SELF-ADJOINT INTERPOLATION ON $AX = Y$ IN A TRIDIAGONAL ALGEBRA $\text{Alg}_L$

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Abstract. Given operators $X$ and $Y$ acting on a separable Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator $A$ such that $AX = Y$. In this article, we investigate self-adjoint interpolation problems for operators in a tridiagonal algebra: Let $\mathcal{L}$ be a subspace lattice acting on a separable complex Hilbert space $\mathcal{H}$ and let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators acting on $\mathcal{H}$. Then the following are equivalent:

1. There exists a self-adjoint operator $A = (a_{ij})$ in $\text{Alg}_L$ such that $AX = Y$.

2. There is a bounded real sequence $\{\alpha_n\}$ such that $y_{ij} = \alpha_n x_{ij}$ for $i, j \in \mathbb{N}$.

1. Introduction

Let $\mathcal{C}$ be a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all operators acting on a Hilbert space $\mathcal{H}$ and let $X$ and $Y$ be operators acting on $\mathcal{H}$. An interpolation question for $\mathcal{C}$ asks for which $X$ and $Y$ is there a bounded operator $A \in \mathcal{C}$ such that $AX = Y$. A variation, the ‘$n$-operator interpolation problems’, asks for an operator $A$ such that $AX_i = Y_i$ for fixed finite collections $\{X_1, X_2, \cdots, X_n\}$ and $\{Y_1, Y_2, \cdots, Y_n\}$. The $n$-operator interpolation problem was considered for a $C^*$-algebra $\mathcal{U}$ by Kadison[4]. In case $\mathcal{U}$ is a nest algebra, the (one-operator) interpolation problem was solved by Lance[5]: his result was extended by Hopenwasser[2] to the case that $\mathcal{U}$ is a CSL-algebra. Munch[6] obtained conditions for interpolation in case $A$ is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser[3] once again extended the

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interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser’s paper also contains a sufficient condition for interpolation $n$-operators, although necessity was not proved in that paper.

We establish some notations and conventions. A commutative subspace lattice $L$, or CSL $L$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $H$. We assume that the projections $0$ and $I$ lie in $L$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $L$ is CSL, $\text{Alg}L$ is mathcalled a CSL-algebra. The symbol $\text{Alg}L$ is the algebra of all bounded operators on $H$ that leave invariant all the projections in $L$. Let $x$ and $y$ be two vectors in a Hilbert space $H$. Then $\langle x, y \rangle$ means the inner product of the vectors $x$ and $y$. Let $M$ be a subset of a Hilbert space $H$. Then $\overline{M}$ means the closure of $M$ and $M^\perp$ the orthogonal complement of $M$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

2. Results

Let $H$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots \}$. Let $x_1, x_2, \cdots, x_n$ be vectors in $H$. Then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \cdots, x_n$. Let $L$ be the subspace lattice generated by the subspaces $[\overline{e_{2k-1}}], [\overline{e_{2k-1}}, e_{2k}, \overline{e_{2k+1}}]$ ($k = 1, 2, \cdots$). Then the algebra $\text{Alg}L$ is mathcalled a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson[1]. These algebras have been found to be useful counterexample to a number of plausible conjectures.

Let $A$ be the algebra consisting of all bounded operators acting on $H$ of the form

\[
\begin{pmatrix}
* & & * \\
* & & \\
* & * & * \\
& & \\
& * & \\
& & \ddots
\end{pmatrix}
\]

with respect to the orthonormal basis $\{e_1, e_2, \cdots \}$, where all non-starred entries are zero. It is easy to see that $\text{Alg}L = A$.

Let $B(H)$ be the set of all bounded operators acting on $H$. 
Lemma 1. Let $A = (a_{ij})$ be an operator in the tridiagonal algebra $\text{AlgL}$. Then the following are equivalent:

(1) $A$ is self-adjoint.

(2) $A$ is diagonal and $a_{ii}$ is real for all $i \in \mathbb{N}$.

Proof. Suppose that $A$ is self-adjoint. Since $A = A^*$, $a_{ij} = 0$ for all $i \neq j$ and $a_{ii}$ is real. So $A$ is a real diagonal matrix.

Conversely, it is clear. □

Theorem 2. Let $\text{AlgL}$ be the tridiagonal algebra and let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators in $\mathcal{H}$. Then the following are equivalent:

(1) There exists a self-adjoint operator $A = (a_{ij})$ in $\text{AlgL}$ such that $AX = Y$.

(2) There is a bounded sequence $\{\alpha_n\}$ of real numbers such that $y_{ij} = \alpha_i x_{ij}$ for all $i, j \in \mathbb{N}$.

Proof. Suppose that $A$ is a self-adjoint operator $A = (a_{ij})$ in $\text{AlgL}$ such that $AX = Y$. By Lemma 1, $A$ is diagonal and $a_{ii}$ is real for all $i \in \mathbb{N}$. Let $\alpha_i = a_{ii}$ for $i = 1, 2, \ldots$. Since $AX = Y$, $y_{ij} = a_{ii} x_{ij} = \alpha_i x_{ij}$ for $i, j = 1, 2, \ldots$.

Conversely, assume that there is a bounded sequence $\{\alpha_n\}$ of real numbers such that $y_{ij} = \alpha_i x_{ij}$ for $i, j = 1, 2, \ldots$. Let $A$ be a diagonal matrix with the diagonal sequence $\{\alpha_n\}$. Since $\{\alpha_n\}$ is bounded, $A$ is a bounded operator. Also $A$ is self-adjoint and $AX = Y$. □

Theorem 3. Let $\text{AlgL}$ be the tridiagonal algebra and let $X_i = (x_{jk}^{(i)})$ and $Y_i = (y_{jk}^{(i)})$ be operators acting on $\mathcal{H}$ for $i = 1, 2, \ldots, n$. Then the following are equivalent:

(1) There exists a self-adjoint operator $A = (a_{ij})$ in $\text{AlgL}$ such that $AX_i = Y_i$ for $i = 1, 2, \ldots, n$.

(2) There is a bounded sequence $\{\alpha_n\}$ of real numbers such that $y_{jk}^{(i)} = \alpha_j x_{jk}^{(i)}$ for all $i = 1, 2, \ldots, n$ and $j, k \in \mathbb{N}$.

Proof. Suppose that there exists a self-adjoint operator $A = (a_{ij})$ in $\text{AlgL}$ such that $AX_i = Y_i$ for $i = 1, 2, \ldots, n$. Then $A$ is diagonal and $a_{ii}$ is real for each $i \in \mathbb{N}$ by Lemma 1. Let $\alpha_i = a_{ii}$ for $i = 1, 2, \ldots$. Then $\{\alpha_n\}$ is bounded. Since $AX_i = Y_i$, $y_{jk}^{(i)} = a_{jj} x_{jk}^{(i)} = \alpha_j x_{jk}^{(i)}$ for $i = 1, 2, \ldots, n$ and $j, k = 1, 2, \ldots$.

Conversely, assume that there is a bounded sequence $\{\alpha_n\}$ of real numbers such that $y_{jk}^{(i)} = \alpha_j x_{jk}^{(i)}$ for $i = 1, 2, \ldots, n$ and $j, k = 1, 2, \ldots$. 

Self-adjoint interpolation on $AX = Y$ in a tridiagonal algebra $\text{AlgL}$
Let $A$ be a diagonal matrix with the diagonal sequence $\{\alpha_n\}$. Since $\{\alpha_n\}$ is bounded, $A$ is a bounded operator. Also $A$ is self-adjoint and $AX_i = Y_i$ for $i = 1, 2, \cdots, n$.

By the similar way with the above, we have the following.

**Theorem 4.** Let $Alg\mathcal{L}$ be the tridiagonal algebra and let $X_i = (x_{jk}^{(i)})$ and $Y_i = (y_{jk}^{(i)})$ be operators acting on $\mathcal{H}$ for $i = 1, 2, \cdots$. Then the following are equivalent:

1. There exists a self-adjoint operator $A = (a_{ij})$ in $Alg\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$.

2. There is a bounded sequence $\{\alpha_n\}$ of real numbers such that $y_{jk}^{(i)} = \alpha_j x_{jk}^{(i)}$ for all $i, j, k \in \mathbb{N}$.

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