REMARKS ON SIMPLY $k$-CONNECTIVITY AND $k$-DEFORMATION RETRACT IN DIGITAL TOPOLOGY

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Abstract. To study a deformation of a digital space from the viewpoint of digital homotopy theory, we have often used the notions of a weak $k$-deformation retract [20] and a strong $k$-deformation retract [10, 12, 13]. Thus the papers [10, 12, 13, 16] firstly developed the notion of a strong $k$-deformation retract which can play an important role in studying a homotopic thinning of a digital space. Besides, the paper [3] deals with a $k$-deformation retract and its homotopic property related to a digital fundamental group. Thus, as a survey article, comparing among a $k$-deformation retract in [3], a strong $k$-deformation retract in [10, 12, 13], a weak deformation $k$-retract in [20] and a digital $k$-homotopy equivalence [5, 24], we observe some relationships among them from the viewpoint of digital homotopy theory. Furthermore, the present paper deals with some parts of the preprint [10] which were not published in a journal (see Proposition 3.1). Finally, the present paper corrects Boxer’s paper [3] as follows: even though the paper [3] referred to the notion of a digital homotopy equivalence (or a same $k$-homotopy type) which is a special kind of a $k$-deformation retract, we need to point out that the notion was already developed in [5] instead of [3] and further corrects the proof of Theorem 4.5 of Boxer’s paper [3] (see the proof of Theorem 4.1 in the present paper). While the paper [4] refers some properties of a deck transformation group (or an automorphism group) of digital covering space without any citation, the study was early done by Han in his paper (see the paper [14]).
1. Introduction

The notions of simply connectivity and a strong deformation retract have often used in algebraic topology [23]. Motivated by them, the notions of simply $k$-connectivity [8] and a strong $k$-deformation retract [10, 12, 13] were firstly developed from the viewpoint of digital topology and digital geometry. Indeed, these concepts play important roles in calculating digital fundamental groups of digital images. Furthermore, the notion of a strong $k$-deformation retract is substantially used to proceed a $k$-homotopic thinning [13, 16]. Even though the paper [6] firstly proposed the notion of a digital $k$-deformation from the viewpoint of digital homotopy theory, it was presented unclearly and finally misprinted at the page 162 (line 10) of [6]. At the moment, the author [6] tried to propose the notion of a strong $k$-deformation retract as shown in the present paper (see Definition 4). Thus it was represented more precisely in the paper [12, 13]. Furthermore, the work [10] contains many tools related to the content of the present paper, it could not be published from a journal at the moment. However, many parts of the manuscript [10] were published in [13, 16] later. Since some meaningful parts in [10] remain unpublished, the present paper deals with the parts.

2. Preliminaries

To study digital topological properties of a digital image $(X, k)$, we have used various tools such as digital fundamental groups [2, 13, 16], digital covering spaces [7, 8, 11, 12, 14, 15, 18, 19], digital homotopy equivalences [5, 24] and digital $k$-surface structures [12]. A (binary) digital image $(X, k)$ can be regarded as a subset $X \subset \mathbb{Z}^n$ with one of the $k$-adjacency relations of $\mathbb{Z}^n$ (or an adjacency graph). For $a, b \in \mathbb{Z}$ with $a \leq b$, the set $[a, b]_{\mathbb{Z}} = \{ n \in \mathbb{Z} | a \leq n \leq b \}$ is called a digital interval [22]. Further, let us recall the following:

Let $p := (p_i)_{i \in [1, n]}_{\mathbb{Z}}$ be a point of $\mathbb{Z}^n$ and $m$ an integer in $[1, n]_{\mathbb{Z}}$. Consider all points $q := (q_i)_{i \in [1, n]}_{\mathbb{Z}} \in \mathbb{Z}^n$ satisfying the property of (2.1) [8] such that $p \neq q$

\[
\left\{ \begin{array}{l}
\bullet \text{there are at most } m \text{ indices } i \text{ such that } |p_i - q_i| = 1 \text{ and} \\
\bullet \text{for all other indices } i, p_i = q_i.
\end{array} \right. \tag{2.1}
\]
The number of such points is the following [16] (for more details, see [17]):

\[ k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \text{ where } C_i^n = \frac{n!}{(n-i)! i!}. \] (2.2)

We will say that two points \( p, q \in \mathbb{Z}^n \) are \( k \)-adjacent if they satisfy the condition (2.1), where \( k := k(m, n) \) of (2.2) [26] (see also [8, 16]).

For instance (see also [8, 12]),

\[ (m, n, k) \in \begin{cases} (1, 2, 4), (2, 2, 8); \\
(1, 3, 6), (2, 3, 18), (3, 3, 26); \\
(1, 4, 8), (2, 4, 32), (3, 4, 64), (4, 4, 80); \\
(1, 5, 10), (2, 5, 50), (3, 5, 130), (4, 5, 210), (5, 5, 242); \text{ and} \\
(1, 6, 12), (2, 6, 72), (3, 6, 232), (4, 6, 472), (5, 6, 664), (6, 6, 728). \end{cases} \] (2.3)

Owing to the phrase “at most \( m \)” from the first bullet of (2.1), we can obviously see that the points \( p = (p_1, p_2, ..., p_n) \) and \( q = (q_1, q_2, ..., q_n) \) in \( \mathbb{Z}^n \) may differ in as many as \( m \) coordinates. Using the \( k \)-adjacency relations of \( \mathbb{Z}^n \) in (2.2), we can study digital topological properties of a set \( X \subset \mathbb{Z}^n \) with a \( k \)-adjacency, \( n \in \mathbb{N} \). This has been often used for representing digital connectivity, a digital isomorphism, a digital homotopy, a digital surface structure, etc. Owing to the digital \( k \)-connectivity paradox in [22], we remind the reader that \( k \neq \bar{k} \) except the case \( (\mathbb{Z}, 2, 2, X) \). However, in this paper we are not concerned with \( k \)-adjacency between two points in \( \mathbb{Z}^n \setminus X \).

Let us now recall basic notions. For an adjacency relation \( k \) of \( \mathbb{Z}^n \), a simple \( k \)-path with \( l + 1 \) elements in \( \mathbb{Z}^n \) is assumed to be an injective sequence \( (x_i)_{i \in [0, l] \mathbb{Z}^n} \subset \mathbb{Z}^n \) such that \( x_i \) and \( x_j \) are \( k \)-adjacent if and only if either \( j = i + 1 \) or \( i = j + 1 \) [22]. If \( x_0 = x \) and \( x_l = y \), then we say that the length of the simple \( k \)-path is \( l \). A simple closed \( k \)-curve with \( l \) elements in \( \mathbb{Z}^n, n \geq 2 \), denoted by \( SC_k^{n,l} \) [8], is the simple \( k \)-path \( (x_i)_{i \in [0, l-1] \mathbb{Z}^n} \), where \( x_i \) and \( x_j \) are \( k \)-adjacent if and only if \( j = i + 1(\mod l) \) or \( i = j + 1(\mod l) \) [22]. Besides, for \( \mathbb{Z}^n \) we remind the following [22]:

\[ \begin{align*}
N_k(x) := \{ x' \mid x \text{ is } k\text{-adjacent to } x' \text{ in } \mathbb{Z}^n \} \\
N_k^e(x) := \{ x' \mid x \text{ is } k\text{-adjacent to } x' \text{ in } \mathbb{Z}^n \} \cup \{ x \}.
\end{align*} \] (4.4)

As a generalization of \( N_k^e(x) \) in \( \mathbb{Z}^n \), for a multi-dimensional digital image \((X, k)\) and a point \( x \in X \subset \mathbb{Z}^n \), the notion of a (digital) \( k \)-neighborhood of a point \( x \) with radius \( \varepsilon \in \mathbb{N} \) was established [5] (see
\( N_k(x_0, \varepsilon) := \{ x \in X \mid l_k(x_0, x) \leq \varepsilon \} \cup \{ x_0 \}, \) \hspace{1cm} (2.5)

where \( l_k(x_0, x) \) is the length of a shortest simple \( k \)-path from \( x_0 \) to \( x \) in \( X \). For instance, for \( x \in (X, k) \) we can observe that \( N_k^* (x) \cap X \) is equal to \( N_k(x, 1) \) \[12\]. Besides, we need to remind that while \( N_k(x, \varepsilon) \) in (2.4) does not contain the point \( x \), the set \( N_k(x_0, \varepsilon) \) has the point \( x_0 \). If a point \( x \) in a digital image \( (X, k) \) is isolated \[22\], then for any \( \varepsilon \in \mathbb{N} \) we can observe that \( N_k^* (x, \varepsilon) \cap X \) is equal to \( N_k(x_0, \varepsilon) \) \[12\].

Definition 1. \[8\] (see also \[12, 15\]) Let \( (X, k_0) \) and \( (Y, k_1) \) be digital images in \( \mathbb{Z}^{n_0} \) and \( \mathbb{Z}^{n_1} \), respectively. A function \( f : X \to Y \) is \((k_0, k_1)\)-continuous if for every \( x \in X \), \( f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1) \).

We have often used the following notion of a \((k_0, k_1)\)-isomorphism instead of a \((k_0, k_1)\)-homeomorphism in \[1\]: for two digital images \( (X, k_0) \) in \( \mathbb{Z}^{n_0} \) and \( (Y, k_1) \) in \( \mathbb{Z}^{n_1} \), a map \( h : X \to Y \) is called a \((k_0, k_1)\)-isomorphism \[25\] (see also \[9\]) if \( h \) is a \((k_0, k_1)\)-continuous bijection and further, \( h^{-1} : Y \to X \) is \((k_1, k_0)\)-continuous \[1\] (see also \[12\]), and we use the notation \( X \approx_{(k_0, k_1)} Y \). If \( n_0 = n_1 \) and \( k_0 = k_1 \), then we call it a \( k_0\)-isomorphism \[25\] (see also \[9\]).

3. Simply \( k \)-connectivity

The following notion of ‘simply \( k \)-connected’ (or ‘simply \( k \)-connectivity’) has been often used in digital topology for calculating digital fundamental groups of some digital spaces \[8\], classifying digital spaces \[14\], studying an automorphism group of a digital covering \[12, 14\] and so forth. The present paper uses the digital \( k \)-fundamental group of a digital image established in \[2\]. Motivated by the notion of a pointed digital
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homotopy [2], the concept of a relative digital homotopy was developed, as follows:

**Definition 2.** [12] (see also [13]) Let $((X, A), k_0)$ and $(Y, k_1)$ be a digital space pair and a digital space, respectively. Let $f, g : X \rightarrow Y$ be $(k_0, k_1)$-continuous functions. Suppose there exist $m \in \mathbb{N}$ and a function $F : X \times [0, m]_{\mathbb{Z}} \rightarrow Y$ such that

- for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbb{Z}} \rightarrow Y$ given by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbb{Z}}$ is $(2, k_1)$-continuous;
- for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : X \rightarrow Y$ given by $F_t(x) = F(x, t)$ for all $x \in X$ is $(k_0, k_1)$-continuous.

Then we say that $F$ is a $(k_0, k_1)$-homotopy between $f$ and $g$ [2].

- Furthermore, for all $t \in [0, m]_{\mathbb{Z}}$, assume that the induced map $F_t$ on $A$ is a constant which follows the prescribed function from $A$ to $Y$. In other words, $F_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0, m]_{\mathbb{Z}}$.

Then we call $F$ a $(k_0, k_1)$-homotopy relative to $A$ between $f$ and $g$, and we say that $f$ and $g$ are $(k_0, k_1)$-homotopic relative to $A$ in $Y$, $f \simeq_{(k_0, k_1)\text{rel}A} g$ in symbols.

Besides, for a digital image $(X, k)$ we denote $\pi^k(X, x_0)$ by its digital $k$-fundamental group, where $x_0 \in X$ [2].

**Definition 3.** [8] A pointed $k$-connected digital space $(X, x_0)$ is called simply $k$-connected if $\pi^k(X, x_0)$ is a trivial group.

For instance, the paper [10] used the simply 2-connectivity of $\mathbb{Z}$ and further, its generalization, i.e. the property of simply $k$-connectedness of $\mathbb{Z}^n$, was proved in the manuscript [10]. Let us now prove the property by using another method as it was in [10], as follows:

**Proposition 3.1.** [10] For each $n \in \mathbb{N}$, $(\mathbb{Z}^n, 0_n)$ is simply $k(m, n)$-connected, where $0_n$ is the origin in $\mathbb{Z}^n$ and $m \in [1, n]_{\mathbb{Z}}$.

**Proof.** First, we need to prove that $(\mathbb{Z}, 0)$ is simply 2-connected. Let $f \in [f] \in \pi^2(\mathbb{Z}, 0)$, $f : [0, m]_{\mathbb{Z}} \rightarrow \mathbb{Z}$ with $f(0) = 0 = f(m)$. Since the domain of $f$ is finite, there is a positive integer $M$ such that $|f(s)| \leq M$ for all $s \in \mathbb{Z}$. Define

$$H : [0, m]_{\mathbb{Z}} \times [0, M]_{\mathbb{Z}} \rightarrow \mathbb{Z}$$

by
It is easily seen that $H$ is a homotopy, holding the endpoints fixed, between $f$ and the loop with image $\{0\}$. Since $[f]$ is chosen arbitrary, $\pi^2(\mathbb{Z},0)$ is trivial.

Second, by applying this argument in each coordinate, we obtain that for each $n \in \mathbb{N}$, $(\mathbb{Z}^n,0_n)$ is simply $2n(\text{or } k(1,n))$-connected. Finally, for any $m \in [1,n]_{\mathbb{Z}}$ since the simply $2n(\text{or } k(1,n))$-connectivity of $\mathbb{Z}^n$ implies the simply $k(m,n)$-connectivity of $\mathbb{Z}^n$, the assertion is proved.

4. Several deformation retracts in digital topology

Since the notions of a weak $k$-deformation retract, a $k$-deformation retract and a strong $k$-deformation retract are very important in digital homotopy theory, many works [3, 5, 6, 10, 12, 13, 16] studied their properties. This section investigates various properties of these concepts and further, shows that a strong $k$-deformation retract is more eligible for proceeding a $k$-homotopic thinning. For a digital space $(X,k)$ and $A \subset X$, $(X,A)$ is called a digital space pair with a $k$-adjacency [12]. Furthermore, if $A$ is a singleton set $\{x_0\}$, then $(X,x_0)$ is called a pointed digital space [22]. Based on the pointed digital homotopy in [2], the notion of a $k$-homotopy relative to a subset $A \subset X$ is often used in studying a $k$-homotopic thinning and a strong $k$-deformation retract of a digital space $(X,k)$ in $\mathbb{Z}^n$ [12, 16]. Let $(A,k) \subset (X,k)$ and let $r : X \to A$ be a $k$-continuous map such that $r(a) = a$ for all $a \in A$ which implies $r \circ i = 1_A$. Such a map $r$ is called a $k$-retraction [1].

The paper [6] proposed a digital $k$-deformation (see the page 162 of [6]) in terms of a relatively unclear expression (or misprinted). Indeed, the author [6] seem to propose the notion of a $k$-deformation retract or a strong $k$-deformation retract in the present paper. Thus the paper [12] makes the concept advanced and clear by using the notion of a strong $k$-deformation retract as follows:
**Definition 4.** [12] (see also [13]) For a digital image pair \(((X, A), k)\), A is said to be a strong \(k\)-deformation retract of \(X\) if there is a \(k\)-homotopy \(H : X \times [0, m] \rightarrow X\) and a \(k\)-retraction \(r : X \rightarrow A\) such that \(i \circ r \simeq_{k, \text{rel}, A} 1_X\), where \(r := H(\cdot, m)\). Then a point \(x \in X \setminus A\) is called strong \(k\)-deformation retractable.

The paper [3] introduces the notion of \(k\)-deformation retract:

**Definition 5.** [3] If there is a digital \(k\)-homotopy \(H : X \times [0, m] \rightarrow X\) such that
- the induced map \(H(\cdot, 0)\) is the identity map \(1_X\) and
- the induced map \(H(\cdot, m)\) is a \(k\)-retraction of \(X\) onto \(H(x, m) \subset X\).

Then the set \(A = H(x, m)\) is called a \(k\)-deformation retract of \(X\).

Let us now recall the notion of a weak \(k\)-deformation retract in [20].

**Definition 6.** [20] For a nonempty subset \((A, k) \subset (X, k)\), \(A\) is a weak \(k\)-deformation retract of \(X\) if there is a \(k\)-homotopy \(H : X \times [0, m] \rightarrow X\) such that
- \(1_X\) is \(k\)-homotopic to \(i \circ r\), and
- \(1_A\) is \(k\)-homotopic to \(r \circ i\),

where \(i : A \rightarrow X\) is an inclusion map and \(r := H(\cdot, m) : X \rightarrow A\) is a \(k\)-retraction.

Then the \(k\)-homotopy is a weak \(k\)-deformation retraction of \(X\) to \(A\). If \(A\) is a weak \(k\)-deformation retract of \(X\), then we say that a point \(x \in X \setminus A\) is weak \(k\)-deformation retractable.

The notion of digital \(k\)-homotopy equivalence was developed in [5] (see also [24]). Hence the paper points out that the paper [3] uses this notion without any citation.

**Definition 7.** [5] (see also [24]) For two digital images \((X, k)\) and \((Y, k)\) in \(\mathbb{Z}^n\), if there are \(k\)-continuous maps \(h : X \rightarrow Y\) and \(l : Y \rightarrow X\) such that \(l \circ h\) is \(k\)-homotopic to \(1_X\) and \(h \circ l\) is \(k\)-homotopic to \(1_Y\), then the map \(h : X \rightarrow Y\) is called a \(k\)-homotopy equivalence. Then we use the notation, \(X \simeq_{k, h, l} Y\).

Since one of the most used preprocessing methods in image processing is a \(k\)-homotopic thinning, the notion of strong \(k\)-deformation retract induces a \(k\)-homotopic thinning.

In relation to the study of a \(k\)-homotopic thinning, the following is a useful notion in comparing with both a weak \(k\)-deformation retract and a \(k\)-deformation retract, which can be essentially used for the calculation of the digital fundamental group.
**Definition 8.** [13] (see [16]) For a digital image \((X, k)\), a deleting all strong \(k\)-deformation retractable points in \(X\) is called a \(k\)-homotopic thinning of \(X\).

To calculate a digital \(k\)-fundamental group of \((X, k)\), we can delete all strong \(k\)-deformation retractable points from \(X\) via a strong \(k\)-deformation retract.

In view of the notions such as a \(k\)-deformation, a \(k\)-deformation retract, a weak \(k\)-deformation retract, a strong \(k\)-deformation retract and a digital \(k\)-homotopy equivalence we obtain the following:

**Theorem 4.1.** Let \(((X, A), k)\) be a space pair with a \(k\)-adjacency.

1. While a \(k\)-deformation retract of \((X, k)\) onto \((A, k)\) implies a weak \(k\)-deformation retract of \((X, k)\) onto \((A, k)\), the converse does not hold.
2. A strong \(k\)-deformation retract of \((X, k)\) onto \((A, k)\) is stronger than a \(k\)-deformation retract of \((X, k)\) onto \((A, k)\).
3. Each of a strong \(k\)-deformation retract, a \(k\)-deformation retract and a weak \(k\)-deformation retract of \((X, k)\) onto \((A, k)\) implies a digital \(k\)-homotopy equivalence between \((X, k)\) and \((A, k)\).

**Proof.** (1) From the hypothesis of a \(k\)-deformation retract of \((X, k)\) onto \((A, k)\), there is a \(k\)-homotopy \(H : X \times [0, m] \rightarrow X\) and a \(k\)-retraction \(H(\cdot, m) := r : X \rightarrow A\) such that \(i \circ r \simeq_k 1_X\) and \(r \circ i = 1_A\).

To be specific, for a nonempty subset \((A, k) \subset (X, k)\), since \(A\) is a \(k\)-deformation retract of \(X\), there is a \(k\)-homotopy \(H : X \times [0, m] \rightarrow X\) such that

- the equation \(r(\cdot) = H(\cdot, m)\) is a \(k\)-retraction from \(X\) onto \(A\); and
- \(H\) is a \(k\)-homotopy between \(1_X = H(\cdot, 0)\) and \(i \circ r\) where \(i : A \rightarrow X\) is an inclusion map (correction of the proof of Theorem 4.5 of [3]).

Thus the \(k\)-homotopy also implies that \(i \circ r \simeq_k 1_X\) and \(r \circ i = 1_A\) which implies a weak \(k\)-deformation retraction of \(X\) onto \(A\).

But, since \(r \circ i \simeq_k 1_A\) does not imply \(r \circ i = 1_A\), a weak \(k\)-deformation retract of \((X, k)\) onto \((A, k)\) need not imply a \(k\)-deformation retract of \((X, k)\) onto \((A, k)\), which makes the assertion valid.

(2) By the hypothesis of the strong \(k\)-deformation retract of \((X, k)\) onto \((A, k)\), there is a \(k\)-homotopy \(H : X \times [0, m] \rightarrow X\) and a \(k\)-retraction \(r : X \rightarrow A\) such that \(i \circ r \simeq_{k, \text{rel}, A} 1_X\) and \(r \circ i = 1_A\), where \(H(\cdot, m) := r : X \rightarrow A\) is a \(k\)-retraction and \(H(x, t) = x\) for all \(x \in A\).

Since the \(k\)-homotopy implies the property \(i \circ r \simeq_k 1_X\), owing to the given \(k\)-retraction \(r\), which implies a \(k\)-deformation retract of \((X, k)\) onto \((A, k)\). But the converse does not hold.

For instance, consider the digital images \((X := \{c_i | i \in [0, 14] \mathbb{Z}\}, 8)\) and
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put \( (A := X \setminus \{c_1, c_6, c_8\}, 8) \subset (X, 8) \) in Figure 1. Then we compare the processes of an \( 8 \)-deformation retract and a strong \( 8 \)-deformation retract of \((X, 8)\) onto \((A, 8)\) via an \( 8 \)-homotopy \( H \) on \( X \).

Let us consider the track (1) in Figure 1. By using an \( 8 \)-homotopy \( H : X \times [0, m] \mathbb{Z} \to X \), we can exchange the point \( c_3 \) into \( c_4 \) from the given \( X \) and further, move the point \( c_6 \) into \( c_5 \). Indeed, this can be considered as a process of an \( 8 \)-homotopy on \( X \) for some \( t \in [0, m] \mathbb{Z} \). Then we obviously observe the point \( c_3 \in A \) was moved into the point \( c_4 \) in terms of the \( 8 \)-homotopy \( H(x, t) \) for some point \( t \in [0, m] \mathbb{Z} \).

Besides, in terms of the process (1-1) we further exchange \( c_3 \) and \( c_4 \) again and move the point \( c_1 \) into \( c_2 \). Furthermore, proceed the step (1-1-1). Concretely, in terms of the process (1) \( \to (1-1) \to (1-1-1) \), we obtain both \( i \circ r \simeq_8 1_X \) and \( r \circ i = 1_A \), which means an \( 8 \)-deformation retract of \((X, 8)\) onto \((A, 8)\), where \( 1_X = H(, 0) \), \( r : X \to A \) and \( i : A \to X \).

This process cannot be a strong \( 8 \)-deformation retract of \((X, 8)\) onto \((A, 8)\) because the two points \( c_3 \) and \( c_4 \) are moved at some moment in the process of the \( 8 \)-homotopy. Meanwhile, let us take the track (2), i.e. the process \( X \to X_2 \to A \) which implies an \( 8 \)-homotopy on \( X \) with the finite interval \([0, m] \mathbb{Z}\) such that \( r \circ i \simeq_8 \text{rel.} 1_A \) and \( i \circ r = 1_X \), where \( H(, m) := r : X \to A \) and \( i : A \to X \) is the inclusion, which means a strong \( 8 \)-deformation retract of \((X, 8)\) onto \((A, 8)\).

(3) Each of a strong \( k \)-deformation retract, a \( k \)-deformation retract and a weak \( k \)-deformation retract \((X, k)\) onto \((A, k)\) implies the property \( i \circ r \simeq_k 1_X \) and \( r \circ i \simeq_k 1_A \). Thus the assertion is valid. \( \square \)

**Remark 4.2.** Comparing among a strong \( k \)-deformation retract, a \( k \)-deformation retract and a weak \( k \)-deformation retract, we observe that a strong \( k \)-deformation retract is eligible to a \( k \)-homotopic thinning.

Since a \( k \)-deformation retract from \((X, k)\) onto \((A, k)\) implies a \( k \)-homotopy equivalence between them, we finally obtain that for a nonempty subset \((A, k) \subset (X, k)\), \( \pi^k(X, a) \) is isomorphic to \( \pi^k(A, a) \), where \((A, k)\) is a \( k \)-deformation retract of \((X, k)\).

**Corollary 4.3.** Each of a strong \( k \)-deformation retract, a \( k \)-deformation retract and a weak \( k \)-deformation retract from \((X, k)\) onto \((A, k)\) implies a digital \( k \)-fundamental group isomorphism between \( \pi^k(X, x) \) onto \( \pi^k(A, x) \).
5. Summary

The property of simply $k$-connectivity of $\mathbb{Z}^n$ can be substantially used to study digital fundamental groups of digital images and automorphism groups of digital covering spaces. Comparing a weak $k$-deformation retract with both a $k$-deformation retract and a strong $k$-deformation retract, we conclude that a strong $k$-deformation retract implies a $k$-deformation retract and further, a $k$-deformation retract implies a weak $k$-deformation retract. However, each of the converses does not hold. To sum up, for a digital image pair $((X, A), k)$ assume that there is a $k$-homotopy $H : X \times [0, m] \mathbb{Z} \to X$ such that $H(\cdot, 0) = 1_X$ and $H(\cdot, m) := r$ as a $k$-retraction. Then if $i \circ r \simeq_{k-rel.A} 1_X$ and $r \circ i = 1_A$, then $(A, k)$ is a strong $k$-deformation retract of $(X, k)$; in case $i \circ r \simeq_k 1_X$ and $r \circ i = 1_A$, then $(A, k)$ is a $k$-deformation retract of $(X, k)$. Recently, the notion of strong $k$-deformation has been used for an establishment of the notion of $k$-homotopic thinning.

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