MODULAR TRANSFORMATION FORMULAE FOR GENERALIZED NON-HOLOMORPHIC EISENSTEIN SERIES

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Abstract. In this paper, we compute transformation formulae for generalized non-holomorphic Eisenstein series.

1. Introduction

J. Lewittes [3] proved transformation formulae for a large class of Eisenstein series which is defined by

\[ G(z, s, r_1, r_2) = \sum_{m,n=-\infty}^{\infty}^\prime ((m + r_1)z + n + r_2)^{-s}, \]

where the dash \( ^\prime \) means \((m,n) \neq (-r_1,-r_2)\), \( \Im z > 0, r_1, r_2 \) are real numbers and \( \Re s > 2 \). However, Lewittes’s results were too complicated to deduce desired results. It was B. C. Berndt [1] who has obtained considerably simpler formulae than Lewittes’s results. Berndt [2] also proved transformation formulae for a more general class of Eisenstein series than Lewittes’s case, which is defined by

\[ G(z, s; r_1, r_2, h_1, h_2) := \sum_{m,n=-\infty}^{\infty}^\prime e^{2\pi i (mh_1 + nh_2)} ((m + r_1)z + n + r_2)^{-s}, \]

where the dash \( ^\prime \) means \((m,n) \neq (-r_1,-r_2)\), \( \Im z > 0, r_1, r_2, h_1, h_2 \) are real and \( \Re s > 2 \).

In this paper, we consider a class of generalized non-holomorphic Eisenstein series. In fact, H. Maass [5] considered non-holomorphic Eisenstein series(when \( r = h = (0,0) \) in section 3) as an important thing of the theory of non-holomorphic modular forms and determined the Fourier
series of this non-holomorphic Eisenstein series. We obtain modular
transformation formulae for generalized non-holomorphic Eisenstein se-
ries.

2. Notations

If \( w \) is a complex, we choose the branch of the argument defined
by \(-\pi \leq \arg w < \pi\). We write \( e(w) \) for \( e^{2\pi i w} \). Let \( \lambda \) denote the
characteristic function of the integers. In the sequel, \( V \tau = V(\tau) = \frac{a \tau + b}{c \tau + d} \)
always denotes a modular transformation with \( c > 0 \) for every complex
\( \tau \). Let \( r = (r_1, r_2) \) and \( h = (h_1, h_2) \) denote real vectors, and define the
associated vectors \( R \) and \( H \) by

\[
R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)
\]

and

\[
H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).
\]

For real \( x, \alpha \) and \( \text{Re } s > 1 \), let

\[
\psi(x, \alpha, s) := \sum_{n+\alpha > 0} \frac{e(nx)}{(n + \alpha)^s}.
\]

For a real number \( x \), \([x]\) denotes the greatest integer less than or equal
to \( x \) and \( \{x\} := x - [x] \). Let \( \text{hypergeometric function defined by} \)

\[
\text{\( 2F1(\alpha, \beta; \gamma; z) \) is a hypergeometric function defined by}
\]

\[
\text{\( 2F1(\alpha, \beta; \gamma; z) \)} := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)n!} z^n,
\]

where \((x)_n\) denotes the rising factorial defined by

\[
(x)_n := x(x + 1) \cdots (x + n - 1) \text{ for } n > 0, \ (x)_0 := 1.
\]

Euler's integral representation of \( 2F1 \)-hypergeometric function says that,
for \( \text{Re } \gamma > \text{Re } \alpha > 0 \) and \( z \in \mathbb{C}\setminus[1, \infty) \),

\[
2F1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-zt)^{-\beta} \, dt.
\]

Thus we see that \([4, 6]\)

\[
\frac{1}{\Gamma(\gamma)} \cdot 2F1(\alpha, \beta; \gamma; z)
\]

can be analytically continued to all \( \alpha, \ \beta, \ \gamma \in \mathbb{C} \) and all \( z \in \mathbb{C}\setminus[1, \infty) \).
3. Generalized non-holomorphic Eisenstein series

Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$, the upper half-plane.

**Definition 3.1.** Let $\tau \in \mathbb{H}$ and $s, s_1 \in \mathbb{C}$. For $\text{Re} \ s > 2$, the generalized non-holomorphic Eisenstein series $G(\tau, \bar{\tau}, s, s_1; r, h)$ is defined to be

$$G(\tau, \bar{\tau}, s, s_1; r, h) := \sum_{m,n=-\infty}^{\infty} \frac{e(mh_1 + nh_2)}{((m + r_1)\tau + n + r_2)^s_1((m + r_1)\bar{\tau} + n + r_2)^{s_1}},$$

where the dash $'$ means $(m,n) \neq (-r_1, -r_2)$.

**Theorem 3.2.** Let $Q = \{\tau \in \mathbb{H} \mid \text{Re} \ \tau > -d/c\}$ and $q = c\{R_2\} - d\{R_1\}$. Then for $\tau \in Q$, $s_1 \in \mathbb{C}$ and $s \in \mathbb{C}$ with $\text{Re} \ s > 2$,

$$\begin{align*}
(c\tau + d)^{-s_1}(c\bar{\tau} + d)^{-s_1}G(V\tau, V\bar{\tau}, s, s_1; r, h) &= G(\tau, \bar{\tau}, s, s_1; R, H) - 2i \sin(\pi s)\lambda(R_1)e(-R_1H_1)\psi(-H_2, -R_2, s) \\
&= \frac{e^{-\frac{c}{2}}}{\Gamma(s_1)\Gamma(s - s_1)}L(\tau, \bar{\tau}, s, s_1; ; R, H),
\end{align*}$$

where

$$L(\tau, \bar{\tau}, s, s_1; R, H) := \sum_{j=1}^{\infty} e(-H_1(j + [R_1] - c) - H_2([R_2] + 1 + [jd + q]/c - d))$$

$$\int_0^1 u^{s_1-1}(1 - v)^{s_1-1} \left( \int_C u^{s-1} e^{-((c\tau + d)v + (c\bar{\tau} + d)(1-v))u/c \frac{e^{(jd+q)/c}u}{e^u - e(-H_2)}} \right) dv,$$

where $C$ is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u = 0$ is the only zero of

$$e^{-((c\tau + d)v + (c\bar{\tau} + d)(1-v))u/c - e(cH_1 + dH_2)}(e^u - e(-H_2))$$

lying “inside” the loop, and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of $u^s$ with $0 < \text{arg} \ u < 2\pi$.

**Proof.** For $M = am + cn$, $N = bm + dn$,

$$((m + r_1)V\tau + n + r_2)^s = \left(\frac{a\tau + b}{c\tau + d} + n + r_2\right)^s$$

$$= \left(\frac{(M + R_1)\tau + N + R_2}{c\tau + d}\right)^s,$$

$$= \left(\frac{(M + R_1)\tau + N + R_2}{c\tau + d}\right)^s,$$

$$= \left(\frac{(M + R_1)\tau + N + R_2}{c\tau + d}\right)^s.$$
If \( m \) and \( n \) run over all integers, then \( M \) and \( N \) run over all integer except for \((M, N) = (-R_1, -R_2)\) since \( ad - bc = 1 \). Using (3.2), we see that

\[
G(\tau, \bar{\tau}, s, s_1; r, h) = \sum_{M,N=-\infty}^{\infty} e^{(M + N)H_2} \left( \frac{MH_1 + NH_2}{e^{\tau + d}} \right)^s \left( \frac{(M + N)H_1 + (M + N)H_2}{e^{\bar{\tau} + d}} \right)^{s-s_1}.
\]

Thus, by Lemma 1 in [1],

\[
(c \tau + d)^{-s_1} (c \bar{\tau} + d)^{-s + s_1} G(\tau, \bar{\tau}, s, s_1; r, h) = e(-s) \sum_{(m,n) \in I} \frac{e(mH_1 + nH_2)}{(m + R_1)\tau + n + R_2)^s ((m + R_1)\bar{\tau} + n + R_2)^{s-s_1} + e(-s) - 1 g(\tau, \bar{\tau}, s, s_1; R, H),
\]

where

\[
g(\tau, \bar{\tau}, s, s_1; R, H) := \sum_{(m,n) \in I} \frac{e(mH_1 + nH_2)}{(m + R_1)\tau + n + R_2)^s ((m + R_1)\bar{\tau} + n + R_2)^{s-s_1},
\]

and

\[I := \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid m + R_1 \leq 0, \ d(m + R_1) > c(n + R_2)\}.
\]

Replacing \( m \) by \(-m\) and \( n \) by \(-n\), and separating the sum for \( m = -R_1 \), we have

\[
g(\tau, \bar{\tau}, s, s_1; R, H) = \lambda(R_1) e \left( \frac{s}{2} \right) e(-R_1 H_1) \psi(-H_2, -R_2, s) + e \left( \frac{s}{2} \right) h(\tau, \bar{\tau}, s, s_1; R, H),
\]

where

\[
h(\tau, \bar{\tau}, s, s_1; R, H) := \sum_{(m,n) \in J} \frac{e(-mH_1 - nH_2)}{(m - R_1)\tau + n - R_2)^s ((m - R_1)\bar{\tau} + n - R_2)^{s-s_1},
\]

where

\[J := \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid m - R_1 > 0, \ d(m - R_1) < c(n - R_2)\}.
\]
We now use Euler’s integral representation of $\Gamma(s)$ to obtain that, for $\tau \in Q$,

$$
\Gamma(s - s_1)\Gamma(s_1) h(\tau, \tilde{\tau}, s, s_1; R, H) = \sum_{(m,n) \in J} e(-mH_1 - nH_2) \cdot \int_0^\infty \int_0^\infty u^{s_1-1}v^{s-1-1}e^{-\tau u + \tau v}dudv.
$$

Replace indices, $m$ by $m + [R_1] + 1$ and $n$ by $n + [R_2 + (d(m - R_1)/c)] + 1$ to find that

$$
\Gamma(s - s_1)\Gamma(s_1) h(\tau, \tilde{\tau}, s, s_1; R, H) = \sum_{m,n=0}^\infty e(-H_1(m + [R_1] + 1) - H_2(n + [R_2] + 1 + [(g + md + d)/c])) \cdot \int_0^\infty \int_0^\infty u^{s_1-1}v^{s-1-1}e^{-(m - [R_1]+1)(\tau u + \tau v) - (n - [R_2]+1 + [(g + md + d)/c])(u + v)}dudv.
$$

Let $m = qc + j - 1, 0 \leq q < \infty, 1 \leq j \leq c$. Involving sum of geometric series and manipulation of integral, we find that

$$
\Gamma(s - s_1)\Gamma(s_1) h(\tau, \tilde{\tau}, s, s_1; R, H) = \sum_{j=1}^c e(-H_1(j + [R_1]) - H_2([R_2] + 1 + [(g + j d)/c])) \cdot \int_0^\infty \int_0^\infty u^{s_1-1}v^{s-1-1}e^{-(j - [R_1]+1)(\tau u + \tau v) - (1 - [R_2]+1 + [(g + jd + d)/c])(u + v)}
$$

$$
\sum_{q=0}^\infty \sum_{n=0}^\infty e^{-q(2\pi i(cH_1 + dH_2) + (c\tau + d)u + (c\tau + d)v) - n(2\pi iH_4 + u + v)}dudv.
$$

$$
\sum_{j=1}^c e(-H_1(j + [R_1]) - H_2([R_2] + 1 + [(g + j d)/c] - d)) \cdot \int_0^\infty \int_0^\infty u^{s_1-1}v^{s-1-1}e^{-(c\tau + d)(j - [R_1])/c - (c\tau + d)(j - [R_1])/c}e^{cH_1 + dH_2} - e^{-(c\tau + d)u - (c\tau + d)v}
$$

$$
\cdot e^{(\tau u + \tau v)}du v/c - e^{(-H_2)}dudv.
$$

$$
\sum_{j=1}^c e(-H_1(j + [R_1]) - H_2([R_2] + 1 + [(g + j d)/c])) \cdot \int_0^1 v^{s_1-1}(1 - v)^{s-1-1} - \int_0^\infty u^{s-1}e^{-(c\tau + d)u - (c\tau + d)(j - [R_1])/c}e^{cH_1 + dH_2} - e^{-(c\tau + d)u + (c\tau + d)(1 - v)u}
$$

$$
\cdot e^{(\tau u + \tau v)}du v/c - e^{(-H_2)}dudv.
$$

The inversion of the order of summations and integrations can be justified by absolute convergence of the original series for $\text{Re } s > 2$. This
Thus, replacing $v$ by $1 - v$, we obtain that
\[
\int_0^1 v^{s_1 - 1} (1 - v)^{-N - s_1 - 1} \int_C u^{-N - 1} e^{-((c\tau + d)v + (c\tau + d)(1-v))(j-(R_1))u/c - e^{(\frac{\varphi + jd}{c})u}} e^{e^{(\frac{\varphi + jd}{c})u} - 1} \, du \, dv
\]
Non-holomorphic Eisenstein series

\[ = 2\pi i \sum_{k=0}^{N+2} \frac{B_k \left( \frac{j-\{R_1\}}{c} \right) \bar{B}_{N+2-k} \left( \frac{\varrho+jd}{c} \right)}{k!(N+2-k)!} (-c\tau - d)^{k-1} \]

\[ \cdot \int_0^1 (1 - v)^{s_1 - 1} v^{-N-s_1-1} \left( 1 - \frac{c(\tau - \bar{\tau})}{c\tau + d} v \right)^{k-1} dv \]

\[ = 2\pi i \frac{\Gamma(s_1)\Gamma(-N-s_1)}{\Gamma(-N)} \sum_{k=0}^{N+2} \frac{B_k \left( \frac{j-\{R_1\}}{c} \right) \bar{B}_{N+2-k} \left( \frac{\varrho+jd}{c} \right)}{k!(N+2-k)!} (-c\tau - d)^{k-1} \]

\[ \cdot \, _2F_1 (-N-s_1, 1-k; -N; \frac{c(\tau - \bar{\tau})}{c\tau + d}) \]

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References


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