CLASSIFICATIONS OF \((\alpha, \beta)\)-FUZZY SUBALGEBRAS OF
\(BCK/BCI\)-ALGEBRAS

YOUNG BAE JUN, SUN SHIN AHN*, AND KYOUNG JA LEE

Abstract. Classifications of \((\alpha, \beta)\)-fuzzy subalgebras of \(BCK/BCI\)-algebras are discussed. Relations between \((\in, \in \lor q)\)-fuzzy subalgebras and \((q, \in \lor q)\)-fuzzy subalgebras are established. Given special sets, so called \(t\)-\(q\)-set and \(t\)-\(\in \lor q\)-set, conditions for the \(t\)-\(q\)-set and \(t\)-\(\in \lor q\)-set to be subalgebras are considered. The notions of \((\in,q)\)\(^{\text{max}}\)-fuzzy subalgebra, \((q,\in)\)\(^{\text{max}}\)-fuzzy subalgebra and \((q,\in \lor q)\)\(^{\text{max}}\)-fuzzy subalgebra are introduced. Conditions for a fuzzy set to be an \((\in,q)\)\(^{\text{max}}\)-fuzzy subalgebra, a \((q,\in)\)\(^{\text{max}}\)-fuzzy subalgebra and a \((q,\in \lor q)\)\(^{\text{max}}\)-fuzzy subalgebra are considered.

1. Introduction

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [7], played a vital role to generate some different types of fuzzy subgroups, called \((\alpha, \beta)\)-fuzzy subgroups, introduced by Bhakat and Das [1]. In particular, \((\in, \in \lor q)\)-fuzzy subgroup is an important and useful generalization of Rosenfeld’s fuzzy subgroup. In \(BCK/BCI\)-algebras, the concept of \((\alpha, \beta)\)-fuzzy subalgebras, which is studied in the
papers [3], [4], [5] and [8], is also important and useful generalization of the well-known concepts, called fuzzy subalgebras.

In this paper, we classify \((\alpha, \beta)-\)fuzzy subalgebras of BCK/BCI-algebras. We establish relations between \((\in, \in v q)-\)fuzzy subalgebras and \((q, \in v q)-\)fuzzy subalgebras. We provide conditions for an \((\in, \in v q)-\)fuzzy subalgebra to be a \((q, \in v q)-\)fuzzy subalgebra. Given special sets, so called \(t-q\)-set and \(t-\in v q\)-set, we provide conditions for the \(t-q\)-set and \(t-\in v q\)-set to be subalgebras. We define \((\in, q)_{\max}\)-fuzzy subalgebra, \((q, \in)_{\max}\)-fuzzy subalgebra and \((q, \in v q)_{\max}\)-fuzzy subalgebra. We consider conditions for a fuzzy set to be an \((\in, q)_{\max}\)-fuzzy subalgebra, a \((q, \in)_{\max}\)-fuzzy subalgebra and a \((q, \in v q)_{\max}\)-fuzzy subalgebra.

2. Preliminaries

By a BCI-algebra we mean an algebra \((X, *, 0)\) of type \((2, 0)\) satisfying the axioms:

(a1) \(((x * y) * (x * z)) * (z * y) = 0,\)

(a2) \((x * (x * y)) * y = 0,\)

(a3) \(x * x = 0,\)

(a4) \(x * y = y * x = 0 \Rightarrow x = y,\)

for all \(x, y, z \in X\). We can define a partial ordering \(\leq\) by \(x \leq y\) if and only if \(x * y = 0\). If a BCI-algebra \(X\) satisfies the axiom

(a5) \(0 * x = 0\) for all \(x \in X,\)

then we say that \(X\) is a BCK-algebra.

A nonempty subset \(S\) of a BCK/BCI-algebra \(X\) is called a subalgebra of \(X\) if \(x * y \in S\) for all \(x, y \in S\). We refer the reader to the books [2] and [6] for further information regarding BCK/BCI-algebras.

A fuzzy set \(\mu\) in a set \(X\) of the form

\[
\mu(y) := \begin{cases} 
  t \in (0, 1] & \text{if } y = x, \\
  0 & \text{if } y \neq x,
\end{cases}
\]
is said to be a fuzzy point with support \( x \) and value \( t \) and is denoted by \( x_t \).

For a fuzzy point \( x_t \) and a fuzzy set \( \mu \) in a set \( X \), Pu and Liu [7] introduced the symbol \( x_t \alpha \mu \), where \( \alpha \in \{ \in, q, \in \lor, \in \land \} \). To say that \( x_t \in \mu \) (resp. \( x_t \in q \mu \)), we mean \( \mu(x) \geq t \) (resp. \( \mu(x) + t > 1 \)), and in this case, \( x_t \) is said to belong to (resp. be quasi-coincident with) a fuzzy set \( \mu \). To say that \( x_t \in \lor q \mu \) (resp. \( x_t \in \land q \mu \)), we mean \( x_t \in \mu \) or \( x_t \in q \mu \) (resp. \( x_t \in \mu \) and \( x_t q \mu \)). To say that \( x_t \in \alpha \mu \), we mean \( x_t \alpha \mu \) does not hold, where \( \alpha \in \{ \in, q, \in \lor, \in \land \} \).

A fuzzy set \( \mu \) in a \( BCK/BCI \)-algebra \( X \) is called a fuzzy subalgebra of \( X \) if it satisfies:

\[
(2.1) \quad \mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}
\]

for all \( x, y \in X \).

**Proposition 2.1 ([4])**. Let \( X \) be a \( BCK/BCI \)-algebra. A fuzzy set \( \mu \) in \( X \) is a fuzzy subalgebra of \( X \) if and only if the following assertion is valid.

\[
(2.2) \quad x_t \in \mu, \ y_s \in \mu \implies (x \ast y)_{\min\{t, s\}} \in \mu
\]

for all \( x, y \in X \) and \( t, s \in (0, 1] \).

### 3. Classifications of \( (\alpha, \beta) \)-fuzzy subalgebras

In what follows, let \( X \) denote a \( BCK/BCI \)-algebra unless otherwise specified.

**Definition 3.1 ([4])**. A fuzzy set \( \mu \) in \( X \) is said to be an \( (\alpha, \beta) \)-fuzzy subalgebra of \( X \), where \( \alpha \neq \in \land q \), if it satisfies the following condition:

\[
(3.1) \quad x_{t_1} \alpha \mu, \ y_{t_2} \alpha \mu \implies (x \ast y)_{\min\{t_1, t_2\}} \beta \mu
\]

for all \( x, y \in X \) and \( t_1, t_2 \in (0, 1] \).
In Definition 3.1, if \( t_1 = t_2 = t \) then we say that \( \mu \) is an \((\alpha, \beta)^t\)-fuzzy subalgebra of \( X \) over \((0, 1]\). In considering \((\alpha, \beta)\)-fuzzy subalgebras in \( X \), we have twelve different types of such structures, that is, \((\alpha, \beta)\) is any one of \((\in, \in), (\in, q), (\in, \in \land q), (\in, \in \lor q), (q, \in), (q, q), (q, q), (q, q), (q, \in \land q), (\in \lor q, q), (\in \lor q, q), (\in \lor q, \in \land q), \) and \((\in \lor q, \in \lor q)\). Clearly, we have relations among these types which are described in the following theorems.

**Theorem 3.2.** We have the following relations:

\[
\begin{array}{c}
(\in, \in) \leftrightarrow (\in, \in \land q) \rightarrow (\in, q) \\
\downarrow \\
(\in, \in \lor q)
\end{array}
\]

and

\[
\begin{array}{c}
(\in \lor q, \in) \leftrightarrow (\in \lor q, \in \land q) \rightarrow (\in \lor q, q)
\end{array}
\]

\[
(3.2)
\]

\[
\begin{array}{c}
(q, \in) \leftrightarrow (q, \in \land q) \rightarrow (q, q) \\
\downarrow \\
(q, \in \lor q)
\end{array}
\]

\[
(3.3)
\]

**Theorem 3.3.** If there exists \( x \in X \) such that \( \mu(x) > 0.5 \), then we have the following relation:

\[
\begin{array}{c}
(\in \land q, \in) \leftrightarrow (\in \land q, \in \land q) \rightarrow (\in \land q, q) \\
\downarrow \\
(\in \land q, \in \lor q)
\end{array}
\]

\[
(3.4)
\]
Remark 3.4. In general, it is not true that \((\in, \in \lor q)\)-type implies \((\in \lor q, \in \lor q)\)-type (see [4, Example 3.2]).

We investigate relations between \((\in, \in \lor q)\)-fuzzy subalgebras and \((q, \in \lor q)\)-fuzzy subalgebras.

Theorem 3.5. Every \((q, \in \lor q)\)-fuzzy subalgebra is an \((\in, \in \lor q)\)-fuzzy subalgebra.

Proof. Let \(\mu\) be a \((q, \in \lor q)\)-fuzzy subalgebra of \(X\) and let \(x, y \in X\) and \(t_1, t_2 \in (0, 1]\) be such that \(x_{t_1} \in \mu\) and \(y_{t_2} \in \mu\). Then \(\mu(x) \geq t_1\) and \(\mu(y) \geq t_2\). Suppose

\[
(x \ast y)_{\min\{t_1, t_2\}} \in \lor q \mu.
\]

Then

\[
\mu(x \ast y) < \min\{t_1, t_2\}, \tag{3.5}
\]

\[
\mu(x \ast y) + \min\{t_1, t_2\} \leq 1. \tag{3.6}
\]

It follows that

\[
\mu(x \ast y) < 0.5. \tag{3.7}
\]

Combining (3.5) and (3.7), we have

\[
\mu(x \ast y) < \min\{t_1, t_2, 0.5\}
\]

and so

\[
1 - \mu(x \ast y) > 1 - \min\{t_1, t_2, 0.5\}
= \max\{1 - t_1, 1 - t_2, 0.5\}
\geq \max\{1 - \mu(x), 1 - \mu(y), 0.5\}.
\]

Hence there exists \(\delta \in (0, 1]\) such that

\[
1 - \mu(x \ast y) \geq \delta > \max\{1 - \mu(x), 1 - \mu(y), 0.5\}. \tag{3.8}
\]

From the right inequality in (3.8), we have \(\mu(x) + \delta > 1\) and \(\mu(y) + \delta > 1\), that is, \(x_\delta q \mu\) and \(y_\delta q \mu\). Since \(\mu\) is a \((q, \in \lor q)\)-fuzzy subalgebra of \(X\),
it follows that \((x \ast y)_\delta \in \vee q \mu\). But, from the left inequality in (3.8), we get 
\[
\mu(x \ast y) + \delta \leq 1,
\]
that is, \((x \ast y)_\delta \in q \mu\), and \(\mu(x \ast y) \leq 1 - \delta < 1 - 0.5 = 0.5 < \delta\), i.e., \((x \ast y)_\delta \in \overline{q} \mu\). Hence \((x \ast y)_\delta \in \overline{q} \mu\), a contradiction. Therefore 
\[
(x \ast y)_{\min(t_1, t_2)} \in \vee q \mu,
\]
and thus \(\mu\) is an \((\in, \in \vee q)\)-fuzzy subalgebra of \(X\).

Combining Theorem 3.5 and (3.3) in Theorem 3.2, we have the following relations.

\[
\begin{align*}
(q, \in) & \quad (q, \in \wedge) \quad (q, q) \\
\downarrow & \quad \downarrow & \quad \downarrow \\
(q, \in \vee) & \quad (\in, \in \vee q) \\
\downarrow & \quad \\
(\in, \in \vee q)
\end{align*}
\]

(3.9)

In general, an \((\in, \in \vee q)\)-fuzzy subalgebra may not be a \((q, \in \vee q)\)-fuzzy subalgebra (see [4, Example 3.2]).

We provide conditions for an \((\in, \in \vee q)\)-fuzzy subalgebra to be a \((q, \in \vee q)\)-fuzzy subalgebra.

**Theorem 3.6.** Assume that every fuzzy point has the value \(t\) in (0, 0.5]. Then every \((\in, \in \vee q)\)-fuzzy subalgebra of \(X\) is a \((q, \in \vee q)\)-fuzzy subalgebra of \(X\).

**Proof.** Let \(\mu\) be an \((\in, \in \vee q)\)-fuzzy subalgebra of \(X\). Let \(x, y \in X\) and \(t_1, t_2 \in (0, 0.5]\) be such that \(x_{t_1} q \mu\) and \(y_{t_2} q \mu\). Then \(\mu(x) + t_1 > 1\) and \(\mu(y) + t_2 > 1\). Since \(t_1, t_2 \in (0, 0.5]\), it follows that \(\mu(x) > 1 - t_1 \geq t_1\) and \(\mu(y) > 1 - t_2 \geq t_2\), that is, \(x_{t_1} \in \mu\) and \(y_{t_2} \in \mu\). It follows from (3.1) that \((x \ast y)_{\min(t_1, t_2)} \in \vee q \mu\). Therefore \(\mu\) is a \((q, \in \vee q)\)-fuzzy subalgebra of \(X\).

\(\square\)

**Corollary 3.7.** Let \(\mu\) be an \((\alpha, \beta)\)-fuzzy subalgebra of \(X\) where \((\alpha, \beta)\) is any one of \((\in, \in), (\in, q), (\in, \in \wedge q), (\in \vee q, \in), (\in \vee q, q), (\in \vee q,\)
Classifications of \((\alpha,\beta)-fuzzy\) subalgebras of \(BCK/BCI\)-algebras

If every fuzzy point has the value \(t\) in \((0,0.5]\), then \(\mu\) is a \((q,\in \lor q)\)-fuzzy subalgebra of \(X\).

**Proof.** It follows from Theorems 3.2 and 3.6.

For a fuzzy set \(\mu\) in \(X\) and \(t \in (0,1]\), consider the \(q\)-set and \(\in \lor q\)-set with respect to \(t\) (briefly, \(t\)-\(q\)-set and \(t\)-\(\in \lor q\)-set, respectively) as follows:

\[
X^t_q := \{x \in X \mid x_t q \mu\} \quad \text{and} \quad X^t_{\in \lor q} := \{x \in X \mid x_t \in \lor q \mu\}.
\]

Note that, for any \(t, r \in (0,1]\), if \(t \geq r\) then every \(r\)-\(q\)-set is contained in the \(t\)-\(q\)-set, that is, \(X^r_q \subseteq X^t_q\), and obviously, \(X^t_{\in \lor q} = U(\mu; t) \cup X^t_q\).

**Theorem 3.8.** If \(\mu\) is an \((\in,\in)\)-fuzzy subalgebra of \(X\), then the \(t\)-\(q\)-set \(X^t_q\) is a subalgebra of \(X\) for all \(t \in (0,1]\) whenever it is nonempty.

**Proof.** Let \(x, y \in X^t_q\). Then \(x_t q \mu\) and \(y_t q \mu\), that is, \(\mu(x) + t > 1\) and \(\mu(y) + t > 1\). It follows that

\[
\mu(x * y) + t \geq \min\{\mu(x), \mu(y)\} + t
\]

\[
= \min\{\mu(x) + t, \mu(y) + t\} > 1
\]

and so that \((x*y)_t q \mu\). Hence \(x*y \in X^t_q\), and therefore \(X^t_q\) is a subalgebra of \(X\). \(\square\)

**Definition 3.9.** A fuzzy set \(\mu\) in \(X\) is said to be an \((\in,q)_{\text{max}}\)-fuzzy subalgebra of \(X\) over \((0,1]\) if it satisfies the following condition:

\[
(3.10) \quad x_{t_1} \in \mu, \ y_{t_2} \in \mu \Rightarrow (x * y)_{\text{max}\{t_1,t_2\}} q \mu.
\]

for all \(x, y \in X\) and \(t_1, t_2 \in (0,1]\).

Obviously, every \((\in,q)\)-fuzzy subalgebra is an \((\in,q)_{\text{max}}\)-fuzzy subalgebra over \((0,1]\).

**Theorem 3.10.** For a fuzzy set \(\mu\) in \(X\), if the nonempty \(t\)-\(q\)-set \(X^t_q\) is a subalgebra of \(X\) for all \(t \in (0.5,1]\), then \(\mu\) is an \((\in,q)_{\text{max}}\)-fuzzy subalgebra of \(X\) over \((0.5,1]\).
Proof. Let \( x, y \in X \) and \( t_1, t_2 \in (0, 1] \) be such that \( x_{t_1} \in \mu \) and \( y_{t_2} \in \mu \). Then \( \mu(x) \geq t_1 > 1 - t_1 \) and \( \mu(y) \geq t_2 > 1 - t_2 \), that is, \( x_{t_1}, q \mu \) and \( y_{t_2}, q \mu \). It follows that \( x, y \in X_q^{\max\{t_1, t_2\}} \) and \( \max\{t_1, t_2\} \in (0, 1] \). By hypothesis, we have \( x \ast y \in X_q^{\max\{t_1, t_2\}} \) and so \( (x \ast y)^{\max\{t_1, t_2\}} q \mu \). Therefore \( \mu \) is an \((\epsilon, q)^{\max}\)-fuzzy subalgebra of \( X \) over \((0, 1] \). \( \square \)

Corollary 3.11. For a fuzzy set \( \mu \) in \( X \), if the nonempty \( t\)-\( q\)-set \( X_q^t \) is a subalgebra of \( X \) for all \( t \in (0, 1] \), then \( \mu \) is an \((\epsilon, q)^t\)-fuzzy subalgebra of \( X \) over \((0, 1] \).

Definition 3.12. A fuzzy set \( \mu \) in \( X \) is said to be a \((q, \epsilon)^{\max}\)-fuzzy subalgebra of \( X \) over \((0, 1] \) if it satisfies the following condition:

\[
(3.11) \quad x_{t_1}, q \mu, \ y_{t_2}, q \mu \Rightarrow (x \ast y)^{\max\{t_1, t_2\}} \in \mu.
\]

for all \( x, y \in X \) and \( t_1, t_2 \in (0, 1] \).

Obviously, every \((q, \epsilon)^{\max}\)-fuzzy subalgebra over \((0, 1] \) is a \((q, \epsilon)^t\)-fuzzy subalgebra.

Theorem 3.13. For a fuzzy set \( \mu \) in \( X \), if the nonempty \( t\)-\( q\)-set \( X_q^t \) is a subalgebra of \( X \) for all \( t \in (0, 0.5] \), then \( \mu \) is a \((q, \epsilon)^{\max}\)-fuzzy subalgebra of \( X \) over \((0, 0.5] \).

Proof. Let \( x, y \in X \) and \( t_1, t_2 \in (0, 0.5] \) be such that \( x_{t_1}, q \mu \) and \( y_{t_2}, q \mu \). Then \( x \in X_q^{t_1} \) and \( y \in X_q^{t_2} \). It follows that \( x, y \in X_q^{\max\{t_1, t_2\}} \) and \( \max\{t_1, t_2\} \in (0, 0.5] \). Thus \( x \ast y \in X_q^{\max\{t_1, t_2\}} \). Hence \( \mu(x \ast y) + \max\{t_1, t_2\} > 1 \), and so \( \mu(x \ast y) > 1 - \max\{t_1, t_2\} \geq \max\{t_1, t_2\} \). Therefore \( (x \ast y)^{\max\{t_1, t_2\}} \in \mu \). Consequently, \( \mu \) is a \((q, \epsilon)^{\max}\)-fuzzy subalgebra of \( X \) over \((0, 0.5] \). \( \square \)

Corollary 3.14. Let \( \mu \) be a fuzzy set in \( X \) and \( t \in (0, 0.5] \). If the \( t\)-\( q\)-set \( X_q^t \) is a subalgebra of \( X \), then \( \mu \) is a \((q, \epsilon)^t\)-fuzzy subalgebra of \( X \) over \((0, 0.5] \).

Theorem 3.15. If \( \mu \) is a \((q, \epsilon \lor q)^t\)-fuzzy subalgebra of \( X \), then the \( t\)-\( q\)-set \( X_q^t \) is a subalgebra of \( X \) for all \( t \in (0.5, 1] \) whenever it is nonempty.
Proof. Let \( x, y \in X_t^t \). Then \( x_t q\mu \) and \( y_t q\mu \). Since \( \mu \) is a \((q, \in \vee q)\)-fuzzy subalgebra of \( X \), we have \((x * y)_t \in \vee q\mu\), that is, \((x * y)_t \in \mu\) or \((x * y)_t q\mu\). If \((x * y)_t q\mu\), then \( x * y \in X_t^t \). If \((x * y)_t \in \mu\), then \( \mu(x * y) \geq t > 1 - t \) since \( t > 0.5 \). Hence \((x * y)_t q\mu\), and so \( x * y \in X_t^t \).

Therefore \( X_t^t \) is a subalgebra of \( X \).

\[ \Box \]

**Corollary 3.16.** If \( \mu \) is an \((\alpha, \beta)\)-fuzzy subalgebra of \( X \) where \((\alpha, \beta)\) is one of \((q, \in \vee q), (q, q)\) and \((q, \in \wedge q)\), then the \( t\)-\( q\)-set \( X_t^t \) is a subalgebra of \( X \) for all \( t \in (0.5, 1] \) whenever it is nonempty.

**Lemma 3.17 ([4]).** For a subalgebra \( S \) of \( X \), let \( \mu \) be a fuzzy set in \( X \) such that

1. \( \mu(x) \geq 0.5 \) for all \( x \in S \),
2. \( \mu(x) = 0 \) for all \( x \in X \setminus S \).

Then \( \mu \) is a \((q, \in \vee q)\)-fuzzy subalgebra of \( X \).

Using Theorem 3.15 and Lemma 3.17, we have the following result.

**Theorem 3.18.** For a subalgebra \( S \) of \( X \), if \( \mu \) is a fuzzy set in \( X \) such that

1. \( \mu(x) \geq 0.5 \) for all \( x \in S \),
2. \( \mu(x) = 0 \) for all \( x \in X \setminus S \),

then the nonempty \( t\)-\( q\)-set \( X_t^t \) is a subalgebra of \( X \) for all \( t \in (0.5, 1] \).

**Definition 3.19.** A fuzzy set \( \mu \) in \( X \) is said to be a \((q, \in \vee q)^{\text{max}}\)-fuzzy subalgebra of \( X \) over \((0, 1]\) if it satisfies the following condition:

\[
(3.12) \quad x_t q\mu, y_t q\mu \Rightarrow (x * y)_{\text{max}(t_1, t_2)} \subseteq \vee q\mu.
\]

for all \( x, y \in X \) and \( t_1, t_2 \in (0, 1] \).

**Theorem 3.20.** For a fuzzy set \( \mu \) in \( X \), if the nonempty \( t\)-\( \in \vee q\)-set \( X_{t \vee q}^t \) is a subalgebra of \( X \) for all \( t \in (0, 1] \), then \( \mu \) is a \((q, \in \vee q)^{\text{max}}\)-fuzzy subalgebra of \( X \) over \((0, 1]\).
Proof. Let \( x, y \in X \) and \( t_1, t_2 \in (0, 1] \) be such that \( x_{t_1} q \mu \) and \( y_{t_2} q \mu \). Then \( x \in X_{t_1}^q \subseteq X_{q \oplus q}^{t_1} \) and \( y \in X_{t_2}^q \subseteq X_{q \oplus q}^{t_2} \). It follows that \( x, y \in X_{q \oplus q}^{\max\{t_1, t_2\}} \) and so from the hypothesis that \( x * y \in X_{q \oplus q}^{\max\{t_1, t_2\}} \). Hence \( (x * y)_{\max\{t_1, t_2\}} \in q \mu \). Therefore \( \mu \) is a \((q, \in \oplus q)\)-max-fuzzy subalgebra of \( X \) over \((0, 1]\).

Lemma 3.21 ([4]). A fuzzy set \( \mu \) in \( X \) is an \((\in, \in \oplus q)\)-fuzzy subalgebra of \( X \) if and only if it satisfies:

(3.13) \((\forall x, y \in X)(\mu(x * y) \geq \min\{\mu(x), \mu(y), 0.5\})\).

Theorem 3.22. If \( \mu \) is an \((\in, \in \oplus q)\)-fuzzy subalgebra of \( S \), then the nonempty \( t-q \)-set \( S_{\mu}^{t} \) is a subalgebra of \( S \) for all \( t \in (0.5, 1] \).

Proof. Assume that \( S_{\mu}^{t} \neq \emptyset \) for all \( t \in (0.5, 1] \). Let \( x, y \in S_{\mu}^{t} \). Then \( x_{t} q \mu \) and \( y_{t} q \mu \), that is, \( \mu(x) + t > 1 \) and \( \mu(y) + t > 1 \). It follows from Lemma 3.21 that

\[
\mu(x * y) + t \geq \min\{\mu(x), \mu(y), 0.5\} + t \\
= \min\{\mu(x) + t, \mu(y) + t, 0.5 + t\} \\
> 1.
\]

So \( (x * y)_{t} q \mu \). Hence \( x * y \in S_{\mu}^{t} \), and therefore \( S_{\mu}^{t} \) is a subalgebra of \( S \).

Theorem 3.23. For a fuzzy set \( \mu \) in \( X \), if the nonempty \( t-q \)-set \( X_{\in \oplus q}^{t} \) is a subalgebra of \( X \) for all \( t \in (0, 1] \), then \( \mu \) is an \((\in, \in \oplus q)\)-fuzzy subalgebra of \( X \).

Proof. Let \( \mu \) be a fuzzy set in \( X \) and \( t \in (0, 1] \) be such that \( X_{\in \oplus q}^{t} \) is a subalgebra of \( X \). If possible, let

\[
\mu(x * y) < t \leq \min\{\mu(x), \mu(y), 0.5\}
\]

for some \( t \in (0, 0.5) \) and \( x, y \in X \). Then \( x, y \in U(\mu; t) \subseteq X_{\in \oplus q}^{t} \), which implies that \( x * y \in X_{\in \oplus q}^{t} \). Hence \( \mu(x * y) \geq t \) or \( \mu(x * y) + t > 1 \), a
contradiction. Therefore
\[\mu(x \ast y) \geq \min\{\mu(x), \mu(y), 0.5\}\]
for all \(x, y \in X\). It follows from Lemma 3.21 that \(\mu\) is an \((\in, \in \lor q)\)-fuzzy subalgebra of \(X\).

**Theorem 3.24.** If \(\mu\) is an \((\in, \in \lor q)\)-fuzzy subalgebra of \(X\), then the nonempty \(t\)-\(\in \lor q\)-set \(X^t_{\in \lor q}\) is a subalgebra of \(X\) for all \(t \in (0, 0.5]\).

*Proof.* Assume that \(X^t_{\in \lor q} \neq \emptyset\) for all \(t \in (0, 0.5]\) and let \(x, y \in X^t_{\in \lor q}\). Then \(x_t \in \lor q \mu\) and \(y_t \in \lor q \mu\). Hence we have the following four cases:

(i) \(x_t \in \mu\) and \(y_t \in \mu\),
(ii) \(x_t \in \mu\) and \(y_t \in q \mu\),
(iii) \(x_t \in q \mu\) and \(y_t \in \mu\),
(iv) \(x_t \in q \mu\) and \(y_t \in q \mu\).

Case (i) implies \((x \ast y)_t \in \lor q \mu\) and so \(x \ast y \in X^t_{\in \lor q}\). For the second case, \(y_t \in q \mu\) induces \(\mu(y) > 1 - t \geq t\), that is, \(y_t \in \mu\). Hence \((x \ast y)_t \in \lor q \mu\) and so \(x \ast y \in X^t_{\in \lor q}\). Similarly, the third case implies \(x \ast y \in X^t_{\in \lor q}\). The last case induces \(\mu(x) > 1 - t \geq t\) and \(\mu(y) > 1 - t \geq t\), that is, \(x_t \in \mu\) and \(y_t \in \mu\). It follows that \((x \ast y)_t \in \lor q \mu\) and so that \(x \ast y \in X^t_{\in \lor q}\). Therefore \(X^t_{\in \lor q}\) is a subalgebra of \(X\) for all \(t \in (0, 0.5]\).

**Theorem 3.25.** If \(\mu\) is a \((q, \in \lor q)\)-fuzzy subalgebra of \(X\), then the nonempty \(t\)-\(\in \lor q\)-set \(X^t_{\in \lor q}\) is a subalgebra of \(X\) for all \(t \in (0.5, 1]\).

*Proof.* Assume that \(X^t_{\in \lor q} \neq \emptyset\) for all \(t \in (0.5, 1]\) and let \(x, y \in X^t_{\in \lor q}\). Then \(x_t \in \lor q \mu\) and \(y_t \in \lor q \mu\). Hence we have the following four cases:

(i) \(x_t \in \mu\) and \(y_t \in \mu\),
(ii) \(x_t \in \mu\) and \(y_t \in q \mu\),
(iii) \(x_t \in q \mu\) and \(y_t \in \mu\),
(iv) \(x_t \in q \mu\) and \(y_t \in q \mu\).

For the first case, we have \(\mu(x) + t \geq 2t > 1\) and \(\mu(y) + t \geq 2t > 1\), that is, \(x_t \in q \mu\) and \(y_t \in q \mu\). It follows that \((x \ast y)_t \in \lor q \mu\) and so that
In the case (ii), $x_t \in \mu$ implies $\mu(x) + t \geq 2t > 1$, that is, $x_t \in q \mu$. Hence $(x * y)_t \in \vee q \mu$ and so $x * y \in X'_{\vee q}$. Similarly, the third case implies $x * y \in X'_{\vee q}$. The last case implies $(x * y)_t \in \vee q \mu$ and so $x * y \in X'_{\vee q}$. Consequently, $t \in q$-set $X'_{\vee q}$ is a subalgebra of $X$ for all $t \in (0.5, 1]$.

Acknowledgments

The authors wish to thank the anonymous reviewers for their valuable suggestions. The first author, Young Bae Jun, is an Executive Research Worker of Educational Research Institute of Teachers College in Gyeongsang National University.

References


Young Bae Jun

Department of Mathematics Education, Gyeongsang National University,
Classifications of $(\alpha, \beta)$-fuzzy subalgebras of $BCK/BCI$-algebras

Jinju 660-701, Korea.
E-mail: skywine@gmail.com

Sun Shin Ahn
Department of Mathematics Education, Dongguk University,
Seoul 100-715, Korea.
E-mail: sunshine@dongguk.edu

Kyoung Ja Lee
Department of Mathematics Education, Hannam University,
Daejeon 306-791, Korea.
E-mail: lsj1109@hotmail.com