(X) \to Ecc(\beta X)$ such that $z_\beta \circ g \circ \beta Ecc(X) = \beta_X \circ z_X$. By Lemma 2.6, there is a unique continuous map $z_X^k : k Ecc(X) \to kX$ such that $z_X^k \circ k Ecc(X) = k_X \circ z_X$. Since $k Ecc(X)$ is a dense embedding, $\beta_X \circ z_X^k = z_\beta \circ g \circ k Ecc(X)$. Hence there is a continuous map $l : k Ecc(X) \to Ecc(kX)$ such that $j \circ l = g \circ \beta k Ecc(X)$ and $z_k \circ l = z_X^k([9])$, where $j : Ecc(kX) \to Ecc(\beta X)$.

**Corollary 2.9.** Let $X$ be a space such that $G(\beta X)$ is a base for closed sets in $\beta X$ and $k Ecc(X) = Ecc(kX)$, that is, $l : k Ecc(X) \to Ecc(kX)$ is a homeomorphism. Then $\beta Ecc(X) = Ecc(\beta X)$, that is, $g : \beta Ecc(X) \to Ecc(\beta X)$ is a homeomorphism ([1]).

**Proof.** Since $l : k Ecc(X) \to Ecc(kX)$ is a homeomorphism, by Theorem 2.8, $k Ecc(X)$ is $C^*$-embedded in $Ecc(\beta X)$. Hence $\beta Ecc(X) = \beta k Ecc(X) = Ecc(\beta X)$. Thus $g$ is a homeomorphism. 

Let $X$ be a space such that $\beta Ecc(X) = Ecc(\beta X)$. Then $g : \beta Ecc(X) \to Ecc(\beta X)$ is a homeomorphism such that $j \circ l = g \circ \beta k Ecc(X)$. Since $g \circ \beta k Ecc(X)$ is an embedding, $l$ is an embedding.

**Theorem 2.10.** Let $X$ be a space such that $G(\beta X)$ is a base for closed sets in $\beta X$ and for any weakly Lindelöf space $Y$ with $X \subseteq Y \subseteq kX$, $kX = Y$. Then $k Ecc(X) = Ecc(kX)$ if and only if $\beta Ecc(X) = Ecc(\beta X)$.

**Proof.** Suppose that $\beta Ecc(X) = Ecc(\beta X)$. Then clearly, $z_\beta^{-1}(X) = Ecc(X)$). Let $m = z_X^k : k Ecc(X) \to kX$.

We first claim that $m(k Ecc(X))$ is a weakly Lindelöf space. Take any open cover $U$ of $m(k Ecc(X))$. Then $\mathcal{V} = \{m^{-1}(U) \mid U \in \mathcal{U}\}$ is an open cover of $k Ecc(X)$. Since $k Ecc(X)$ is a weakly Lindelöf space, there is a countable subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $\cup \{m^{-1}(U) \mid U \in \mathcal{U}_0\}$ is dense in $m(k Ecc(X))$. Since $m$ is continuous, $\cup \{U \mid U \in \mathcal{U}_0\}$ is dense in $m(k Ecc(X))$. Hence $m(k Ecc(X))$ is weakly Lindelöf.

Since $X \subseteq m(k Ecc(X)) \subseteq kX$, by the assumption, $m(k Ecc(X)) = kX$ and so $m$ is onto. Take any $x \in kX$. Since $m$ is an onto map and $z_X$ is a covering map, $m(k Ecc(X) - Ecc(X)) = kX - X([9])$. Since $\beta_X \circ m = z_\beta \circ g \circ kAX$, $m^{-1}(x) = (z_\beta \circ g)^{-1}(x) \subseteq k Ecc(X) - Ecc(X)$. 


Since \( z_\beta \circ g \) is a covering map, \( m^{-1}(x) \) is a compact subset of \( kE_{cc}(X) \) and hence \( m \) is a compact map.

Let \( F \) be a closed set in \( kE_{cc}(X) \) and \( x \in kX - m(F) \). Then \( m^{-1}(x) \cap F = \emptyset \). Since \( m^{-1}(x) \) is a compact space and \( E_{cc}(\beta X) \) is the Stone space of \( \mathcal{G}(\beta X) \), there is a \( B \in \mathcal{G}(\beta X) \) such that \( m^{-1}(x) \subseteq \Sigma_B \) and \( F \subseteq \Sigma_{B'} \). Since \( z_\beta(\Sigma_B') = B' \) and \( z_\beta^{-1}(x) \cap \Sigma_{B'} = m^{-1}(x) \cap \Sigma_{B'} = \emptyset \), \( x \notin B' \). Since \( \text{cl}_{kX}(m(F)) \subseteq B' \), \( x \notin \text{cl}_{kX}(m(F)) \). Thus \( m \) is a closed map and so \( m \) is a perfect map.

Since \( z_\beta \circ g \circ \beta_{kE_{cc}(X)} = \beta_{kX} \circ m \) and \( z_\beta \circ g \) is a covering map, \( m \) is a covering map. Since \( kE_{cc}(X) \) is a cloz-space, there is a covering map \( t : kE_{cc}(X) \to E_{cc}(kX) \) such that \( z_k \circ t = m \). Since \( kE_{cc}(X) \) is \( C^* \)-embedded in \( \beta E_{cc}(X) \) and \( z_\beta \circ g : \beta E_{cc}(X) \to \beta X \) is \( z^* \)-irreducible, \( m \) is \( z^* \)-irreducible. Hence by Lemma 2.7, \( t \) is \( z^* \)-irreducible.

Take any \( \delta_1 \neq \delta_2 \) in \( kE_{cc}(X) \). Note that \( \beta E_{cc}(X) = S(\mathcal{G}(\beta X)) \) and \( kE_{cc}(X) \subseteq \beta E_{cc}(X) \). Then there are \( A, B \in \mathcal{G}(\beta X) \) such that \( \delta_1 \in \sigma_A, \delta_2 \in \sigma_B \) and \( \sigma_A \cap \sigma_B = \emptyset \). Since \( m \) is \( z^* \)-irreducible, \( m(\sigma_A) = z_k(t(\sigma_A)) \in \mathcal{G}(kX) \). Hence \( \text{cl}_{E_{cc}(kX)}(z_k^{-1}(z_k(t(\sigma_A)))) = t(\sigma_A) \) is a clopen set in \( E_{cc}(kX) \). Similarly, \( t(\sigma_B) \) is a clopen set in \( E_{cc}(kX) \). Since \( t(\sigma_A) \land t(\sigma_A) = \emptyset \), \( t(\sigma_A) \cap t(\sigma_A) = \emptyset \). Since \( t(\delta_1) \in t(\sigma_A) \) and \( t(\delta_2) \in t(\sigma_B), t \) is one-to-one and hence \( t \) is a homeomorphism. \( \square \)

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