**NOTE ON THE CLASSICAL WATSON’S THEOREM FOR THE SERIES \( _3F_2 \)**

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**Abstract.** Summation theorems for hypergeometric series \( _2F_1 \) and generalized hypergeometric series \( _pF_q \) play important roles in themselves and their diverse applications. Some summation theorems for \( _2F_1 \) and \( _pF_q \) have been established in several or many ways. Here we give a proof of Watson’s classical summation theorem for the series \( _3F_2 \) by following the same lines used by Rakha [7] except for the last step in which we applied an integral formula introduced by Choi et al. [3].

1. Introduction and Preliminaries

We begin by introducing a response of Michael Atiyah [9] when Michael Atiyah and Isadore Singer were interviewed which took place in Oslo on May 24, 2004, during the Abel Prize celebrations: Any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalize in different directions—they are not just repetitions of each other ···. If you cannot look at a problem from different directions, it is probably not very interesting; the more perspectives, the better.

We recall the well known classical Watson’s summation theorem for the generalized hypergeometric series \( _3F_2 \) (see, e.g., [1, p.16, Eq. (1)]):

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\[ 3F_2 \left[ \begin{array}{c} a, b, c; \\ \frac{1}{2}(a + b + 1), 2c; 1 \end{array} \right] \]

\[ = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(c + \frac{1}{2}) \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right) \Gamma\left(c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}b + \frac{1}{2}\right) \Gamma\left(c - \frac{1}{2}a + \frac{1}{2}\right) \Gamma\left(c - \frac{1}{2}b + \frac{1}{2}\right)} \]

provided that \( \Re(2c - a - b) > -1 \). This Watson’s summation theorem (1.1) has been established in many different ways (see, e.g., [1, 2, 5, 7, 8, 11, 12]). For concise outlines of various proofs of (1.1), see [7, 8].

Here we present a proof of Watson’s summation theorem (1.1) for the series \( {}_3F_2(1) \) by following the same lines used by Rakha [7] except for the last step in which we applied an integral formula introduced by Choi et al. [3].

For our purpose, we need to recall some known functions and earlier works. The well known Beta function \( \beta(\alpha, \beta) \) is defined by

\[ \beta(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt \quad (\Re(\alpha) > 0; \ Re(\beta) > 0) \]

or, equivalently,

\[ \beta(\alpha, \beta) = 2 \int_0^{\pi/2} \sin^\alpha \theta \cos^\beta \theta d\theta \quad (\Re(\alpha) > 0; \ Re(\beta) > 0). \]

An integral representation for \( {}_3F_2 \) is given as follows (see [4]):

\[ {}_3F_2 \left[ \begin{array}{c} a, b, c; \\ d, e; z \end{array} \right] = \frac{\Gamma(d)}{\Gamma(d-c)\Gamma(c)} \int_0^1 t^{c-1}(1-t)^{d-c-1} {}_2F_1 \left[ \begin{array}{c} a, b; \\ e; zt \end{array} \right] dt, \]

provided \( \Re(c) > 0, \ Re(d-c) > 0 \) and \( \Re(d-a-b) > 0 \).

A transformation formula for \( {}_2F_1 \) is as follows (see, e.g., [6, p. 65, Theorem 24]):

\[ {}_2F_1 \left[ \begin{array}{c} a, b; \\ 2b; 2y \end{array} \right] = (1-y)^{-a} {}_2F_1 \left[ \begin{array}{c} \frac{1}{2}a + \frac{1}{2}; \\ \frac{1}{2}b + \frac{1}{2}; \left( \frac{y}{1-y} \right)^2 \end{array} \right] \]

provided \( |y| < \frac{1}{2} \) and \( \left| \frac{y}{1-y} \right| < 1 \).
Euler’s integral representation for the hypergeometric function \( _2F_1 \) is given as follows (see \([10, p. 65]\)):

\[
(1.6) \quad _2F_1 \left[ a, b; \frac{1}{2} \right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt,
\]

provided that \( \Re(c) > \Re(b) > 0 \) and \( |z| < 1 \).

An integral representation for \( _2F_1(1/2) \) is given as follows (see, e.g., \([3, p. 510, Eq. (8)]\)):

\[
(1.7) \quad _2F_1 \left[ a, b; \frac{1}{2} \right] = \frac{2^a \Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \int_0^{\pi/2} (\cos \theta)^{b-1} \left( \frac{\sin \theta}{2} \right)^{2c-2b-1} \left( \frac{\cos \theta}{2} \right)^{2a-2c+1} d\theta.
\]

Gauss’s second summation theorem is given as follows (see, e.g., \([1, p. 10, Eq. (2)]\)):

\[
(1.8) \quad _2F_1 \left[ \frac{1}{2}(a+b+1); \frac{1}{2} \right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2}) \Gamma(\frac{1}{2}b + \frac{1}{2})}.
\]

2. Derivation of Watson’s summation theorem (1.1)

Setting \( e = 2b \) in (1.4), we have

\[
(2.1) \quad _3F_2 \left[ \frac{a}{d}, b, c; \frac{1}{2} \right] = \frac{\Gamma(d)}{\Gamma(d-c)\Gamma(c)} \int_0^1 t^{c-1}(1-t)^{d-c-1} _2F_1 \left[ a, b; \frac{1}{2} \right] dt.
\]

Replacing \( y \) by \( \frac{1}{2}zt \) in (1.5) and applying the resulting identity to the \( _2F_1 \) in (2.1), after a little simplification, we obtain

\[
(2.2) \quad _3F_2 \left[ \frac{a}{d}, b, c; \frac{1}{2} \right] = \frac{\Gamma(d)}{\Gamma(d-c)\Gamma(c)} \cdot \int_0^1 t^{c-1}(1-t)^{d-c-1} (1-\frac{1}{2}zt)^{-a} _2F_1 \left[ \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}; \frac{b}{2} + \frac{1}{2}; \left( \frac{zt}{2} - \frac{z}{2} \right)^2 \right] dt.
\]
Expressing the $2F_1$ in (2.2) as a series and changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series on the interval $(0, 1)$, after a little algebra, we have

\[(2.3) \quad 3F_2 \left[ \begin{array}{c} a, b, c; \\ d, 2b; \\ z \end{array} \right] = \frac{\Gamma(d)}{\Gamma(d-c)\Gamma(c)} \cdot \sum_{n=0}^{\infty} \frac{(\frac{1}{2}a)_n(\frac{1}{2}a + \frac{1}{2})_n}{(b + \frac{1}{2})_n n!} \left( \frac{z}{2} \right)^{2n} \int_0^1 t^{c+2n-1}(1-t)^{d-c-1} \left( 1 - \frac{1}{2}zt \right)^{-(a+2n)} dt.
\]

Using (1.6) to evaluate the integral in (2.3), after a little simplification, we get

\[(2.4) \quad 3F_2 \left[ \begin{array}{c} a, b, c; \\ d, 2b; \\ z \end{array} \right] = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}a)_n(\frac{1}{2}a + \frac{1}{2})_n (c)_n}{(b + \frac{1}{2})_n (d)_n n!} \left( \frac{z}{2} \right)^{2n} 2F_1 \left[ \begin{array}{c} a + 2n, c + 2n; \\ d + 2n; \\ \frac{z}{2} \end{array} \right].
\]

Interchanging $b$ and $c$ and taking $d = \frac{1}{2}(a + b + 1)$ in (2.4), we have

\[(2.5) \quad 3F_2 \left[ \begin{array}{c} \frac{1}{2}(a + b + 1), 2c; \\ d, 2b; \\ z \end{array} \right] = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}a)_n(\frac{1}{2}a + \frac{1}{2})_n (b)_n}{(c + \frac{1}{2})_n (\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})_n n!} \left( \frac{z}{2} \right)^{2n} 2F_1 \left[ \begin{array}{c} a + 2n, b + 2n; \\ \frac{1}{2}(a + b + 1) + 2n; \\ \frac{z}{2} \end{array} \right].
\]

Taking $z = 1$ in (2.5) and using (1.7) in the resulting equation, we obtain

\[(2.6) \quad 3F_2 \left[ \begin{array}{c} \frac{1}{2}(a + b + 1), 2c; \\ d, 2b; \\ 1 \end{array} \right] = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}a)_n(\frac{1}{2}a + \frac{1}{2})_n (b)_n}{(c + \frac{1}{2})_n (\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})_n n!} \left( \frac{1}{2} \right)^{2n} \frac{2^{a+2n} \Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(b + 2n)\Gamma(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2})} \cdot 2^{b-a} \int_0^{\pi/2} (\cos \theta)^{b+2n-1} (\sin \theta)^{a-b} d\theta,
\]

where we used an elementary trigonometric identity: $\sin \theta = 2 \sin(\theta/2) \cos(\theta/2)$. Applying (1.3) to evaluate the integral in (2.6) and using Legendre’s duplication formula for the Gamma function (see [10, p. 6, Eq. (29)]) in the resulting identity, we get
Note on The Classical Watson’s Theorem for the Series $3F_2$

\[ \begin{align*}
3F_2 & \left[ \frac{1}{2}(a + b + 1), 2c; 1 \right] \\
& = \frac{\Gamma \left( \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c \right) \Gamma \left( \frac{1}{2}a + \frac{1}{2}b \right) \Gamma \left( \frac{1}{2}a \right)}{\Gamma \left( \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c \right) \Gamma \left( \frac{1}{2}a + \frac{1}{2}b \right) \Gamma \left( \frac{1}{2}a \right)} 2F_1 \left[ \frac{1}{2}a, \frac{1}{2}b; c + \frac{1}{2}; 1 \right],
\end{align*} \]

which, upon using the well known Gauss’s summation theorem (see, e.g., [10, p. 64, Eq. (7)]), yields (1.1). This completes the proof of Watson’s summation theorem (1.1).

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**References**


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