PRIMITIVE/SEIFERT KNOTS WHICH ARE NOT TWISTED TORUS KNOT POSITION

Sungmo Kang

Abstract. The twisted torus knots and the primitive/Seifert knots both lie on a genus 2 Heegaard surface of $S^3$. In [5], J. Dean used the twisted torus knots to provide an abundance of examples of primitive/Seifert knots. Also he showed that not all twisted torus knots are primitive/Seifert knots. In this paper, we study the other inclusion. In other words, it shows that not all primitive/Seifert knots are twisted torus knot position. In fact, we give infinitely many primitive/Seifert knots that are not twisted torus knot position.

1. Introduction

Two types of knots in $S^3$, the twisted torus knots and the primitive/Seifert knots, both lie on a genus 2 Heegaard surface of $S^3$. In [5], J. Dean defined primitive/Seifert knots, and to find primitive/Seifert knots he used the twisted torus knots. Furthermore he gave the criteria for twisted torus knots to be primitive/Seifert knots. Also he showed that not all twisted torus knots are primitive/Seifert knots. On the other hand, in [3] Berge and the author give the complete list of hyperbolic primitive/Seifert knots in $S^3$. Thus one natural question is whether or not all hyperbolic primitive/Seifert knots belong to the twisted torus knots. This paper gives an implication for a negative answer. In other words, we provide infinitely many primitive/Seifert knots which are not the twisted torus knot position.

Received November 5, 2013. Accepted November 19, 2013.

2010 Mathematics Subject Classification. Primary 57M25.

Key words and phrases. knots, twisted torus knots, primitive curves, Seifert curves, proper power curves, primitive/Seifert knots.

This study was financially supported by Chonnam National University, 2012.
Figure 1. The (7, 3) torus knot $T(7, 3)$ and 3 parallel copies $3T(1, 1)$ of the (1, 1) torus knot.

Since the precise definitions of the twisted torus knots and the primitive/Seifert knots can both be found in [5], we give brief explanation on how to construct them.

First regarding the twisted torus knots, let $T(p, q)$ be the $(p, q)$-torus knot which lies in the boundary of the standardly embedded solid torus $V$ in $S^3$. Let $rT(m, n)$ be the $r$ parallel copies of $T(m, n)$ which lies in the boundary of another standardly embedded solid torus $V'$ in $S^3$. Let $D$ be the disk in $\partial V$ so that $T(p, q)$ intersects $D$ in $r$ disjoint parallel arcs, where $0 \leq r \leq p+q$, and $D'$ the disk in $\partial V'$ so that $rT(m, n)$ intersects $D'$ in $r$ disjoint parallel arcs, one for each component of $rT(m, n)$. See Figure 1. We excise the disks $D$ and $D'$ from their respective tori and glue the punctured tori together along their boundaries so that the orientations of $T(p, q)$ and $rT(m, n)$ align correctly. The resulting one must result in a knot and is called a twisted torus knot, which is denoted by $K(p, q, r, m, n)$. Figure 2 shows $K(7, 3, 3, 1, 1)$. It is obvious from the construction that the twisted torus knots lie on a standard genus 2 Heegaard surface of $S^3$. Let $K$ be a simple closed curve in a genus 2 Heegaard surface $\Sigma$ of $S^3$. Then we say that $K$ is a twisted torus knot position if there exists a homeomorphism of $S^3$ sending $(\Sigma, K)$ to $(F, K')$, where $K'$ is a twisted torus knot lying in a standard genus 2 Heegaard surface $F$ of $S^3$.

Now we describe primitive/Seifert knots. If $H$ is a genus two handlebody and $c$ is an essential simple closed curve in $\partial H$, $H[c]$ will denote the 3-manifold obtained by adding a 2-handle to $H$ along $c$. The curve $c$ in $\partial H$ is primitive in $H$ if $H[c]$ is a solid torus. We say $c$ is Seifert in $H$ if $H[c]$ is a Seifert-fibered space and not a solid torus. Note that
since $H$ is a genus two handlebody, that $c$ is Seifert in $H$ implies that $H[c]$ is an orientable Seifert-fibered space over $D^2$ with two exceptional fibers, or an orientable Seifert-fibered space over the Möbius band with at most one exceptional fiber.

Suppose $K$ is a simple closed curve in a genus two Heegaard surface $\Sigma$ of $S^3$ bounding handlebodies $H$ and $H'$. $K$ in $\Sigma$ is primitive/Seifert if it is primitive with respect to one of $H$ or $H'$, say $H'$, and Seifert with respect to $H$.

2. Backgrounds on R-R diagrams

R-R diagrams are a type of planar diagram related to Heegaard diagrams. These diagrams were originally introduced by Osborne and Stevens in [6]. They enable us to analyze curves which lie very complicatedly on the boundary of a genus two handlebody. They are particularly useful for describing embeddings of simple closed curves in the boundary of a handlebody so that the embedded curves represent certain conjugacy classes in $\pi_1$ of the handlebody.

The basics of Heegaard diagrams and R-R diagrams of simple closed curves in the boundary of a genus two handlebody are well explained in the paper [1] of Berge. Here, we describe briefly terminologies related to R-R diagrams and show how to transform simple closed curves in the boundary of a genus two handlebody into R-R diagrams.

We start with a genus two handlebody $H$ with a complete set of cutting disks $\{D_A, D_B\}$. Suppose $C$ is a set of pairwise disjoint simple closed curves in the boundary $\Sigma$ of $H$. Figure 3 shows two simple closed curves in the boundary of a genus two handlebody. Consider two parallel
simple closed curves which separate the two cutting disks $D_A$ and $D_B$. These two curves decompose $\Sigma$ into two once-punctured tori $F_A$ and $F_B$, and one annulus $A$. These two curves were originally introduced by Zieschang [7] as belt curves, and the two once-punctured tori $F_A$ and $F_B$ are called handles. See Figures 4 and 5 for this decomposition.

With this decomposition of $\Sigma$ set, we analyze each curve in $C$ in each component of the decomposition as follows. We may assume after isotopy each curve $c \in C$ is either disjoint from $\partial F_A \cup \partial F_B$, or $c$ is cut by its intersections with $\partial F_A \cup \partial F_B$ into arcs, each properly embedded and essential in one of $A, F_A, F_B$. A properly embedded essential arc in $F_A$ or $F_B$ is called a connection. Two connections in $F_A$ or $F_B$ are parallel if they are isotopic in $F_A$ or $F_B$ via an isotopy keeping their endpoints in $\partial F_A$ or $\partial F_B$. A collection of pairwise disjoint connections on a given handle can be partitioned into bands of pairwise parallel connections.

Note that since each handle is a once-punctured torus, there can be at most three nonparallel bands of connections on a given handle. Figure 5 shows that there are three nonparallel bands of connections in $F_A$ and there are two in $F_B$.

In each handle, we merge parallel connections in one band into a single connection. This also merges the endpoints of connections on the boundary of each handle. With these endpoints merged in each handle, now we merge properly embedded essential arcs in $A$ such that if $n$ parallel arcs in $A$ have the same endpoints in each handle after merging connections, then we merge these arcs into one edge and label this edge by $n$ indicating $n$ parallel arcs and call this edge with label $n$ as a band of width $n$. 
With all of the above information provided, we are ready to transform simple closed curves in the boundary of a genus two handlebody into R-R diagrams, which are type of a planar graph in $S^2$. First, we embed the annulus $\mathcal{A}$ in $S^2$ by deleting two disks from $S^2$ as shown in Figure 6. Next we immerse arc components of curves lying in the two handles $F_A$ and $F_B$ into $S^2$ as follows. Since each boundary of $\mathcal{A}$ is also the boundary of the handles $F_A$ and $F_B$, and there are at most three connections after merger, put the six endpoints of the three connections in each boundary circle of $\mathcal{A}$ and connect two endpoints of a connection by a diameter. Figure 6 shows this transformation. We put the capital letters $A$ and $B$ to indicate correspondence to the two handles $F_A$ and $F_B$ respectively and we call them as $A$–handle and $B$–handle respectively.

We encode the endpoints of each band of connections by integers as follows. Orient the boundaries of the cutting disks $D_A$ and $D_B$ and each simple closed curve in $\mathcal{C}$. The orientation of a simple closed curve gives
the orientation of connections in a handle distinguishing its endpoints as an initial point and a terminal point. If a connection intersects a cutting disk $s$ times positively, we label an initial point by $s$ and a terminal point by $-s$, and we say the band of connections with endpoints labeled by $s$ and $-s$ as $s$-connection or $(-s)$-connection. Note that the labels of endpoints satisfy the following conditions:

1. If $s$ and $t$ are consecutive labels of endpoints of two connections, then $\gcd(s, t) = 1$.
2. If $s, t,$ and $u$ are consecutive labels of endpoints of three connections, then $t = s + u$.

By disregarding the boundary circles of $F_A$ and $F_B$ in Figure 6, we finally obtain the corresponding R-R diagram. Figure 7 shows the transformation of two curves $c_1$ and $c_2$ in the boundary of a genus two handlebody into R-R diagram.

As mentioned at the beginning in this section, R-R diagram gives sufficient information about conjugacy classes of the element represented by $c$ in $\pi_1(H)$. $\pi_1(H)$ is a free group $F(A, B)$ which is generated by $A$ and $B$ dual to the cutting disks $D_A$ and $D_B$ respectively. In Figure 7, $c_1$ and $c_2$ represent the conjugacy classes of $AB$ and $A^3B^2A^2B^2$ respectively in $\pi_1(H)$.

3. R-R diagrams of twisted torus knots and their properties

In this section, we describe the R-R diagram of twisted torus knots. From the construction or Figures 1 and 2, a twisted torus knot lies on
the boundary of a standardly embedded genus two handlebody $H$ in $S^3$, which is obtained from two solid tori by gluing the disks $D$ and $D'$. Let $H'$ be the closure of $S^3 - H$, which is also a genus two handlebody, and $\Sigma$ the common boundary of $H$ and $H'$. Then $(\Sigma; H, H')$ is a standard genus two Heegaard splitting of $S^3$.

Let $\Gamma$ be a separating simple closed curve in $\Sigma$ which bounds in both $H$ and $H'$ (in the construction of twisted torus knots, $\Gamma$ can be chosen as the boundary of the disk $D$ or $D'$). Let $\{D_A, D_B\}$ and $\{D_X, D_Y\}$ be complete sets of cutting disks of $H$ and $H'$ respectively, disjoint from $\Gamma$, with $\partial D_A$ and $\partial D_X$ on one side of $\Gamma$, and $\partial D_B$ and $\partial D_Y$ on the other side of $\Gamma$. See Figure 8. With this setup given, we can make R-R diagrams of twisted torus knots with respect to both $H$ and $H'$. First, we regard $\Gamma$ as the belt curve separating the boundary of $H$ ($H'$, respectively) into the two handles $A$–handle and $B$–handle ($X$–handle and $Y$–handle, respectively). According to the construction of twisted torus knots, since the intersection of a twisted torus knot $K(p, q, r, m, n)$ and the $B$–handle in $H$ consists of $r$ parallel arcs, there is only one band of connections on the $B$–handle. Similarly, there is only one band of
Figure 8. The $A-$ and $B-$ handles in $H$ and the $X-$ and $Y-$ handles in $H'$. 

Figure 9. R-R diagrams of a twisted torus knot $K(p,q,r,m,n)$ with respect to $H$ (in (a)) and $H'$ (in (b)).

collections on the $Y-$ handle. Therefore we have the R-R diagram of twisted torus knots as shown in Figure 9.
Proposition 3.1. If $K$ is a twisted torus knot which lies on a standard genus two Heegaard surface $\Sigma$ of $S^3$ bounding two handlebodies $H$ and $H'$, then there exists a simple closed curve $\beta$ on $\Sigma$ which either bounds a disk, is primitive, or is a proper power in each handlebody.

Proof. In Figure 9(a), the dotted curve $\beta$ bounds a disk if $s = 0$, is primitive if $s = 1$, and is a proper power if $s > 1$ in $H$. Similarly in Figure 9(b), the curve $\beta$ bounds a disk, is primitive, or is a proper power in $H'$.

That $\beta$ is a proper power curve in $H$ means that $\beta$ is disjoint from a separating disk in $H$, does not bound a disk in $H$, and is not primitive in $H$. Equivalently, $[\beta]$ is conjugate to $w^n$, $n \geq 2$, of $\pi_1(H)$, where $w$ is a free generator of $\pi_1(H)$.

Remark 3.2. If the curve $\beta$ in Proposition 3.1 bounds a disk in one of the handlebody, then the twisted torus knot $K$ is a torus knot.

The existence of the curve $\beta$ in Proposition 3.1 plays a crucial role in determining if a hyperbolic primitive/Seifert knot is not a twisted torus knot. In other words, due to Berge [2] the existence of the curve $\beta$ enables us to make the following procedure for the determination:

1. Find all candidates for $\beta$ which is primitive or a proper power in $H$.
2. Check if $\beta$ is also primitive or a proper power in $H'$. If not, it is not a twisted torus knot.
3. Locate the unique cutting disks $D_A^*$ and $D_X^*$ disjoint from $\beta$ of $H$ and $H'$ respectively.
4. Check if $\partial D_A^*$ and $\partial D_X^*$ intersect exactly once. If not, it is not a twisted torus knot.

4. Primitive/Seifert knots which are not twisted torus knot position

In this section, we give an infinite family of primitive/Seifert knots which are not a twisted torus knot position. According to the procedure, the first two steps are to find all candidates for $\beta$ which is primitive or a proper power in both $H$ and $H'$. The following theorem due to Cohen, Metzler, and Zimmerman provides the necessary condition for a simple closed curve being a primitive or a proper power curve in a genus two handlebody once one knows a cyclically reduced word of the curve in a free group of rank two $F(A, B)$, which is $\pi_1(H)$. 

Figure 10. The $(7,3)$-torus knot and one unknotting tunnel $t$ which lie in the boundary of a standardly embedded solid torus $V$ in $S^3$.

**Theorem 4.1.** ([4]) Suppose a cyclic conjugate of 

$$W = A^{n_1} B^{m_1} \cdots A^{n_p} B^{m_p}$$

is a member of a basis of $F(A,B)$ or a proper power of a member of a basis of $F(A,B)$, where $p \geq 1$ and each indicated exponent is nonzero. Then, after perhaps replacing $A$ by $A^{-1}$ or $B$ by $B^{-1}$, there exists $e > 0$ such that:

$$n_1 = \cdots = n_p = 1, \quad \text{and} \quad \{m_1, \ldots, m_p\} \subseteq \{e, e + 1\},$$

or

$$\{n_1, \ldots, n_p\} \subseteq \{e, e + 1\}, \quad \text{and} \quad m_1 = \cdots = m_p = 1.$$
Figure 11. An infinite family of primitive/Seifert knots $K$ in the genus 2 Heegaard splitting $(\Sigma, H, H')$ of $S^3$, where $K = \alpha_0 \mathbb{T}(\tau_1^{J_1}, \tau_2^{J_2})$ with $J_1, J_2 > 1$ and $J_1 \neq J_2$ and $J_1 \neq J_2 + 1$. Here the handlebody $H'$ is the closure of a regular neighborhood of an $(h,k)$ torus knot $\kappa$ with $h,k > 1$ and an unknotting tunnel $t$ of $\kappa$ embedded in the boundary $T$ of a standardly embedded solid torus $V$ in $S^3$ as in Figure 10, and $H = S^3 - H'$.

$D_R$ be the meridian of $\kappa$, which appears as in Figure 11. Note that $D_M$ and $D_R$ form a complete set of cutting disks of $H'$. Let $M$ and $R$ be the boundaries of $D_M$ and $D_R$ respectively with the orientation given as in Figure 11. Finally we let $K = \alpha_0 \mathbb{T}(\tau_1^{J_1}, \tau_2^{J_2})$ with $J_1, J_2 > 1$, $J_1 \neq J_2$ and $J_1 \neq J_2 + 1$, i.e. $K$ is the curve obtained from $\alpha_0$ twisted about $\tau_1$ and $\tau_2$ $J_1$ and $J_2$ times respectively.

In the following theorem, we will show that the curve $K$ is primitive in $H'$ and Seifert in $H$ so that $K$ is a primitive/Seifert knot in $S^3$.

**Theorem 4.2.** The knot $K$ described in Figure 11 is a hyperbolic primitive/Seifert knot in $S^3$.

**Proof.** It follows from Figure 11 that since $K$ intersects $D_M$ in a single point, $K$ is primitive in $H'$. In order to show that $K$ is Seifert in $H$, we use R-R diagram of $K$ with respect to $H$.

To find R-R diagram of $K$, we need to determine a complete set of cutting disks $\{D_A, D_B\}$ of $H$ for the R-R diagram. Let $A = \overline{T - N(\kappa)}$, $D' = \overline{\mathbb{A} - \mathbb{N}(t)}$ and $D = \mathbb{A} - D'$. Note that $\partial A$ consists of two copies of an
To find a complete set of cutting disks of $H$, we perform an isotopy on $A$ which switches the positions of $D'$ and $D$. See Figure 12 for the isotopy performed. Figure 13 shows $M$ and $R$ after the isotopy. Here $M$ is pushed onto $\partial N(\kappa)$.

We consider $V \cup T'W$ as $V' \cup T \times [-\frac{1}{2}, \frac{1}{2}] \cup W'$, where $V'$ and $W'$ are solid tori inside $V$ and $W$ respectively such that they have the same cores. Then it follows that $H \cong V' \cup D \times [-\frac{1}{2}, \frac{1}{2}] \cup W'$, where $D = D \times \{0\}$. Let $D_{V'}$ and $D_{W'}$ be meridians of $V'$ and $W'$, which are induced from the meridians $D_V$ and $D_W$ of $V$ and $W$ respectively. See Figure 14.
Primitive/Seifert knots which are not twisted torus knot position

$D_V$ and $D_W$ form a complete set of cutting disks of $H$ and thus we take $(D_V', D_W')$ as $(D_A, D_B)$ of $H$. Now we can consider R-R diagram where $A$ and $B$-handles are associated with the two once-punctured tori which are created by cutting $\partial H$ along the separating curve $\partial D$ and the labels on connections on both handles are induced from $\partial D_A$ and $\partial D_B$ with the orientation given as in Figure 14.

Now it is easy to find corresponding R-R diagrams of the curves $\alpha_0, M, R, \tau_1,$ and $\tau_2$. First for $M$, since $M$ intersects $D_A h$ times and then $D_B k$ times, $M$ has exactly the same R-R diagram as in Figure 15. Similarly, $R$ intersects $D_A h'$ times and then $D_B k'$ times for some positive integers $h'$ and $k'$ with $0 < h' < h$ and $0 < k' < k$, which satisfy the equation $hk' - h'k = 1$. Thus, $R$ has R-R diagram as in Figure 15. For the curves $\tau_1$ and $\tau_2$, since they do not intersect the separating disk $D$ of $H$, and intersect the cutting disks $D_A$ and $D_B h$ and $k$ times respectively, their R-R diagrams appear as in Figure 15. By figuring out the intersections of $\alpha_0$ with the other curves $M, R, \tau_1,$ and $\tau_2$ we can obtain R-R diagram of $\alpha_0$ as in Figure 15.

Let $\pi_1(H) = \langle A, B \rangle$ and $\pi_1(H') = \langle X, Y \rangle$, where $A$ and $B$ are dual to the cutting disks $D_A$ and $D_B$ respectively, and $X$ and $Y$ are dual to the cutting disks $D_M$ and $D_R$ respectively. From the R-R diagram in Figure 15 one has, in $\pi_1(H)$ and $\pi_1(H')$ respectively:

$$K = A^{(2J_1 - 1)h + 2h'k}B^{J_2k + k'},$$
$$K = XY^{-2J_1 + J_2 + 1},$$

which shows that $K$ is Seifert in $H$ by Lemma 2.2 in [5] and primitive in $H'$. Thus $K$ represents primitive/Seifert knots in $S^3$. Furthermore, by [3] these knots are hyperbolic unless $J_1 = J_2$ or $J_1 = J_2 + 1$. \qed
Figure 15. The R-R diagrams of the curves $\alpha_0, M, R, \tau_1,$ and $\tau_2$.

Figure 16. An alternative diagram of $\alpha$ and $M$ obtained by performing the twists about $\tau_1$ and $\tau_2$ in Figure 15.

Theorem 4.3. The knot $K$ described in Figure 11 are not a twisted torus knot position.

Proof. After applying for the twists about the curves $\tau_1$ and $\tau_2$ $J_1$ and $J_2$ times respectively, we obtain an alternative R-R diagram with the form of Figure 16, where the $A-$handle contains the bands of connections labeled by $h' + J_1h, 2h' + (2J_1 - 1)h, h' + (J_1 - 1)h,$ and the $B-$handle contains the bands of connections labeled by $k' + J_2k, k$.

Now we apply for the procedure given at the last part in Section 3. First, we try to find a primitive or proper power curve $\beta$ disjoint from $K$ in $H$. If $\beta$ has connections on both $A-$ and $B-$handle, then Theorem 4.1 implies that one of the labels of connections of $\beta$ must be 1.
However, we will show that none of the labels of the possible band of connections of $\beta$ is 1. In the $A$-handle, $h' + J_1 h, 2h' + (2J_1 - 1) h$, and $h' + J_1 h + n(2h' + (2J_1 - 1) h)$, where $n \in \mathbb{Z}$, are possible labels, and in the $B$-handle $k, k' + J_2 k$ and $k + m(k' + J_2 k)$, where $m \in \mathbb{Z}$, are possible labels for $\beta$. Since $0 < h' < h$, $0 < k' < k$, and $J_1, J_2 > 1$, none of these labels can be equal to 1.

Since there are no bands of connections labeled by 1 on both handles, only candidates of a primitive or proper power $\beta$ disjoint from $K$ are the two regular fiber curves $\beta_1$ and $\beta_2$ as shown in Figure 16. Algebraically, in $\pi_1(H)$ $\beta_1$ and $\beta_2$ are $A^{2h'}(2J_1 - 1)h$ and $B^{k' + J_2 k}$ respectively.

Next, we need to check if $\beta_1$ and $\beta_2$ are primitive or a proper power in $H'$. However it follows from Figures 15 and 16 that $\beta_1 = Y^{-J_1}X^{1-J_1}X$ and $\beta_2 = X^{-1}Y^{-J_2}$. Thus they are primitive in $H'$.

According to the procedure, next test is to locate the unique cutting disks of $H$ and $H'$ disjoint from $\beta_i$, $i = 1, 2$, and check if the boundaries of the two unique cutting disks intersect exactly in a single point.

For the curve $\beta_1$, it follows immediately from the R-R diagram that the cutting disk $D_B$ in $H$ is disjoint from $\beta_1$ and thus is the unique cutting disk in $H$ disjoint from $\beta_1$. In order to find the unique cutting disk in $H'$, since $\beta_1 = Y^{-J_1}X^{1-J_1}X$, we perform change of cutting disks twice, the first of which induces the automorphism $X \mapsto Y^{-J_1}X$ and the second $Y \mapsto YX^{-2}$. Then after performing, $\beta_1 = Y$. So there are a set of cutting disks of $H'$, one of which is dual to $\beta_1 = Y$ and the other, say $D_{X^*}$, is the unique cutting disk in $H'$ disjoint form $\beta_1$.

To see how many times $\partial D_B$ intersects $\partial D_{X^*}$, we need to figure out to which element in $H_1(H') = \langle Y, X^* \rangle$ $\partial D_B$ is carried under the changes of cutting disks performed above. The following explains this;

$$\partial D_B = kX + k'Y$$

$$\xrightarrow{X \mapsto J_1 Y + X} k(J_1 Y + X) + k'Y = (kJ_1 + k')Y + kX$$

$$\xrightarrow{Y \mapsto 2X + Y} (kJ_1 + k')(2X + Y) + kX^*$$

$$= -(2J_1 - 1)k + 2k')X^* + (kJ_1 + k')Y.$$

Therefore $\partial D_B$ intersects $\partial D_{X^*}$ $((2J_1 - 1)k + 2k')$ times, which is greater than 1.

For the curve $\beta_2$, the similar argument applies. The cutting disk $D_A$ in $H$ is disjoint from $\beta_2$ and thus is the unique cutting disk in $H$ disjoint from $\beta_2$. In order to find the unique cutting disk in $H'$, we perform change of cutting disks inducing the automorphism $X^{-1} \mapsto X^{-1}Y^{-J_2}$. Then after performing, $\beta_2 = X^{-1}$ and there are a set of cutting disks
of $H'$, one of which is dual to $\beta_2 = X^{-1}$ and the other, say $D_Y^*$, is the unique cutting disk in $H'$ disjoint form $\beta_2$.

The following shows that $\partial D_A$ intersects $\partial D_Y^*$ $(J_2 h + h')$ times, which is greater than 1.

$$\partial D_A = hX + h'Y$$

$$\xrightarrow{X \mapsto J_2 Y + X} h(J_2 Y^* + X) + h'Y^*$$

$$= hX + (J_2 h + h')Y^*.$$ This completes the proof. 

From Theorems 4.2 and 4.3, we have the following.

**Corollary 4.4.** There exists an infinite family of primitive/Seifert knots which are not a twisted torus knot position.

Theorem 4.3 shows that the knot $K$ described in Figure 11 are not a twisted torus knot position. However it does not imply that the knot $K$ described in Figure 11 are not a twisted torus knot because they might lie in a different genus 2 Heegaard splitting of $S^3$ which is a twisted torus knot position. Thus it is worth to work on the following question whose answer is expected be affirmative.

**Question:** Are the knot $K$ described in Figure 11 not a twisted torus knot?

**Acknowledgements**

The author would like to thank the referees for pointing out errors in the original texts and helpful comments, and John Berge for useful conversations.

**References**


Sungmo Kang
Department of Mathematics Education, Chonnam National University,
77 Yongbong-ro, Buk-gu, Gwangju 500-757, Republic of Korea.
E-mail: skang4450@chonnam.ac.kr