THE BOUNDARIES OF DIPOLE GRAPHS AND THE COMPLETE BIPARTITE GRAPHS $K_{2,n}$

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Abstract. We study the Seifert surfaces of a link by relating the embeddings of graphs with induced graphs. As applications, we prove that every link $L$ is the boundary of an oriented surface which is obtained from a graph embedding of a complete bipartite graph $K_{2,n}$, where all voltage assignments on the edges of $K_{2,n}$ are 0. We also provide an algorithm to construct such a graph diagram of a given link and demonstrate the algorithm by dealing with the links $4_1^t$ and $5_2$.

1. Introduction

A link $L$ is an embedding of $n$ copies of $S^1$ into $S^3$. If a link has only one copy of $S^1$, the link is called a knot. Throughout the article, we will assume all links are tame, which means all links can be deformed in a form of a finite union of line segments. In the language of graph theory, a knot can be considered as a spatial graph of a cycle graph $C_n$ on $n$ vertices. Two links are equivalent if there is an isotopy between them. In the case of prime knots, this equivalence is the same as the existence of an orientation preserving homeomorphism on $S^3$, which sends a knot to the other knot. Although the equivalent class of a link $L$ is called a link type, throughout the article, a link really means the equivalent class of the link $L$. Additional terms in the knot theory can be found in [3].

A graph $\Gamma$ is an ordered pair $\Gamma = (V(\Gamma), E(\Gamma))$ comprising a set $V(\Gamma)$ of vertices together with a set $E(\Gamma)$ of edges. Two graphs $\Gamma_1 = (V(\Gamma_1), E(\Gamma_1))$ and $\Gamma_2 = (V(\Gamma_2), E(\Gamma_2))$ are equivalent if there exists a bijective
function $\phi : V(\Gamma_1) \longrightarrow V(\Gamma_2)$ such that $e = \{u, v\} \in E(\Gamma_1)$ if and only if $\{\phi(u), \phi(v)\} \in E(\Gamma_2)$.

Although these two subjects are easily considered independently, there are a few branches of graph theory and knot theory which overlap each other [4,9,10,16,17,20]. One of the branches we are interested in is about the Seifert surfaces of the links. A compact orientable surface $F$ in $S^3$ is called a Seifert surface of a link $L$ if the boundary of $F$ is isotopic to $L$. The existence of such a surface was first proven by Seifert using an algorithm on a diagram of $L$. This algorithm was named after him as Seifert’s algorithm [16]. A Seifert surface $F$ gives rise to a natural signed graph, which is called the induced graph $\Gamma(F)$ by collapsing a disc to a vertex and a twisted band to a signed edge as illustrated in Figure 1 (b). On the other hand, a graph embedding into an orientable surface naturally produces a Seifert surface by taking a tubular neighborhood of the graph. These graph embeddings are completely determined by a rotation scheme and a voltage assignment, as explained in Section 2. However, the objects of having these surfaces are very different. Knot theorists are concerned about the isotopy classes of surfaces whereas graph theorists only enumerate the homeomorphic classes of graphs in the surfaces. Besides the sphere, the mapping class groups of orientable surfaces, which are the groups of homeomorphisms on the surface quotient out by the component containing the identity homeomorphism, are infinite. However, as known as a versatile tool for many different areas, graphs have been used in many articles related with Seifert surfaces [7, 10–12, 19].

In particular, we are interested in banded surfaces and plumbing surfaces. We will relate these surfaces as graph embeddings of bouquets of circles and dipoles as well as the complete bipartite graphs with integral voltage assignments. Each surface obtained from the graph embeddings of the bouquets of circles, dipoles and the complete bipartite graphs is called bouquet of $n$-circles surface, $n$-dipole surface and the complete bipartite graph surface, respectively. If the voltage assignments on all edges are zero, they are said to be flat.

A banded surface originally introduced by Kauffman used to study Seifert pairings, Alexander polynomials of links [8]. The author, Kwon and Lee demonstrated the existence of flat banded surfaces from braid representatives and canonical Seifert surfaces of a link by using the induced graph of the link [12].

Since the induced graphs of the Seifert surfaces of links are bipartite, all canonical Seifert surfaces of links might be considered as embeddings.
of bipartite graphs, where the voltage assignment on the edges of bipartite graphs are ±1. However, these graphs are not complete in general.

In a very recent paper, Baader introduced ribbon diagrams for strongly quasipositive links [1] to show that every \((m, n)\) torus link is a boundary of a surface which is obtained from the 0 voltage assignment on all the edges of the complete bipartite graphs \(K_{m,n}\), where the diagram of the complete bipartite graph \(K_{m,n}\) is chosen to be in a very special form, as explained as the standard diagram. For instance, the boundary of the standard diagram of \(K_{2,3}\) is the trefoil knot, which is the \((2,3)\) torus knot; yet if we change a crossing in the standard diagram of \(K_{2,3}\), the boundary of this non-standard diagram of \(K_{2,3}\) becomes the figure eight knot, which is not a torus knot; in fact, it is hyperbolic. This phenomenon naturally raises a question whether every link is a boundary of a complete bipartite graph \(K_{m,n}\), as stated in Question 3.1 [13]. A weaker version of Question 3.1 was proven immediately without a difficult theory, as in [7]. Let us remark that these edges in the graph embeddings are allowed to be linked, but not knotted. In the present article, we positively answer Question 3.1 that, for a given link \(L\), there exists a graph diagram \(D(K_{2,m})\) of a complete bipartite graph \(K_{2,m}\) such that the link \(L\) is a boundary of \(D(K_{2,m})\) where all voltage assignments on the edges of \(K_{2,m}\) are 0. If we flip two discs corresponding to the vertices in one of the bipartition sets whose cardinality is 2, we obtain a graph diagram \(D(K_{2,m})\) of a complete bipartite graph \(K_{2,m}\) whose boundary is the link \(L\), where all voltage assignments on the edges of \(K_{2,m}\) are ±1.

The outline of this paper is as follows. We first provide some preliminary definitions and results in Section 2. In Section 3, we investigate the complete bipartite graph \(K_{m,n}\) surface by allowing the bands represented by the edges of the complete bipartite graph \(K_{m,n}\) to be linked but not knotted. In particular, the complete bipartite graph \(K_{2,n}\) can be seen as a subdivision of \(n\)-dipole graphs. We show that every link is a boundary of dipole surface where the signs of the edges are all 0 (and ±1). We provide a few examples of such presentations of links. In Section 4, we provide an algorithm to construct such a graph diagram of a given link and demonstrate the algorithm by dealing with the links 4_1 and 5_2.

2. Preliminaries

A Seifert surface \(F_L\) of an oriented link \(L\) is produced by applying Seifert’s algorithm to a link diagram \(D(L)\) of \(L\) as shown in Figure 1.
(a); it is called a **canonical Seifert surface**. A canonical Seifert surface $F$ gives rise to a natural signed graph, which is referred to as the *induced graph* $\Gamma(F)$ by collapsing a disc to a vertex and a twisted band to a signed edge, as illustrated in Figure 1 (b). These processes can also be performed on arbitrary Seifert surfaces. Since the link $L$ is tame and its Seifert surface $F_L$ is compact, the induced graph $\Gamma(F_L)$ is finite. By separating the discs by local orientation as indicated $\pm$ on each vertices in Figure 1 (b), the induced graph $\Gamma(F_L)$ can be considered as a bipartite graph. If the Seifert surface is connected, then its induced graph is also connected.

It is fairly easy to see that the number of Seifert circles (half twisted bands), denoted by $s(F_L)(c(F_L))$, is the cardinality of the vertex set, $V(\Gamma(F_L))$ (edge set $E(\Gamma(F_L))$, respectively).

Next, let us provide the definitions of graphs which are used in the article. A **bouquet of $n$-circles**, denoted by $B_n$, is a graph with a single vertex and $n$ self loops as illustrated in Figure 2 (a). These bouquets of circles are fundamental building blocks in topological graph theory because any connected graph can be reduced to a bouquet of circles by contracting a spanning tree to a point as all coedges become generators of the fundamental group of the graph. An **$n$-dipole**, denoted by $D_n$, is a graph with two vertices and $n$ edges joining these two vertices as depicted in Figure 2 (b). A **complete bipartite graph**, $\Gamma = (V_1 \amalg V_2, E)$, is a bipartite graph such that for any two vertices $v_1 \in V_1$ and $v_2 \in V_2$, $\{v_1, v_2\}$ is an edge in $\Gamma$. The complete bipartite graph with partitions of size $|V_1| = m$ and $|V_2| = n$ is denoted $K_{m,n}$, as shown in Figure 2 (c).
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Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph. A rotation scheme is a set of cyclic orders of edges which are adjacent to $v$ for all vertex $v \in V(\Gamma)$. A voltage assignment is a map from $E(\Gamma)$ to $\mathbb{Z}_2$. These information are often called a map. Next, we will explain how the rotation scheme and voltage assignment of a graph $\Gamma$ determine the embedding of $\Gamma$ into an orientable surface. The main idea of the construction of a surface is a band decomposition, as follows: each vertex $v \in V(\Gamma)$ is replaced by a disc as 0-band, 1-bands representing an edge are glued in order to respect the rotation scheme (cyclic order on each vertices) and the voltage assignment (if the value on an edge is 0, the band is flat, otherwise, the band is half twisted) and 2-bands are capping off the boundaries [6]. Figure 3 provides an example where $\Gamma$ is the complete graph $K_4$. Let us remark that the boundary of the flat $K_4$ surface in Figure 3 (c) is the hope link.

As described earlier, a Seifert surface produces the induced graph. Conversely, a graph produces a surface which may not be orientable. In order to have the orientability of a resulting surface, we will assume that the voltage assignment is zero for all edges. In graph embeddings, graph theorists are mainly concerned about the enumerations of non equivalent embeddings of a graph into a surface. In contrast, knot theorists are concerned about the isotopy classes instead of homeomorphism classes of surfaces. In fact, there exist non equivalent links which are boundaries of homeomorphic surfaces obtained from the two cell embeddings of the same graph. Figure 6 and Figure 7, which are two cell embeddings of the graph $K_{2,3}$, have different boundaries, the trefoil and the figure eight knot. The relation between graph embeddings and the Seifert surface are studied extensively with topological graph theorists [2].

Figure 2. (a) A bouquet $n$-circles graph $B_n$, (b) an $n$-dipole graph $D_n$ and (c) the complete bipartite graph $K_{m,n}$.
Figure 3. (a) The complete bipartite graph $K_4$ with a rotation scheme and $\mathbb{Z}_2$ voltage assignment, (b) a band decomposition of $K_4$ surface with respect to the rotation scheme and $\mathbb{Z}_2$ voltage assignment in (a), (c) a flat $K_4$ surface obtained from (b) by flipping discs $\gamma$ and $\delta$ and (d) an embedding of the complete bipartite graph $K_4$ into the torus which is obtained from (c) by attaching the two discs along the boundary of a flat $K_4$ surface.

For fixed types of graphs, considering the graph embedding as Seifert surfaces is not new. Kauffman obtained a banded surface $S$ from an $n$-band $B$ by attaching disc $D$ with a blackboard framing as depicted in Figure 4 (b). One may find that these are embeddings of a bouquet of circles, and we call these banded surfaces bouquet of $n$-circles surfaces. An example of a 3-dipole and its flat 3-dipole surface are given in Figure 5. In particular, if all bands in a bouquet of $n$-circles surface are flat,
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Figure 4. (a) A 2-band (b) its banded surface with a blackboard framing, (c) a bouquet of 2-circles presentation of (a) and (d) its bouquet of circles surfaces with a blackboard framing.

Figure 5. (a) a 3-dipole presentation of 3-bands and (b) its 3-dipole surface with a blackboard framing.

then we call it a flat bouquet of $n$-circles surface. Similarly, we define $n$-dipole surfaces, flat $n$-dipole surfaces, $K_{n,m}$ surfaces and flat $K_{n,m}$ surfaces. Let us remark that 1-bands in these surfaces may be linked but not knotted. The author, Kwon and Lee [12] showed the existence of a flat unknotted banded surface whose boundary is the given link. They also provided a few classification theorems and some upperbound on the minimal numbers of bands required to present the given link. As we have mentioned and illustrated in Figure 4, these flat unknotted
banded surfaces with \( n \) bands may be considered as bouquet of \( n \)-circles surfaces.

**Theorem 2.1.** ([12]) For a given link \( L \), there exists a flat unknotted banded surface \( F \) whose boundary is the link \( L \).

Furthermore, if 1-bands in an \( n \)-dipole surface are braided, it is called a braidzel surface and Nakamura showed that every link is a boundary of a braidzel surface [15]. His result was improved by Miura [14] that every link is a boundary of a flat braidzel surface.

Bands in a braidzel surface are allowed to be twisted for any integral number of times. We will only consider an even number of twists. Similarly, a framed banded surface can be produced by replacing each arc by a band (each arc in the middle of the band is called a core), where a prefixed framing on the band represents \( m_i \) full twists; precisely, \( m_i \) is the linking number between a closed path \( \alpha \), which is a path sum of the core of the \( i \)-th band of \( S \), and any path joining both ends of the core in \( D \) and its push up \( \alpha^+ \) towards to the positive normal direction (as indicated “+” in Figure 4 (b)). The linking number discussed here does not depend on the choice of a path on \( D \).

### 3. Dipole surfaces and \( K_{2,n} \) surfaces

A motivation to consider the dipole surfaces initiated from an article by Baader [1] which introduced ribbon diagrams for strongly quasipositive links in order to show that every \((m, n)\) torus link is a boundary of a surface which is obtained from the 0 voltage assignment on all edges of the complete bipartite graphs \( K_{m,n} \), where the diagram of the complete bipartite graph \( K_{m,n} \) is chosen to be in a very special form as explained the standard diagram as depicted in Figure 6 for \( K_{2,3} \).

However, if we use a non-standard diagram of \( K_{2,3} \), as depicted in Figure 7, the boundary of this surface is the figure eight knot. Since the figure eight knot is hyperbolic, it must not be a torus knot. This phenomenon motivates us to find an answer for the question as to whether every link is a boundary of a complete bipartite graph \( K_{n,m} \). In [13], the following question was raised in order to settle this problem.

**Question 3.1.** ([13]) For a given link \( L \), is there a graph diagram \( D(\Gamma) \) of a complete bipartite graph \( \Gamma \) such that the link \( L \) is the boundary of \( D(\Gamma) \) where all voltage assignments on the edges of \( \Gamma \) are 0?

A weaker version of Question 3.1 was proven by Kim and et al. in [7].
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Figure 6. A diagram of the complete bipartite graph $K_{2,3}$ whose boundary is the torus knot $T(2,3)$.

Figure 7. A different diagram of the complete bipartite graph $K_{2,3}$ whose boundary is the figure-eight knot.

Theorem 3.2. ([7]) For a given link $L$, there exists a graph diagram $D(K_{2,n})$ of a complete bipartite graph $K_{2,n}$ such that the link $L$ is the boundary of $D(K_{2,n})$ where all voltage assignments on the edges of $K_{2,n}$ are either $0, 1$ or $-1$.

Now, we will prove Theorem 3.3. First, we explain the moves which will be used in the proof of the theorem. From a flat bouquet of $n$-circles surface $\mathcal{F}$, we can obtain a sequence of length $2n$ which represents the connection of bands as follows. The disc $\mathcal{D}$ can be presented as unit disc, the intervals which are the intersection of bands and the disc $\mathcal{D}$ can be labeled by $\{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\}$ clockwise from a fixed point on the boundary of the disc $\mathcal{D}$ which is not in the intervals. By moving the fixed point, we will get different labels on the intervals, this is called a relabeling. Then each band connecting two intervals can be presented by a pair of labels. The set of all such pairs of labels is denoted by $L(\mathcal{F})$. For example, $L(\mathcal{F})$ for a flat banded surface with 5 bands in Figure 11 (b) is $L(\mathcal{F}) = \{\{1, \overline{1}\}, \{2, \overline{3}\}, \{3, \overline{5}\}, \{4, \overline{2}\}, \{5, \overline{3}\}\}$. Although all $i$ is paired with $\overline{j}$ in this example, this is not true in general. Indeed, showing such $L(\mathcal{F})$ can be chosen by a suitable relabeling and a finite sequence of band slides is the key idea of Theorem 3.3.

Theorem 3.3. Every link is a boundary of a flat $n$-dipole surface.
Figure 8. A slide of the band connects $i$ and $j$ along the band $j+1$ and $l$ if $1 \leq i < j < n < k, l \leq n$.

Proof. For a given link $L$, there exists a flat bouquet of $n$-circles surface $B$ such that the boundary of $B$ is $L$ by Theorem 2.1. From a flat bouquet of $n$-circles surface $B$ such that the boundary of $B$ is $L$, as described above, by fixing a point $P$ on the boundary of $D$, we obtain a sequence $L(B)$. First we prove the following claim.

Claim: Any flat bouquet of $n$-circles surface $B$ can be transformed to a flat bouquet of $n$-circles surface $F$ by a finite sequence of band slides such that $L(F)$ consists only of the shape $\{i, j\}$.

To prove the claim we induct on $n$.

If $n = 1$, $L(F)$ is clearly the desired form. For $n = 2$, there are three possible cases for $L(F): \{\{1, 2\}, \{1, 2\}\}, \{\{1, 2\}, \{2, 3\}\}$ and $\{\{1, 2\}, \{2, 3\}\}$.
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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9}
\caption{A slide of the band connects $i$ and $j$ along the band $j+1$ and $\bar{l}$ if $1 \leq i < j < n < 1 < k < l \leq n$.}
\end{figure}

\{2, \bar{T}\}, but the last two of them are already in the desired form. For \{\{1,2\}, \{\bar{T}, \bar{\bar{T}}\}\}, we slide the first bands presented by \{1,2\} along the band presented by \{\bar{T}, \bar{\bar{T}}\} to have \{\{1, \bar{T}\}, \{2, \bar{\bar{T}}\}\}.

Now we assume $n \geq 3$. Suppose there exists a flat bouquet of $n$-circles surface $B$ which cannot be transformed to a flat bouquet of $n$-circles surface $F$ by a finite sequence of band slides such that $L(F)$ consists only of the shape $\{i, \bar{j}\}$ for a suitable fixed point which determines the relabeling. Then, for any fixed point $R$ on the boundary of the disc $D$ and a flat bouquet of $n$-circles surface $F$ which is obtained from $B$ by a finite sequence of band slides, the set $\Omega(F, R) = \{\{i, j\} \in L(F) | 1 \leq i < j \leq n\}$ is nonempty.

We pick one $F_0$ which has the minimal cardinality of the set $\Omega(F_0, P)$ for a point $P$ among all counterexamples. We choose a pair $\{i, j\}$ for which $j$ is the largest among such pairs in $\Omega(F_0, P)$. If $j \neq n$, then by the maximality of $j$, $j+1$ (possibly $j+1$ can be $n$) must be connected to $\bar{l}$. We divide cases into 1) $\bar{l}$ is connected to $\bar{k}$ and 2) $i \bar{l}$ is connected to $k$.

For the first case 1) as shown in Figure 8, we slide the band presented by $\{i, j\}$ along the band presented by $\{j+1, \bar{l}\}$. The resulting flat bouquet of $n$-circles surface $F_1$ has a new $L(F, P)$ which has the property that $\Omega(F_1, P) < \Omega(F_0, P)$ where we used the same fixed point $P$ for labeling. But it contradicts the minimum hypothesis of $\Omega(F_0, P)$.

For the second case 2), even if we slide the band presented by $\{i, j\}$ along the band presented by $\{j+1, \bar{l}\}$, the resulting flat bouquet of $n$-circles surface $F_1$ has the property that $\Omega(F_1, P) = \Omega(F_0, P)$ but $\{k, n\} \in \Omega(F_1, P)$. Then this case can be handled as follows.
Now without the loss of generality, we may assume that there exists a pair \(\{i, n\}\) in \(\Omega(F_0, P)\). Similarly, we divide cases into \(2 - i\) \(\bar{T}\) is connected to \(T\) and \(2 - ii\) \(\bar{T}\) is connected to \(k\).

For the first case \(2 - i\), if \(\bar{T}\) is connected to \(k\), then we slide the band presented by \(\{i, n\}\) along the band presented by \(\{\bar{T}, k\}\) and it leads us the similar contradiction for the case \(1 - i\).

For the second case \(2 - ii\) if \(\bar{T}\) is connected to \(k\), then we move \(P\) to a point \(Q\) between \(n - 1\) and \(\pi\). Then \(\Omega(F_0, P) \geq \Omega(F_0, Q)\) but by the minimum hypothesis of \(\Omega(F_0, P) = \Omega(F_0, Q)\). To obtain such an equality, one can see that \(\{1, \pi\} \in \Omega(F_0, P)\). Now, we consider the set \(\Omega(F_0, P) = \{\{i, \pi\}| 1 \leq i < j \leq n\}\). One can see that the cardinality of two sets \(\Omega(F, P)\), \(\Omega(F_0, P)\) must be the same. Therefore, there exists \(\{l, m\} \in \Omega(F, P)\) and we further choose one \(\{l, m\}\) for which \(m\) is the largest among such pairs in \(\Omega(F_0, P)\) as illustrated in Figure 9. If we slide the band presented by \(\{l, m\}\) along the band presented by \(\{i, m + 1\}\). The resulting flat bouquet of \(n\)-circles surface \(F_1\) has a new \(L(F_1)\) with the same fixed point \(P\) which has the property that \(|\Omega(F_1, P)| < |\Omega(F_0, P)|\). It contradicts the minimum hypothesis of \(\Omega(F_0, P)\) and completes the proof of the claim.

By the claim, the resulting flat bouquet of \(n\)-circles surface \(\overline{B}\) has a shape as illustrated in Figure 10 (a). The final modification is to make the disc \(\mathcal{D}\) into two discs connected by a single flat band as depicted in Figure 10 (b). The moves used in this process do not change the link.
type of the boundaries of the surfaces. This completes the proof of the theorem.

Let us remark that every braidzel surface is a dipole surface; however, the converse is not true in general. However, the dipole surfaces dealt in the article, which are obtained from a flat plumbing basket surface are braidzel surfaces. Moreover, they are completely determined by two permutations on \( n \) letters, describing how bands are connected and the order that explains the layer of bands.

The following corollary positively answer the Question 3.1.

**Corollary 3.4.** For a given link \( L \), there exists a graph diagram \( D(K_{2,n}) \) of a complete bipartite graph \( K_{2,n} \) such that link \( L \) is the boundary of \( D(K_{2,n}) \) where all voltage assignments on the edges of \( K_{2,n} \) are 0.

**Proof.** For a given link \( L \), there exists an \( n \)-dipole flat surface \( B \) whose boundary is \( L \) by Theorem 3.3. By subdividing each band in \( B \) into two flat bands, the resulting surface can be considered a complete bipartite graph surface, where the bipartition of the vertex set is (the set of vertices presented by two vertices in \( n \)-dipole, the set of vertices produced by the subdivision).

Let us remark that by flipping two discs corresponding to vertices in one of the bipartition sets whose cardinality is 2, Corollary 3.4 can be restated that for a given link \( L \), there exists a graph diagram \( D(K_{2,n}) \) of a complete bipartite graph \( K_{2,n} \) such that the link \( L \) is a boundary of \( D(K_{2,n}) \) where all voltage assignments on the edges of \( K_{2,n} \) are \( \pm 1 \).

4. Algorithm and examples

In this section, for a given link we provide how to find a graph diagram \( D(K_{2,n}) \) of a complete bipartite graph \( K_{2,n} \) such that link \( L \) is a boundary of \( D(K_{2,n}) \). We also demonstrate our algorithm by exhibiting two examples, the links \( 5_2 \) and \( 4_1^2 \).

**Algorithm**

- **Step 1.** For a give link \( L \), we find its braid representation \( \beta \), the closed braid \( \beta = L \).
- **Step 2.** Apply the method in [5] in order to obtain a flat plumbing basket surface \( F \) which is obtained from a disc by successively plumbing flat annuli.
Figure 11. (a) The link $4_2^2$, (b) a flat plumbing basket surface of the link $4_1^2$ with 5 flat plumbings, (c) a flat 5-dipole surface of the link $4_2^2$ and (d) a graph diagram $D(K_{2,6})$ of a complete bipartite graph $K_{2,6}$ whose boundary is the link $4_2^2$, where the voltage assignments are all zero.

- **Step 3.** Apply the claim in Theorem 3.3 in order to find a flat plumbing basket surface $F$ of $n$ annuli whose boundary is $L$, which can be presented by two sets of $n$-tuples composed of $\{1, 2, \ldots, n\}$, the first $n$-tuples representing how the bands in the flat dipole surface are connected and the second $n$-tuples representing the order of bands from the top to the bottom.
- **Step 4.** Apply the move in Figure 10, and we obtain a flat $(n+1)$-dipole surface.
- **Step 5.** Subdivide the bands by adding a disc in the middle of each band, as described in Corollary 3.4; we obtain the desired graph diagram $D(K_{2,n+1})$ whose boundary is the given link $L$.

Now, we apply the algorithm for the links $5_2$ and $4_1^2$ in the following examples.
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Figure 12. (a) The knot $5_2$ as a closed braid, (b) a Seifert surface of $5_2$ in order to apply the algorithm in [5], (c) a flat plumbing basket surface of $5_2$, (d) a flat 6-dipole surface of $5_2$ and (e) a graph diagram $D(K_{2,7})$ whose boundary is $5_2$ where the voltage assignments are all zero.
Example 4.1. A graph diagram $D(K_{2,6})$ of a complete bipartite graph $K_{2,6}$ whose boundary is the link $4_2^1$ where the voltage assignments are all zero as illustrated in Figure 11 (d).

Proof. The link $4_2^1$ is a closed braid $(\sigma_1)^4$ of two strings, as drawn in Figure 11 (a). In [5], it was shown that $4_1^2$ has a flat plumbing surface with 3 flat plumbings and a flat plumbing basket surface with 5 flat plumbings as illustrated in Figure 11 (b). This one already satisfies the hypothesis of a flat dipole surface; thus, we have $< (1, 4, 5, 2, 3)| (1, 2, 3, 4, 5) >$ to present it as described in Step 3. We shrink the disc $D$ into the union of the two discs to have a flat 6-dipole surface, as depicted as the shaded region in Figure 11 (c). By following Step 4, we obtain a graph diagram $D(K_{2,6})$ of a complete bipartite graph $K_{2,6}$ whose boundary is the link $4_2^1$ where the voltage assignments are all zero as shown in Figure 11 (d).

Example 4.2. A graph diagram $D(K_{2,7})$ of a complete bipartite graph $K_{2,7}$ whose boundary is the knot $5_2$ where the voltage assignments are all zero as illustrated in Figure 12.

Proof. The link $5_2$ is a closed braid $\sigma_2^{-1}\sigma_1(\sigma_2)^3\sigma_1$ of three strings as drawn in Figure 12 (a). In [5] an algorithm to find a link’s flat plumbing surface as depicted in Figure 12 (b) is provided. Let us explain the algorithm. First we choose the disc $D$ which is the union of Seifert discs connected by two half twisted bands which are represented by $\sigma_5^{-1}\sigma_1$ as drawn in the figure by the dashed red line. For possibility of flat plumbing, we add two extra annuli. The numbers 1, 2, …, 12 in the figure were chosen by reading the order of flat plumbing from the point along the direction of the arrow in the figure. The resulting flat plumbing basket surface with 6 flat plumbings are given in Figure 12 (c). This one already satisfies the hypothesis of a flat dipole surface which can be read as $< (4, 5, 1, 2, 3, 6)|(1, 2, 3, 4, 5, 6) >$ in order to present it as described in Step 3. We shrink the disc $D$ to the union of two discs $D_1$ and $D_2$ to have a flat 6-dipole surface as depicted as the shaded region in Figure 12 (d). By following Step 4, we obtain a graph diagram $D(K_{2,6})$ of a complete bipartite graph $K_{2,7}$ whose boundary is the knot $5_2$ where the voltage assignments are all zero as shown in Figure 12 (e).

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